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New Generating Functions for a Class of Generalized Hermite Polynomials

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The main object of this paper is to present several (presumably new) families of linear, bilinear, and mixed multilateral generating functions for a certain interesting generalization of the classical Hermite (and Laguerre) polynomials. Some of these generating functions are associated with the Stirling numbers of the second kind. Numerous known or new consequences of the results derived here also considered.

Key Words: generating functions; Hermite polynomials; Laguerre polynomials; Stirling numbers; Rodrigues formula; Lagrange expansion.

1. INTRODUCTION, DEFINITIONS, AND PRELIMINARIES

A considerably large number of special functions (including, for example, all of the classical orthogonal polynomials) are known to possess generating functions which fit easily into the Singhal–Srivastava generating

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function [5, p. 755, Eq. (1)]:

$$\sum_{k=0}^{\infty} A_{n,k}\ S_{n+k}(x)t^k = f(x,t)\{g(x,t)\}^{-n}\ S_n(h(x,t))$$
$$(n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}; \mathbb{N} := \{1,2,3,\ldots\}), \tag{1.1}$$

where the coefficients $A_{n,k}$ are constants, real or complex, and f, g, h are suitable functions of x and t.

Recently, Srivastava [6] investigated a widely applicable *special* case of the Singhal–Srivastava generating function (1.1) when

$$A_{n,k} = \binom{n+k}{k} \qquad (n, k \in \mathbb{N}_0). \tag{1.2}$$

In fact, for the sequence $\{\mathscr{S}_n(x)\}_{n=0}^{\infty}$ generated by

$$\sum_{k=0}^{\infty} \binom{n+k}{k} \mathscr{S}_{n+k}(x)t^k = f(x,t)\{g(x,t)\}^{-n}\ \mathscr{S}_n(h(x,t)) \quad (n \in \mathbb{N}_0), \tag{1.3}$$

he derived the following general result on generating functions associated with the Stirling numbers $S(n,k)$ of the second kind, defined by

$$S(n,k) := \frac{1}{k!}\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j} j^n, \tag{1.4}$$

so that

$$S(n,0) = \begin{cases} 1 & (n = 0) \\ 0 & (n \in \mathbb{N}) \end{cases} \tag{1.5}$$

and

$$S(n,1) = S(n,n) = 1 \qquad \text{and} \qquad S(n,n-1) = \binom{n}{2}. \tag{1.6}$$

THEOREM 1 (Srivastava [6, p. 754, Theorem 1]). *Let the sequence* $\{\mathscr{S}_n(x)\}_{n=0}^{\infty}$ *be generated by* (1.3).

Then, in terms of the Stirling numbers $S(n,k)$ *defined by* (1.4), *the following family of generating functions holds true*:

$$\sum_{k=0}^{\infty} k^n \mathscr{S}_k(h(x,-z))\left(\frac{z}{g(x,-z)}\right)^k$$
$$= \{f(x,-z)\}^{-1} \sum_{k=0}^{n} k!\ S(n,k)\mathscr{S}_k(x)z^k \qquad (n \in \mathbb{N}_0), \tag{1.7}$$

provided that each member of (1.7) *exists*.

Srivastava [6] applied Theorem 1 (*as well as* its multivariable extension also given by him [6, p. 765, Theorem 2]) in order to derive generating functions of the class (1.7) for a remarkably large number of special functions and polynomials in one, two, and more variables. In this sequel to the work of Srivastava [6], we aim at presenting analogous and other families of linear, bilinear, and mixed multilateral generating functions for a certain generalization of the classical Hermite (and classical Laguerre) polynomials, defined by Gould and Hopper [2] by means of the following Rodrigues formula [2, p. 52, Eq. (2.1)]:

$$H_n^r(x, \alpha, \beta) := (-1)^n x^{-\alpha} \exp(\beta x^r) D_x^n\{x^\alpha \exp(-\beta x^r)\}$$
$$\left(D_x := \frac{d}{dx}\right) \tag{1.8}$$

or, alternatively, by means of the following generating function [2, p. 54, Eq. (3.8)]:

$$\sum_{k=0}^{\infty} H_k^r(x, \alpha, \beta) \frac{t^k}{k!} = \left(1 - \frac{t}{x}\right)^\alpha \exp\left(\beta x^r \left[1 - \left(1 - \frac{t}{x}\right)^r\right]\right)$$
$$(|t| < |x|), \tag{1.9}$$

where the parameters α, β, and r are unrestricted, in general. Here $H_n^r(x, \alpha, \beta)$ is a polynomial of degree n in x^{r-1} (*and* also in α). Obviously, the definition (1.8) would yield the following relationships with the classical Hermite and the classical Laguerre polynomials (cf., e.g., [8, Chap. 5]):

$$H_n^2(x, 0, 1) = H_n(x) := (-1)^n \exp(x^2) D_x^n\{\exp(-x^2)\}, \tag{1.10}$$

$$\frac{(-x)^n}{n!} H_n^1(x, \alpha + n, 1) = L_n^{(\alpha)}(x) := \frac{x^{-\alpha} e^x}{n!} D_x^n\{x^{\alpha+n} e^{-x}\}, \tag{1.11}$$

and

$$\frac{(-x)^n}{n!} H_n^1(x, \alpha, 1) = L_n^{(\alpha-n)}(x) := \frac{x^{n-\alpha} e^x}{n!} D_x^n\{x^\alpha e^{-x}\}, \tag{1.12}$$

each of which will be needed in our present investigation.

