

# ON POLYNOMIALS RELATED TO DERIVATIVES OF THE GENERATING FUNCTION OF CATALAN NUMBERS

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## 1. INTRODUCTION AND SUMMARY

In [3] it has been shown that powers of the generating function  $c(x)$  of Catalan numbers  $\{C_n\}_{n \in \mathbb{N}_0} = \{1, 1, 2, 5, 14, 42, \dots\}$ , where  $\mathbb{N}_0 := \{0, 1, 2, \dots\}$  (nr. 1459 and A000108 of [8] and references of [3]) can be expressed in terms of a linear combination of 1 and  $c(x)$  with coefficients replaced by certain scaled Chebyshev polynomials of the second kind. In this paper, derivatives of  $c(x)$  are studied in a similar manner. The starting point is the following expression for the first derivative:

$$\frac{d c(x)}{d x} \equiv c'(x) = \frac{1}{x(1-4x)} (1 + (-1+2x)c(x)). \quad (1)$$

This equation is equivalent to the simple recurrence relation valid for  $C_n$ :

$$(n+2)C_{n+1} - 2(2n+1)C_n = 0, \quad n = -1, 0, 1, \dots, \quad \text{with } C_{-1} = -1/2. \quad (2)$$

Equation (1) can, of course, also be found from the explicit form  $c(x) = (1 - \sqrt{1-4x}) / (2x)$ . The result for the  $n^{\text{th}}$  derivative is of the form

$$\frac{1}{n!} \frac{d^n c(x)}{d x^n} = \frac{1}{(x(1-4x))^n} (a_{n-1}(x) + b_n(x)c(x)), \quad (3)$$

with certain polynomials  $a_{n-1}(x)$  of degree  $n-1$  and  $b_n(x)$  of degree  $n$ . These polynomials are found to be

$$b_n(x) = \sum_{m=0}^n (-1)^m B(n, m) x^{n-m}$$

with

$$B(n, m) := \binom{2n}{n} \binom{n}{m} / \binom{2m}{m}, \quad (4)$$

which defines a triangle of numbers for  $n, m \in \mathbb{N}$ ,  $n \geq m \geq 0$ , where  $\mathbb{N} := \{1, 2, 3, \dots\}$ . The first terms are depicted in Table 1 with  $B(n, m) = 0$  for  $n < m$ . Another representation for the polynomials  $b_n(x)$  is also found, i.e.,

$$b_n(x) = -2 \sum_{k=0}^n C_{k-1} x^k (4x-1)^{n-k}. \quad (5)$$

Equating both forms of  $b_n(x)$  leads to a formula involving convolutions of Catalan numbers with powers of an arbitrary constant  $\lambda := (4x-1)/x$ . This formula is given in (31). Equation (5) reveals the generating function of the polynomials  $b_n(x)$  because it is a convolution of two functional sequences. The result is

$$g_b(x; z) := \sum_{n=0}^{\infty} b_n(x)z^n = \frac{\sqrt{1-4xz}}{1+(1-4x)z}. \tag{6}$$

TABLE 1.  $B(n, m)$  Central Binomial Triangle

$n \backslash m$	0	1	2	3	4	5	6	7	8	9	10
0	1	0	0	0	0	0	0	0	0	0	0
1	2	1	0	0	0	0	0	0	0	0	0
2	6	6	1	0	0	0	0	0	0	0	0
3	20	30	10	1	0	0	0	0	0	0	0
4	70	140	70	14	1	0	0	0	0	0	0
5	252	630	420	126	18	1	0	0	0	0	0
6	924	2772	2310	924	198	22	1	0	0	0	0
7	3432	12012	12012	6006	1716	286	26	1	0	0	0
8	12870	51480	60060	36036	12870	2860	390	30	1	0	0
9	48620	218790	291720	204204	87516	24310	4420	510	34	1	0
10	184756	923780	1385670	1108536	554268	184756	41990	6460	646	38	1

The other family of polynomials is

$$a_n(x) = \sum_{k=0}^n (-1)^k A(n+1, k+1)x^{n-k}$$

with the triangular array  $A(n, m)$  defined for  $m = 0$  by  $A(n, 0) = C_n$ , and for  $n, m \in \mathbb{N}$  with  $n \geq m > 0$  by the numbers

$$A(n, m) = \frac{1}{2} \binom{n}{m-1} \left[ 4^{n-m+1} - \binom{2n}{n} / \binom{2(m-1)}{m-1} \right]. \tag{7}$$

The first terms of this triangular array of numbers are shown in Table 2 with  $A(n, m) = 0$  for  $n < m$ . Both results (4) and (7) are solutions to recurrence relations which hold for  $b_n(x)$  and  $a_n(x)$  and their respective coefficients  $B(n, m)$  and  $A(n, m)$ .