2. MIXED GENERATING FUNCTIONS

We begin by considering the generating function:

$$\Delta_n^{(\lambda, \mu)}(x, t) := \sum_{k=0}^{\infty} H_{n+k}^r(x, \alpha + \lambda k, \beta + \mu k) \frac{\{t x^\lambda \exp(-\mu x^r)\}^k}{k!}$$
$$(\lambda, \mu \in \mathbb{C}; n \in \mathbb{N}_0), \tag{2.1}$$

which, in view of the definition (1.8), yields

$$\Delta_n^{(\lambda,\mu)}(x,t) = (-1)^n x^{-\alpha} \exp(\beta x^r) \cdot D_x^n \sum_{k=0}^{\infty} D_x^k \Big\{ x^{\alpha+\lambda k} \exp(-[\beta+\mu k]x^r) \Big\} \frac{(-t)^k}{k!}. \tag{2.2}$$

By appealing appropriately to the familiar Lagrange expansion (cf. [9, p. 133]; see also [7, p. 354, Eq. 7.1 (3)]):

$$\sum_{k=0}^{\infty} D_x^k \Big\{ f(x)[\varphi(x)]^k \Big\} \frac{t^k}{k!} = \frac{f(\zeta)}{1 - t\varphi'(\zeta)} \qquad (\zeta = x + t\varphi(\zeta);\ \varphi(x) \neq 0), \tag{2.3}$$

we find from (2.2) that

$$\Delta_n^{(\lambda,\mu)}(x,t) = (-1)^n x^{-\alpha} \exp(\beta x^r) \cdot D_x^n \left\{ \frac{\zeta^\alpha \exp(-\beta\zeta^r)}{1 + t\zeta^{\lambda-1}(\lambda - \mu r \zeta^r)\exp(-\mu\zeta^r)} \right\} \qquad (\zeta = x - t\zeta^\lambda \exp(-\mu\zeta^r); n \in \mathbb{N}_0), \tag{2.4}$$

which, for

$$t \longmapsto t x^{-\lambda} \exp(\mu x^r),$$

reduces finally to the form:

$$\sum_{k=0}^{\infty} H_{n+k}^r(x, \alpha+\lambda k, \beta+\mu k) \frac{t^k}{k!} = (-1)^n x^{-\alpha} \exp(\beta x^r) \cdot D_x^n \left\{ \frac{\zeta^\alpha \exp(-\beta\zeta^r)}{1 + (x-\zeta)\zeta^{-1}(\lambda - \mu r\zeta^r)} \right\}$$
$$\left(n \in \mathbb{N}_0; \lambda, \mu \in \mathbb{C}; \zeta = x - t\left(\frac{\zeta}{x}\right)^\lambda \exp(\mu[x^r - \zeta^r]) \right). \tag{2.5}$$

In its special case when $n = 0$, the generating function (2.5) immediately yields

$$\sum_{k=0}^{\infty} H_k^r(x, \alpha+\lambda k, \beta+\mu k) \frac{t^k}{k!} = \frac{(\zeta/x)^\alpha \exp(\beta[x^r - \zeta^r])}{1 + (x-\zeta)\zeta^{-1}(\lambda - \mu r\zeta^r)}$$
$$\left(\lambda, \mu \in \mathbb{C}; \zeta = x - t\left(\frac{\zeta}{x}\right)^\lambda \exp(\mu[x^r - \zeta^r]) \right), \tag{2.6}$$

which, for $\zeta \longmapsto \zeta x$, assumes the *simpler* form:

$$\sum_{k=0}^{\infty} H_k^r(x, \alpha + \lambda k, \beta + \mu k)\frac{t^k}{k!} = \frac{\zeta^\alpha \exp(\beta x^r[1 - \zeta^r])}{1 + (1 - \zeta)\zeta^{-1}(\lambda - \mu r\zeta^r x^r)}$$

$$\left(\lambda, \mu \in \mathbb{C}; \zeta = 1 - \left(\frac{t}{x}\right)\zeta^\lambda \exp(\mu x^r[1 - \zeta^r])\right). \tag{2.7}$$

The familiar generating function (1.9) is an obvious *further* special case of (2.7) when

$$\lambda = \mu = 0.$$

Next we consider the following particular case of our result (2.5) when $\mu = 0$:

$$\sum_{k=0}^{\infty} H_{n+k}^r(x, \alpha + \lambda k, \beta)\frac{t^k}{k!} = (-1)^n x^{-\alpha} \exp(\beta x^r) D_x^n \left\{\frac{\zeta^\alpha \exp(-\beta\zeta^r)}{1 + \lambda(x - \zeta)\zeta^{-1}}\right\}$$

$$\left(n \in \mathbb{N}_0; \lambda \in \mathbb{C}; \zeta = x - t\left(\frac{\zeta}{x}\right)^\lambda\right), \tag{2.8}$$

which, for $t \longmapsto tx^\lambda$, becomes

$$\sum_{k=0}^{\infty} H_{n+k}^r(x, \alpha + \lambda k, \beta)\frac{(x^\lambda t)^k}{k!} = (-1)^n x^{-\alpha} \exp(\beta x^r) D_x^n \left\{\frac{\zeta^\alpha \exp(-\beta\zeta^r)}{1 + \lambda(x - \zeta)\zeta^{-1}}\right\}$$

$$(n \in \mathbb{N}_0; \lambda \in \mathbb{C}; \zeta = x - t\zeta^\lambda). \tag{2.9}$$