Another representation for the polynomials  $a_n(x)$  is found to be

$$a_n(x) = \sum_{k=0}^n C_k x^k (4x-1)^{n-k}, \tag{8}$$

which shows that the generating function of these polynomials is

$$g_a(x; z) := \sum_{n=0}^{\infty} a_n(x)z^n = \frac{c(xz)}{1+(1-4x)z}. \tag{9}$$

Comparing (5) with (8) yields the following relation between these two types of polynomials

$$b_n(x) = (4x-1)^n - 2xa_{n-1}(x), \quad n \in \mathbb{N}_0, \text{ with } a_{-1}(x) \equiv 0, \tag{10}$$

and between the coefficients

$$B(n, m) = \binom{n}{m} 4^{n-m} - 2A(n, m+1). \tag{11}$$

**TABLE 2.  $A(n, m)$  Catalan Triangle**

$n \backslash m$	0	1	2	3	4	5	6	7	8	9	10
0	1	0	0	0	0	0	0	0	0	0	0
1	1	1	0	0	0	0	0	0	0	0	0
2	2	5	1	0	0	0	0	0	0	0	0
3	5	22	9	1	0	0	0	0	0	0	0
4	14	93	58	13	1	0	0	0	0	0	0
5	42	386	325	110	17	1	0	0	0	0	0
6	132	1586	1686	765	178	21	1	0	0	0	0
7	429	6476	8330	4746	1477	262	25	1	0	0	0
8	1430	26333	39796	27314	10654	2525	362	29	1	0	0
9	4862	106762	185517	149052	69930	20754	3973	478	33	1	0
10	16796	431910	848830	781725	428772	152946	36646	5885	610	37	1

The triangle of numbers  $A(n, m)$  is related to a rectangular array of integers  $\hat{A}(n, m)$  with  $\hat{A}(0, m) \equiv 1$ ,  $\hat{A}(n, 0) = -C_n$  for  $n \in \mathbb{N}$ , and for  $n \geq m \geq 1$  by

$$A(n, m) = -\hat{A}(n-m, m) + 2^{2(n-m)+1} \binom{n-1}{m-1}, \tag{12}$$

or with (7) for  $m \in \mathbb{N}, n \in \mathbb{N}_0$ , by

$$\hat{A}(n, m) = \frac{1}{2} \binom{n+m}{n+1} \left[ \binom{2(n+m)}{n+m} / \binom{2(m-1)}{m-1} - 4^{n+1} \frac{m-1}{n+m} \right]. \tag{13}$$

Part of the array  $\hat{A}(n, m)$  is shown in Table 3, where it is called  $C4(n, m)$ .

**TABLE 3.  $C4(n, m)$  Catalan Array**

$n \backslash m$	0	1	2	3	4	5	6
0	1	1	1	1	1	1	1
1	-1	3	7	11	15	19	23
2	-2	10	38	82	142	218	310
3	-5	35	187	515	1083	1955	3195
4	-14	126	874	2934	7266	15086	27866
5	-42	462	3958	15694	44758	105102	216566
6	-132	1716	17548	80324	259356	679764	1546028
7	-429	6435	76627	397923	1435347	4154403	10338515
8	-1430	24310	330818	1922510	7663898	24281510	65635570
9	-4862	92378	1415650	9105690	39761282	136887322	399429602
10	-16796	352716	6015316	42438076	201483204	749032492	2346750900

It turns out that the  $m^{\text{th}}$  column of the triangle of numbers  $A(n, m)$  for  $m = 0, 1, \dots$  is determined by the generating function

$$c(x) \left( \frac{x}{1-4x} \right)^m.$$

The  $m^{\text{th}}$  column of the triangle of numbers  $B(n, m)$  for  $m = 0, 1, \dots$  is generated by

$$\frac{1}{\sqrt{1-4x}} \left( \frac{x}{1-4x} \right)^m.$$

This fact identifies the infinite dimensional matrices **A** and **B** as examples of Riordan matrices in the terminology of [7]. The matrix  $\hat{A}$  associated with  $\hat{A}(n, m)$  is an example of a Riordan array.

Because differentiation of  $c(x) = \sum_{k=0}^{\infty} C_k x^k$  leads to

$$\frac{1}{n!} \frac{d^n c(x)}{dx^n} = \sum_{k=0}^{\infty} C(n, k) x^k, \text{ with } C(n, k) := \frac{1}{n!} \prod_{j=1}^n (k+j) C_{n+k} = \frac{(2(n+k))!}{n! k! (n+k+1)!}, \quad (14)$$

where  $C(0, k) = C_k$ , one finds, together with (3), the following identities for  $n \in \mathbb{N}$ ,  $p \in \{0, 1, \dots, n-1\}$ ,

$$(D1): \sum_{k=0}^p (-1)^k C_k \binom{n}{p-k} / \binom{2(n-p+k)}{n-p+k} = \frac{1}{2} \binom{n}{p+1} \left\{ 2^{2(p+1)} / \binom{2n}{n} - 1 / \binom{2(n-p-1)}{n-p-1} \right\} \\ = A(n, n-p) / \binom{2n}{n}, \quad (15)$$

and for  $n \in \mathbb{N}$ ,  $k \in \mathbb{N}_0$ ,

$$(D2): \sum_{j=0}^n (-1)^j \binom{n}{j} / \binom{2j}{j} \sum_{l=0}^k 4^l \binom{n+l-1}{n-1} C_{k+j-l} = C(n, k) / \binom{2n}{n}. \quad (16)$$

The remainder of this paper provides proofs for the above statements.

## 2. DERIVATIVES

The starting point is equation (1) which can either be verified from the explicit form of the generating function  $c(x)$  or by converting the recursion relation (2) for Catalan numbers into an equation for their generating function. A computation of

$$\frac{1}{(n+1)!} \frac{d^{n+1} c(x)}{dx^{n+1}} = \frac{1}{n+1} \frac{d}{dx} \left( \frac{1}{n!} \frac{d^n c(x)}{dx^n} \right),$$

with (3) taken as granted and equation (1), produces the following mixed relations between the quantities  $a_n(x)$  and  $b_n(x)$  and their first derivatives, valid for  $n \in \mathbb{N}_0$ ,

$$(n+1)a_n(x) = x(1-4x)a'_{n-1}(x) + b_n(x) + n(8x-1)a_{n-1}(x), \quad (17)$$

$$(n+1)b_{n+1}(x) = x(1-4x)b'_n(x) + (-(n+1) + 2(1+4n)x)b_n(x), \quad (18)$$

with inputs  $a_{-1}(x) \equiv 0$  and  $b_0(x) \equiv 1$ .

From (18), it is clear by induction that  $b_n(x)$  is a polynomial of degree  $n$ . Again by induction, the same statement holds for  $a_n(x)$  in (17). Therefore, we write, for  $n \in \mathbb{N}_0$ ,

$$a_n(x) = \sum_{k=0}^n (-1)^k a(n, k) x^{n-k}, \tag{19}$$

$$b_n(x) = \sum_{k=0}^n (-1)^k B(n, k) x^{n-k}, \tag{20}$$

with the triangular arrays of numbers  $a(n, k)$  and  $B(n, k)$  with row number  $n$  and column number  $k \leq n$ . The triangular array  $a(n, k)$  will later be enlarged to another one which will then be called  $A(n, k)$ .

We first solve  $b_n(x)$  in (18) by inserting (20) and deriving the recursion relation for the coefficients  $B(n, m)$  after comparing coefficients of  $x^{n+1}$ ,  $x^0$ , and  $x^{n-k}$  for  $k = 0, 1, \dots, n-1$ .

$$x^{n+1}: (n+1)B(n+1, 0) = 2(2n+1)B(n, 0), \tag{21}$$

$$x^0: B(n+1, n+1) = B(n, n), \tag{22}$$

$$x^{n-k}: (n+1)B(n+1, k+1) = (k+1)B(n, k) + 2(2(n+k)+3)B(n, k+1). \tag{23}$$

With the input  $B(0, 0) = 1$ , one deduces from (21) for the leading coefficient of  $b_n(x)$

$$B(n, 0) = 2^n \frac{(2n-1)!!}{n!} = \frac{(2n)!}{n!n!} = \binom{2n}{n}, \tag{24}$$

and from (22)

$$B(n, n) \equiv 1, \text{ i.e., } b_n(0) = (-1)^n. \tag{25}$$

The double factorial  $(2n-1)!! := 1 \cdot 3 \cdot 5 \cdot \dots \cdot (2n-1)$  appeared in (24).