Since

$$x - \zeta = t\zeta^\lambda$$

and

$$D_x = \frac{d\zeta}{dx} D_\zeta = \frac{1}{1 + \lambda t\zeta^{\lambda-1}} D_\zeta,$$

we can rewrite (2.9) in the form:

$$\sum_{k=0}^{\infty} H_{n+k}^r(x, \alpha + \lambda k, \beta)\frac{(x^\lambda t)^k}{k!} = (-1)^n x^{-\alpha} \exp(\beta x^r) \left(\frac{1}{1 + \lambda t\zeta^{\lambda-1}} D_\zeta\right)^n \left\{\frac{\zeta^\alpha \exp(-\beta\zeta^r)}{1 + \lambda t\zeta^{\lambda-1}}\right\}$$

$$\left(n \in \mathbb{N}_0; \lambda \in \mathbb{C}; \zeta = x - t\zeta^\lambda\right). \tag{2.10}$$

Two *further* special cases of the generating function (2.10) are worthy of note. First of all, a special case of (2.10) when $\lambda = 0$ yields the *known* generating function [2, p. 57, Eq. (5.3)]:

$$\sum_{k=0}^{\infty} H_{n+k}^{r}(x, \alpha, \beta)\frac{t^k}{k!} = \left(1-\frac{t}{x}\right)^{\alpha} \exp\left(\beta x^r\left[1-\left(1-\frac{t}{x}\right)^{r}\right]\right) \cdot H_n^r(x-t, \alpha, \beta) \quad (n \in \mathbb{N}_0; |t| < |x|). \tag{2.11}$$

On the other hand, (2.10) *with* $\lambda = 1$ (and *with* $t \longmapsto t/x$) furnishes the following (*presumably new*) generating function:

$$\sum_{k=0}^{\infty} H_{n+k}^{r}(x, \alpha+k, \beta)\frac{t^k}{k!} = \left(1+\frac{t}{x}\right)^{-\alpha-n-1} \exp\left(\beta x^r\left[1-\left(1+\frac{t}{x}\right)^{-r}\right]\right) \cdot H_n^r\left(x\left(1+\frac{t}{x}\right)^{-1}, \alpha, \beta\right) \quad (n \in \mathbb{N}_0; |t| < |x|). \tag{2.12}$$

The above method of derivation of the generating function (2.5), which is based essentially upon the Lagrange expansion (2.3), can indeed be applied *mutatis mutandis* in order to obtain yet another family of mixed generating functions given by

$$\sum_{k=0}^{\infty} H_k^r((x+ky)^{1/r}, \alpha+\lambda k, \beta)\frac{\{t(x+ky)^{1/r}\}^k}{k!} = \frac{(1-\zeta)^{\alpha}\exp(\beta x[1-(1-\zeta)^r])}{1+\zeta(1-\zeta)^{-1}[\lambda - r\beta y(1-\zeta)^r]} \quad \left(r \neq 0; \lambda, y \in \mathbb{C}; \zeta = t(1-\zeta)^{\lambda}\exp(\beta y[1-(1-\zeta)^r])\right), \tag{2.13}$$

which readily yields the generating functions:

$$\sum_{k=0}^{\infty} H_k^r((x+ky)^{1/r}, \alpha, \beta)\frac{\{t(x+ky)^{1/r}\}^k}{k!} = \frac{(1-\zeta)^{\alpha}\exp(\beta x[1-(1-\zeta)^r])}{1-r\beta y\zeta(1-\zeta)^{r-1}} \quad (r \neq 0; y \in \mathbb{C}; \zeta = t\exp(\beta y[1-(1-\zeta)^r])) \tag{2.14}$$

and

$$\sum_{k=0}^{\infty} H_k^r(x, \alpha+\lambda k, \beta)\frac{t^k}{k!} = \frac{(1-\zeta)^{\alpha}\exp(\beta x^r[1-(1-\zeta)^r])}{1+\lambda\zeta(1-\zeta)^{-1}} \quad \left(\lambda \in \mathbb{C}; \zeta = \left(\frac{t}{x}\right)(1-\zeta)^{\lambda}\right) \tag{2.15}$$

when $\lambda = 0$ and $y = 0$, respectively.

This last mixed generating function (2.15) would result also from (2.7) (with, of course, $\mu = 0$) upon making the following simple notational change:

$$\zeta \longmapsto 1-\zeta.$$

Since $\zeta = 0$ when $t = 0$, two *further* special cases of the mixed generating function (2.15) are worthy of note. First of all, if in (2.15) we set $\lambda = -1$, we readily observe that

$$\zeta = \frac{1-\sqrt{1-4(t/x)}}{2} \qquad \text{and} \qquad 1-\zeta = \frac{1+\sqrt{1-4(t/x)}}{2}, \tag{2.16}$$

so that we have

$$\sum_{k=0}^{\infty} H_k^r(x,\alpha-k,\beta)\frac{t^k}{k!} = \frac{1}{\sqrt{1-4(t/x)}}\left(\frac{1+\sqrt{1-4(t/x)}}{2}\right)^{\alpha+1} \cdot \exp\left(\beta x^r\left[1-\left\{\frac{1}{2}\left(1+\sqrt{1-4(t/x)}\right)\right\}^r\right]\right) \left(|t|<\frac{1}{4}|x|;\ x\neq 0\right). \tag{2.17}$$

Similarly, by setting $\lambda = 2$ in (2.15), we obtain the generating function:

$$\sum_{k=0}^{\infty} H_k^r(x,\alpha+2k,\beta)\frac{t^k}{k!} = \frac{1}{\sqrt{1+4(t/x)}}\left(\frac{1+\sqrt{1+4(t/x)}}{2}\right)^{-\alpha} \cdot \exp\left(\beta x^r\left[1-\left\{\frac{1}{2}\left(1+\sqrt{1+4(t/x)}\right)\right\}^{-r}\right]\right) \left(|t|<\frac{1}{4}|x|;\ x\neq 0\right). \tag{2.18}$$