In order to solve (23), we conjecture from Table 1 that, for  $n, m \in \mathbb{N}$ ,

$$B(n, m) = 4B(n-1, m) + B(n-1, m-1), \tag{26}$$

with input  $B(n, 0) = \binom{2n}{n}$  from (24).

If we use this conjecture in (23), written with  $n \rightarrow n-1$ ,  $k \rightarrow m-1$ , we are led to consider the simple recursion

$$B(n, m) = \frac{n+1-m}{2(2m-1)} B(n, m-1). \tag{27}$$

The solution of this recursion is, for  $n, m \in \mathbb{N}_0$ ,

$$B(n, m) = \frac{1}{2^m(2m-1)!!} \frac{n!}{(n-m)!} \binom{2n}{n} = \frac{m!n!}{(2m)!(n-m)!} \binom{2n}{n} = \binom{2n}{n} \binom{n}{m} / \binom{2m}{m}. \tag{28}$$

With the Pochhammer symbol  $(a)_n := \Gamma(n+a) / \Gamma(a)$ , this result can also be written as

$$B(n, m) = ((2m+1)/2)_{n-m} 4^{m-n} / (n-m)!.$$

This result satisfies (21), i.e., (24), as well as (22), i.e., (25). It is also the solution to (23) provided we prove the conjecture (26) using  $B(n, m)$  in (28). This can be done by inserting

$$B(n, m) = \frac{(2n)!m!}{(2m)!n!(n-m)!}$$

in (26). Thus, we have proved the following proposition.

**Proposition 1:** We have

$$b_n(x) = \sum_{k=0}^n (-1)^k B(n, k) x^{n-k}, \text{ where } B(n, k) = \binom{2n}{n} \binom{n}{k} / \binom{2k}{k}.$$

This triangle of numbers as shown in Table 1 appears as A046521 in the database [8].

One can derive another explicit representation for the polynomials  $b_n(x)$  by using (27) in (20):

$$(1-4x)b'_n(x) + 2(2n-1)b_n(x) + 2 \binom{2n}{n} x^n = 0. \tag{29}$$

This leads, together with (18), to the following inhomogeneous recursion relation for  $b_n(x)$ :

$$b_{n+1}(x) = (4x-1)b_n(x) - 2C_n x^{n+1}, \quad b_0(x) \equiv 1. \tag{30}$$

Equation (29) can also be solved as first-order linear and inhomogeneous differential equation for  $b_n(x)$ .

**Proposition 2:** We have

$$b_n(x) = -2 \sum_{k=0}^n C_{k-1} x^k (4x-1)^{n-k},$$

where the  $C_k$ 's are the Catalan numbers for  $k \in \mathbb{N}_0$  and  $C_{-1} = -1/2$ .

*Proof:* Iteration of (30).  $\square$

**Proposition 3:** The generating function  $g_b(x; z) := \sum_{n=0}^{\infty} b_n(x) z^n$  for  $\{b_n(x)\}$  is given by (6).

*Proof:* The alternative form of  $b_n(x)$  given by equation (5) is a convolution of the functional sequences  $\{-2C_{k-1}x^k\}_{n \in \mathbb{N}_0}$  and  $\{(4x-1)^n\}_{n \in \mathbb{N}_0}$ , with generating functions  $1-2xz c(xz) = \sqrt{1-4xz}$  and  $1/(1+(1-4x)z)$ , respectively. Therefore,  $g_b(x; z)$  is the product of these two generating functions.  $\square$

Comparing this alternative form (5) for  $b_n(x)$  with the one given by (20), together with (28), proves the following identity in  $n$  and  $\lambda := (4x-1)/x$ . The term  $k=0$  in the sum (5) has been written separately.