For the classical Laguerre polynomials $L_n^{(\alpha)}(x)$, involved in the relationships (1.11) and (1.12), the special cases of the generating functions (2.17) and (2.18) when

$$r = \beta = 1 \qquad \text{and} \qquad t \longmapsto -xt$$

yield the following (fairly well-known) results:

$$\sum_{k=0}^{\infty} L_k^{(\alpha-2k)}(x)t^k = \frac{1}{\sqrt{1+4t}}\left(\frac{1+\sqrt{1+4t}}{2}\right)^{\alpha+1} \cdot \exp\left(-\frac{2xt}{1+\sqrt{1+4t}}\right) \qquad \left(|t|<\frac{1}{4}\right) \tag{2.19}$$

and

$$\sum_{k=0}^{\infty} L_k^{(\alpha+k)}(x)t^k = \frac{1}{\sqrt{1-4t}}\left(\frac{1+\sqrt{1-4t}}{2}\right)^{-\alpha} \cdot \exp\left(-x\left(\frac{1-\sqrt{1-4t}}{1+\sqrt{1-4t}}\right)\right) \qquad \left(|t| < \frac{1}{4}\right), \tag{2.20}$$

respectively.

More generally, by setting $r = \beta = 1$ in (2.13) and (2.15), and letting $\lambda \longmapsto \lambda + 1$, we obtain Carlitz's generating functions for the classical Laguerre polynomials (cf., e.g., [7, p. 378, Eq. 7.5 (21); p. 375, Eq. 7.5 (1)]):

$$\sum_{k=0}^{\infty} L_k^{(\alpha+\lambda k)}(x+ky)t^k = \frac{(1+\zeta)^{\alpha+1}\exp(-x\zeta)}{1-\zeta[\lambda - y(1+\zeta)]} \qquad (\lambda, y \in \mathbb{C};\ \zeta = t(1+\zeta)^{\lambda+1}\exp(-y\zeta)); \tag{2.21}$$

$$\sum_{k=0}^{\infty} L_k^{(\alpha+\lambda k)}(x)t^k = (1+\zeta)^{\alpha+1}(1-\lambda\zeta)^{-1}\exp(-x\zeta) \qquad (\lambda \in \mathbb{C};\ \zeta = t(1+\zeta)^{\lambda+1}), \tag{2.22}$$

which obviously follows from (2.21) in its special case when $y = 0$.

For the classical Hermite polynomials $H_n(x)$ involved in the relationship (1.10), our result (2.13) with

$$\alpha = \lambda = 0 \qquad \text{and} \qquad r - 1 = \beta = 1 \tag{2.23}$$

immediately yields Srivastava's generating function (cf., e.g., [7, p. 398, Problem 29 (ii)]):

$$\sum_{k=0}^{\infty} H_k(\sqrt{x+ky})\frac{(t\sqrt{x+ky})^k}{k!} = \frac{\exp\bigl(x(2\zeta-\zeta^2)\bigr)}{1-2y\zeta(1-\zeta)} \qquad \bigl(y \in \mathbb{C};\ \zeta = t\exp\bigl(y(2\zeta-\zeta^2)\bigr)\bigr). \tag{2.24}$$

3. GENERATING FUNCTIONS ASSOCIATED WITH THE STIRLING NUMBERS OF THE SECOND KIND

Each of the generating functions (2.11) and (2.12) belongs to the family given by (1.3). Indeed, by comparing (2.11) with (1.3), it is readily seen that

$$f(x,t) = \left(1-\frac{t}{x}\right)^{\alpha}\exp\left(\beta x^r\left[1-\left(1-\frac{t}{x}\right)^r\right]\right),$$

$$g(x,t) = 1, \qquad h(x,t) = x - t,$$

and

$$\mathscr{S}_k(x) \longmapsto \frac{1}{k!} H_k^r(x, \alpha, \beta) \qquad (k \in \mathbb{N}_0).$$

Thus the assertion (1.7) of Theorem 1 leads us to the generating function:

$$\sum_{k=0}^{\infty} \frac{k^n}{k!} H_k^r(x+z, \alpha, \beta) z^k = \left(1+\frac{z}{x}\right)^{-\alpha} \exp\left(-\beta x^r\left[1-\left(1+\frac{z}{x}\right)^r\right]\right) \cdot \sum_{k=0}^{n} S(n,k) H_k^r(x, \alpha, \beta) z^k \quad (n \in \mathbb{N}_0;\ |z| < |x|) \tag{3.1}$$

in terms of the Stirling numbers $S(n, k)$ defined by (1.4).

The other generating function (2.12) would similarly yield

$$\sum_{k=0}^{\infty} \frac{k^n}{k!} H_k^r\left(x\left(1-\frac{z}{x}\right)^{-1}, \alpha+k, \beta\right)\left(\frac{z}{[1-(z/x)]^2}\right)^k = \left(1-\frac{z}{x}\right)^{\alpha+1} \exp\left(-\beta x^r\left[1-\left(1-\frac{z}{x}\right)^{-r}\right]\right) \cdot \sum_{k=0}^{n} S(n, k) H_k^r(x, \alpha+k, \beta) z^k \qquad (n \in \mathbb{N}_0; |z| < |x|), \tag{3.2}$$

which, for $z \longmapsto -zx$, simplifies to the form:

$$\sum_{k=0}^{\infty} \frac{k^n}{k!} H_k^r\left(\frac{x}{1+z}, \alpha+k, \beta\right)\left(-\frac{zx}{(1+z)^2}\right)^k = (1+z)^{\alpha+1} \exp(-\beta x^r[1-(1+z)^{-r}]) \cdot \sum_{k=0}^{n} S(n, k) H_k^r(x, \alpha+k, \beta)(-zx)^k \qquad (n \in \mathbb{N}_0; |z| < 1). \tag{3.3}$$

In view of the relationship (1.10), a special case of the generating function (3.1) when

$$r = 2, \qquad \alpha = 0, \qquad \text{and} \qquad \beta = 1$$

leads us immediately to the following generating function for the classical Hermite polynomials:

$$\sum_{k=0}^{\infty} \frac{k^n}{k!} H_k(x+z) z^k = \exp(2xz+z^2) \sum_{k=0}^{n} S(n,k) H_k(x) z^k \quad (n \in \mathbb{N}_0), \tag{3.4}$$

which, for $x \longmapsto x - z$, yields

$$\sum_{k=0}^{\infty} \frac{k^n}{k!} H_k(x) z^k = \exp(2xz-z^2) \sum_{k=0}^{n} S(n,k) H_k(x-z) z^k \quad (n \in \mathbb{N}_0). \tag{3.5}$$

An obvious special case of this last generating function (3.5) when $n = 0$ corresponds to a classical result (cf., e.g., [8, p. 106, Eq. (5.5.7)]).

For the classical Laguerre polynomials, by appealing to the relationship (1.11), we similarly find from (3.3) that

$$\sum_{k=0}^{\infty} k^n L_k^{(\alpha)}\Big(\frac{x}{1+z}\Big)\Big(\frac{z}{1+z}\Big)^k = (1+z)^{\alpha+1}\exp\Big(-\frac{xz}{1+z}\Big) \cdot \sum_{k=0}^{n} k!\, S(n,k) L_k^{(\alpha)}(x) z^k \qquad (n \in \mathbb{N}_0;\, |z| < 1). \tag{3.6}$$

which, for

$$z \longmapsto \frac{z}{1-z} \qquad \text{and} \qquad x \longmapsto \frac{x}{1-z},$$

assumes the elegant form:

$$\sum_{k=0}^{\infty} k^n L_k^{(\alpha)}(x) z^k = (1-z)^{-\alpha-1}\exp\Big(-\frac{xz}{1-z}\Big) \cdot \sum_{k=0}^{n} k!\, S(n,k) L_k^{(\alpha)}\Big(\frac{x}{1-z}\Big)\Big(\frac{z}{1-z}\Big)^k \qquad (n \in \mathbb{N}_0;\, |z| < 1). \tag{3.7}$$

Finally, by virtue of either of the *equivalent* relationships (1.11) and (1.12), the generating function (3.1) can also be specialized to the form:

$$\sum_{k=0}^{\infty} k^n L_k^{(\alpha-k)}(x(1-z))\Big(\frac{z}{1-z}\Big)^k = (1-z)^{-\alpha}\exp(-xz) \cdot \sum_{k=0}^{n} k!\, S(n,k) L_k^{(\alpha-k)}(x) z^k \qquad (n \in \mathbb{N}_0;\, |z| < 1), \tag{3.8}$$

which, for

$$z \longmapsto \frac{z}{1+z} \qquad \text{and} \qquad x \longmapsto x(1+z),$$

assumes the form:

$$\sum_{k=0}^{\infty} k^n L_k^{(\alpha-k)}(x) z^k = (1+z)^{\alpha}\exp(-xz) \cdot \sum_{k=0}^{n} k!\, S(n,k) L_k^{(\alpha-k)}(x(1+z))\Big(\frac{z}{1+z}\Big)^k \qquad (n \in \mathbb{N}_0;\, |z| < 1). \tag{3.9}$$

The generating function (3.7) is due to Gabutti and Lyness [1, p. 211, Eq. (5.7)]; its special case when $z = 1/2$ was proven directly by Mathis and Sismondi [3, p. 187, Eq. (5)] (see also the aforementioned work by Srivastava [6, p. 760]). The generating functions (3.4) and (3.5) are believed to be new.

4. MULTILINEAR AND MULTILATERAL GENERATING FUNCTIONS

In this section we first state and prove each of the following general results (Theorem 2 and Theorem 3 below) on multilinear and multilateral generating functions for the Gould–Hopper polynomials $H_n^r(x, \alpha, \beta)$ defined by (1.8).

THEOREM 2. *Corresponding to a nonvanishing function $\Omega_\mu(y_1, \ldots, y_s)$ of s (real or complex) variables $y_1, \ldots, y_s$ ($s \in \mathbb{N}$) and of (complex) order μ, let*

$$\Lambda_{m,p,q}^{(1)}[x; y_1, \ldots, y_s; z] := \sum_{n=0}^{\infty} a_n H_{m+qn}^r(x, \alpha + \rho q n, \beta) \cdot \Omega_{\mu+pn}(y_1, \ldots, y_s) \frac{z^n}{(qn)!} \qquad (a_n \neq 0; m \in \mathbb{N}_0; p, q \in \mathbb{N}), \tag{4.1}$$

where ρ is a suitable complex parameter. Suppose also that

$$\Theta_{n,m,p}^{\alpha,q,\rho}(x; y_1, \ldots, y_s; z) := \sum_{k=0}^{[n/q]} \binom{n}{qk} a_k H_{m+n}^r(x, \alpha + \rho q k, \beta) \cdot \Omega_{\mu+pk}(y_1, \ldots, y_s) z^k. \tag{4.2}$$

Then

$$\sum_{n=0}^{\infty} \Theta_{n,m,p}^{\alpha,q,\rho}(x; y_1, \ldots, y_s; z) \frac{t^n}{n!} = \left(1 - \frac{t}{x}\right)^{\alpha} \exp\left(\beta x^r \left[1 - \left(1 - \frac{t}{x}\right)^r\right]\right) \cdot \Lambda_{m,p,q}^{(1)}\left[x - t; y_1, \ldots, y_s; z t^q \left(1 - \frac{t}{x}\right)^{\rho q}\right] \qquad (|t| < |x|; m \in \mathbb{N}_0), \tag{4.3}$$

provided that each member of (4.3) *exists.*

THEOREM 3. *Under the hypotheses of Theorem* 2, *let*

$$\Lambda^{(2)}_{m,\,p,\,q}[x; y_1, \ldots, y_s; z] := \sum_{n=0}^{\infty} a_n H^r_{m+qn}(x, \alpha + (\rho+1)qn, \beta) \cdot \Omega_{\mu+pn}(y_1, \ldots, y_s) \frac{z^n}{(qn)!} \qquad (a_n \neq 0; m \in \mathbb{N}_0; p, q \in \mathbb{N}), \tag{4.4}$$

where ρ *is a suitable complex parameter. Suppose also that*

$$\Phi^{\alpha,\,q,\,\rho}_{n,\,m,\,p}(x; y_1, \ldots, y_s; z) := \sum_{k=0}^{[n/q]} \binom{n}{qk} a_k H^r_{m+n}(x, \alpha + n + \rho qk, \beta) \cdot \Omega_{\mu+pk}(y_1, \ldots, y_s) z^k. \tag{4.5}$$

Then

$$\sum_{n=0}^{\infty} \Phi^{\alpha,\,q,\,\rho}_{n,\,m,\,p}(x; y_1, \ldots, y_s; z) \frac{t^n}{n!} = \left(1 + \frac{t}{x}\right)^{-\alpha-m-1} \exp\left(\beta x^r \left[1 - \left(1 + \frac{t}{x}\right)^{-r}\right]\right) \cdot \Lambda^{(2)}_{m,\,p,\,q}\left[x\left(1 + \frac{t}{x}\right)^{-1}; y_1, \ldots, y_s; zt^q \left(1 + \frac{t}{x}\right)^{-(\rho+1)q}\right] \qquad (|t| < |x|; m \in \mathbb{N}_0), \tag{4.6}$$

provided that each member of (4.6) *exists.*

Proof of Theorem 2. Denote, for convenience, the first member of the assertion (4.3) of Theorem 2 by $\Xi(x, z; t)$. Then, upon substituting for the polynomials

$$\Theta^{\alpha,\,q,\,\rho}_{n,\,m,\,p}(x; y_1, \ldots, y_s; z)$$

from the definition (4.2) into the left-hand side of (4.3), we get

$$\begin{aligned} \Xi(x, z; t) &= \sum_{n=0}^{\infty} \frac{t^n}{n!} \sum_{k=0}^{[n/q]} \binom{n}{qk} a_k H^r_{m+n}(x, \alpha + \rho qk, \beta) \Omega_{\mu+pk}(y_1, \ldots, y_s) z^k \\ &= \sum_{k=0}^{\infty} a_k \Omega_{\mu+pk}(y_1, \ldots, y_s) \frac{(zt^q)^k}{(qk)!} \cdot \sum_{n=0}^{\infty} H^r_{m+qk+n}(x, \alpha + \rho qk, \beta) \frac{t^n}{n!}, \end{aligned} \tag{4.7}$$

by inverting the order of the double summation involved.

The inner series in (4.7) can be summed in a closed form by applying the known generating function (2.11) in its *equivalent* form:

$$\sum_{n=0}^{\infty} H^r_{m+n}(x,\alpha,\beta)\frac{t^n}{n!} = \left(1-\frac{t}{x}\right)^{\alpha}\exp\left(\beta x^r\left[1-\left(1-\frac{t}{x}\right)^r\right]\right) \cdot H^r_m(x-t,\alpha,\beta) \qquad (m\in\mathbb{N}_0;|t|<|x|) \quad (4.8)$$

with, of course, m and α replaced by $m+qk$ and $\alpha+\rho qk$, respectively ($q\in\mathbb{N}; k\in\mathbb{N}_0$). We thus find from (4.7) and (4.8) that

$$\Xi(x,z;t) = \left(1-\frac{t}{x}\right)^{\alpha}\exp\left(\beta x^r\left[1-\left(1-\frac{t}{x}\right)^r\right]\right) \cdot \sum_{k=0}^{\infty}\frac{a_k}{(qk)!}H^r_{m+qk}(x-t,\alpha+\rho qk,\beta) \cdot \Omega_{\mu+pk}(y_1,\ldots,y_s)\left\{zt^q\left(1-\frac{t}{x}\right)^{\rho q}\right\}^k \qquad (|t|<|x|; m\in\mathbb{N}_0). \quad (4.9)$$

Upon interpreting this last infinite series in (4.9) by means of the definition (4.1), we arrive immediately at the right-hand side of the assertion (4.3) of Theorem 2.

This evidently completes the proof of Theorem 2 under the assumption that the double series involved in the first two steps of our proof are absolutely convergent. Thus, in general, Theorem 2 holds true (at least as a relation between formal power series) for those values of the various parameters and variables involved for which each member of the assertion (4.3) exists.

Proof of Theorem 3. Our proof of Theorem 3 is much akin to that of Theorem 2, which we have already detailed here fairly adequately. In place of the generating function (4.8) used in proving Theorem 2, we shall require the generating function (2.12) in its *equivalent* form:

$$\sum_{n=0}^{\infty} H^r_{m+n}(x,\alpha+n,\beta)\frac{t^n}{n!} = \left(1+\frac{t}{x}\right)^{-\alpha-m-1}\exp\left(\beta x^r\left[1-\left(1+\frac{t}{x}\right)^{-r}\right]\right) \cdot H^r_m\left(x\left(1+\frac{t}{x}\right)^{-1},\alpha,\beta\right) \qquad (m\in\mathbb{N}_0;|t|<|x|) \quad (4.10)$$

in proving Theorem 3.

For $\rho=0$, Theorem 2 immediately yields a result due to Srivastava (cf., e.g., [7, p. 430, Corollary 15]). And, by appealing to the relationship (1.9), Theorem 2 with

$$r=2, \qquad \alpha=\rho=0, \qquad \text{and} \qquad \beta=1$$

would lead us readily to another result of Srivastava (cf., e.g., [7, p. 422, Corollary 3]). Moreover, in view of the relationships (1.11) and (1.12) with the classical Laguerre polynomials, by setting

$$r = \beta = 1, \qquad \alpha \longmapsto \alpha + m, \qquad \text{and}$$

$$a_n \longmapsto \frac{(qn)!}{(m+qn)!} a_n \qquad (m, n \in \mathbb{N}_0),$$

Theorems 2 and 3 can easily be reduced to some known results on multilinear and multilateral generating functions for the Laguerre polynomials, which were given earlier by Rassias and Srivastava [4, p. 534, Theorem 1; p. 535, Theorem 2]. Many *further* special cases and consequences of Theorem 2 as well as Theorem 3 can indeed be found scattered in the mathematical literature on this subject.

Next we consider the following yet another (*presumably new*) generating function for the Gould–Hopper polynomials $H_n^r(x, \alpha, \beta)$:

$$\sum_{k=0}^{\infty} H_n^r(x, \alpha + rk, \beta) \frac{t^k}{k!} = e^t \left[1 - \left(\frac{t}{\beta x^r}\right)\right]^{n/r} H_n^r\left(\left[x^r - \left(\frac{t}{\beta}\right)\right]^{1/r}, \alpha, \beta\right)$$

$$(n \in \mathbb{N}_0; |\frac{t}{\beta}| < |x|^r; \beta r x \neq 0). \tag{4.11}$$

For an *elementary* proof of (4.11), *without* using the techniques detailed in Section 2, we set $t = xu$ in the generating function (1.9), so that we have

$$(1-u)^{\alpha} \exp(\beta x^r[1-(1-u)^r]) = \sum_{n=0}^{\infty} H_n^r(x, \alpha, \beta) \frac{(xu)^n}{n!} \qquad (|u|<1). \tag{4.12}$$

Upon replacing α in (4.12) by $\alpha + rk$ $(r \neq 0; k \in \mathbb{N}_0)$, if we multiply each member of (4.12) by

$$\frac{t^k}{k!} \qquad (k \in \mathbb{N}_0),$$

we find by summing both sides of the resulting equation from $k = 0$ to $k = \infty$ that

$$(1-u)^{\alpha} \exp(\beta x^r[1-(1-u)^r]) \sum_{k=0}^{\infty} \frac{[(1-u)^r t]^k}{k!}$$

$$= \sum_{n=0}^{\infty} \frac{(xu)^n}{n!} \sum_{k=0}^{\infty} H_n^r(x, \alpha + rk, \beta) \frac{t^k}{k!} \qquad (r \neq 0)$$

or, equivalently, that

$$e^t (1-u)^{\alpha} \exp(\beta \left[x^r - \left(\frac{t}{\beta}\right)\right] [1-(1-u)^r])$$

$$= \sum_{n=0}^{\infty} \frac{(xu)^n}{n!} \sum_{k=0}^{\infty} H_n^r(x, \alpha + rk, \beta) \frac{t^k}{k!} \qquad (\beta r \neq 0), \tag{4.13}$$

which, by virtue of the generating function (4.12) again, assumes the *convenient* form:

$$e^t \sum_{n=0}^{\infty} \frac{\{u[x^r - (t/\beta)]^{1/r}\}^n}{n!} H_n^r\left(\left[x^r - \left(\frac{t}{\beta}\right)\right]^{1/r}, \alpha, \beta\right)$$

$$= \sum_{n=0}^{\infty} \frac{(xu)^n}{n!} \sum_{k=0}^{\infty} H_n^r(x, \alpha + rk, \beta)\frac{t^k}{k!} \qquad (\beta r \neq 0). \tag{4.14}$$

The generating function (4.11) would now result immediately upon equating the coefficients of u^n on both sides of (4.14).

In terms of the classical Laguerre polynomials involved in the relationships (1.11) and (1.12), an interesting special case of the generating function (4.11) when

$$r = \beta = 1 \qquad \text{and} \qquad \alpha \longmapsto \alpha + n \qquad (n \in \mathbb{N}_0)$$

yields the well-known (*rather classical*) result:

$$\sum_{k=0}^{\infty} L_n^{(\alpha+k)}(x)\frac{t^k}{k!} = e^t L_n^{(\alpha)}(x - t), \tag{4.15}$$

which incidentally is an immediate consequence of the Taylor expansion of

$$e^t L_n^{(\alpha)}(x - t)$$

in powers of t.

By making use of the generating function (4.11), the method of proof of Theorem 2 (and of Theorem 3) can be applied *mutatis mutandis* in order to establish an unusual family of multilinear and multilateral generating functions for the Gould–Hopper polynomials $H_n^r(x, \alpha, \beta)$, which is given by

THEOREM 4. *Under the hypotheses of Theorem* 2, *let*

$$\Lambda_{p,q}^{(3)}[x; y_1, \ldots, y_s; z] := \sum_{k=0}^{\infty} a_k H_n^r(x, \alpha + (\rho + r)qk, \beta) \cdot \Omega_{\mu+pk}(y_1, \ldots, y_s)\frac{z^k}{(qk)!}$$

$$(a_k \neq 0; n \in \mathbb{N}_0; p, q \in \mathbb{N}; r \neq 0), \tag{4.16}$$

where ρ is a suitable complex parameter. Suppose also that

$$\Psi_{k,p,q}^{\alpha,\mu,\rho}(x; y_1, \ldots, y_s; z) := \sum_{l=0}^{[k/q]} \binom{k}{ql} a_l H_n^r(x, \alpha + \rho q l + rk, \beta) \cdot \Omega_{\mu+pl}(y_1, \ldots, y_s) z^l. \tag{4.17}$$

Then

$$\sum_{k=0}^{\infty} \Psi_{k,p,q}^{\alpha,\mu,\rho}(x; y_1, \ldots, y_s; z)\frac{t^k}{k!}$$
$$= e^t\left[1-\left(\frac{t}{\beta x^r}\right)\right]^{n/r} \Lambda_{p,q}^{(3)}\left[\left[x^r-\left(\frac{t}{\beta}\right)\right]^{1/r}; y_1, \ldots, y_s; zt^q\right] \tag{4.18}$$
$$\left(\left|\frac{t}{\beta}\right| < |x|^r; n \in \mathbb{N}_0; \beta r x \neq 0\right),$$

provided that each member of (4.18) *exists*.

A special case of Theorem 4 when

$$r = \beta = 1, \qquad \alpha \longmapsto \alpha + n, \qquad \text{and} \qquad a_k \longmapsto \frac{a_k}{n!} \qquad (n, k \in \mathbb{N}_0),$$

involving the classical Laguerre polynomials occurring in the relationships (1.11) and (1.12), was proven earlier by Rassias and Srivastava [4, p. 537, Theorem 3].

For *each* suitable choice of the coefficients a_k $(k \in \mathbb{N}_0)$, if we express the multivariable function

$$\Omega_\mu(y_1, \ldots, y_s) \qquad (s \in \mathbb{N}\backslash\{1\})$$

as an appropriate product of several relatively *simpler* (known or new) functions, each of our results (Theorems 2, 3, and 4 above) can be shown to yield various families of multilinear and multilateral generating functions for the Gould–Hopper polynomials $H_n^r(x, \alpha, \beta)$ (and also for its aforementioned relatives). The detailed demonstration of this observation may be left as an exercise for the interested reader.

We conclude this paper by observing that yet another family of generating functions for the Gould–Hopper polynomials $H_n^r(x, \alpha, \beta)$, associated with the Stirling numbers $S(n, k)$ defined by (1.4), can be derived by appropriately applying Theorem 1 to the obviously *unusual* generating function (4.11). First of all, by letting

$$n \longmapsto N \qquad \text{and} \qquad \alpha \longmapsto \alpha + rn \qquad (n, N \in \mathbb{N}_0),$$

we rewrite the generating function (4.11) in its *equivalent* form:

$$\sum_{k=0}^{\infty} H_N^r(x, \alpha + r(n+k), \beta)\frac{t^k}{k!}$$
$$= e^t\left[1-\left(\frac{t}{\beta x^r}\right)\right]^{N/r} H_N^r\left(\left[x^r-\left(\frac{t}{\beta}\right)\right]^{1/r}, \alpha + rn, \beta\right)$$
$$\left(n, N \in \mathbb{N}_0; \left|\frac{t}{\beta}\right| < |x|^r; \beta r x \neq 0\right), \tag{4.19}$$

which does indeed belong to the family (1.3) with, of course

$$f(x,t) = e^t\left[1-\left(\frac{t}{\beta x^r}\right)\right]^{N/r}, \qquad g(x,t)=1, \qquad h(x,t)=\left[x^r-\left(\frac{t}{\beta}\right)\right]^{1/r},$$

and

$$\mathcal{S}_k(x) \longmapsto \frac{1}{k!}H_N^r(x,\alpha+rk,\beta) \qquad (k\in\mathbb{N}_0).$$

Thus, by appealing to Theorem 1, we obtain the following *addition* to the various results presented already in Section 3:

$$\sum_{k=0}^{\infty}\frac{k^n}{k!}H_N^r\left(\left[x^r+\left(\frac{z}{\beta}\right)\right]^{1/r},\alpha+rk,\beta\right)z^k$$
$$= e^z\left[1+\left(\frac{z}{\beta x^r}\right)\right]^{-N/r}\sum_{k=0}^{n}S(n,k)H_N^r(x,\alpha+rk,\beta)z^k$$
$$\left(n,N\in\mathbb{N}_0;\left|\frac{z}{\beta}\right|<|x|^r;\beta r x\neq 0\right). \tag{4.20}$$

For the classical Laguerre polynomials $L_n^{(\alpha)}(x)$ given by the relationships (1.11) and (1.12), a special case of (4.20) when

$$r=\beta=1, \qquad \alpha\longmapsto\alpha+N, \qquad \text{and} \qquad x\longmapsto x-z$$

yields the generating function:

$$\sum_{k=0}^{\infty}\frac{k^n}{k!}L_N^{(\alpha+k)}(x)z^k = e^z\sum_{k=0}^{n}S(n,k)L_N^{(\alpha+k)}(x-z)z^k$$
$$(n,N\in\mathbb{N}_0). \tag{4.21}$$

This last generating function (4.21) can also be derived alternatively by applying Theorem 1 *directly* to the well-known (*rather classical*) result (4.15).

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