**Corollary 1 (convolution of Catalan sequence and the sequence of powers of  $\lambda$ ):** For  $n \in \mathbb{N}$  and  $\lambda \neq \infty$ ,

$$s_{n-1}(\lambda) := \lambda^{n-1} \sum_{k=0}^{n-1} \frac{C_k}{\lambda^k} = \frac{1}{2} \left( \lambda^n - \binom{2n}{n} \sum_{k=0}^n (-1)^k (4-\lambda)^k \binom{n}{k} / \binom{2k}{k} \right). \tag{31}$$

Therefore, the generating function for the sequence  $s_n(\lambda)$  is

$$g(\lambda; x) := \sum_{n=0}^{\infty} s_n(\lambda) x^n = c(x) / (1-\lambda x).$$

From the generating function, the recurrence relation is found to be  $s_n(\lambda) = \lambda s_{n-1}(\lambda) + C_n$ ,  $s_{-1}(\lambda) \equiv 0$ . The connection with the polynomial  $b_n(x)$  is

$$s_n(\lambda) = \frac{1}{2} (\lambda^{n+1} - (4-\lambda)^{n+1} b_{n+1}(1/(4-\lambda))).$$

The case  $\lambda = 0$  ( $x = 1/4$ ) is also covered by this formula. It produces from  $s_n(0) = C_n$  the following identity.

**Example 1:** Case  $\lambda = 0$  ( $x = 1/4$ ),

$$\sum_{k=0}^n (-1)^{k+1} \binom{n}{k} 4^k / \binom{2k}{k} = \frac{1}{2n-1}. \quad (32)$$

This identity occurs in one of the exercises 2.7, 2, page 32 of [4].

We note that from (5) one has  $-2b_{n+1}(1/4) = C_n / 4^n$ . The large  $n$  behavior of this sequence is known to be (see [2], Exercise 9.60):

$$C_n / 4^n \sim \frac{1}{\sqrt{\pi}} \frac{1}{n^{3/2}}.$$

If one puts  $4x - 1 = x$ , i.e.,  $x = 1/3$ , in (5), one can identify the partial sum  $s_n(1)$  of Catalan numbers:

$$s_n(1) := \sum_{k=0}^n C_k = \frac{1}{2} (1 - 3^{n+1} b_{n+1}(1/3)). \quad (33)$$

This sequence  $\{1, 2, 4, 9, 23, 65, 197, 626, 2056, \dots\}$  appears as A014137 in the web encyclopedia [8]. If one puts  $\lambda - 1$  in Corollary 1, one also finds the following example.

**Example 2:**

$$2s_{n-1}(1) = 1 + \binom{2n}{n} \sum_{k=0}^n (-1)^{k+1} \binom{n}{k} 3^k / \binom{2k}{k}. \quad (34)$$

Another interesting example is the case  $\lambda = 4$  ( $x = \infty$ ). Here one finds a simple result for the convolution of Catalan's sequence with powers of 4.

**Example 3:**  $\lambda = 4$  ( $x = \infty$ ),

$$2s_{n-1}(4) = 4^n - \binom{2n}{n}. \quad (35)$$

This sequence  $\{1, 5, 22, 93, 386, 1586, 6476, \dots\}$  appears in the book [8] as Nr. 3920 and as A000346 in the web encyclopedia [8]. It will show up again in this work as  $A(n+1, 1)$ , the second column in the  $A(n, m)$  triangle (see Table 2).

The sequence for  $\lambda = -1$  ( $x = 1/5$ ) is also nonnegative, as can be seen by writing

$$s_{2k}(-1) = C_2 + \sum_{l=2}^k (C_{2l} - C_{2l-1}) \text{ for } k \in \mathbb{N}$$

and

$$s_{2k+1}(-1) = \sum_{l=1}^k (C_{2l+1} - C_{2l}),$$

and using

$$\Delta C_n := C_n - C_{n-1} = 3 \frac{n-1}{n+1} C_{n-1} \geq 0.$$

This is the sequence  $\{1, 0, 2, 3, 11, 31, 101, 328, 1102, 3760, \dots\}$  which appears now as A032357 in the web encyclopedia [8].

Recursion (26) for  $B(n, m)$  can be transformed into an equation for the generating function for the sequence appearing in the  $m^{\text{th}}$  column of the  $B(n, m)$  triangle

$$G_B(m, x) := \sum_{n=m}^{\infty} B(n, m)x^n, \tag{36}$$

with input

$$G_B(0, x) = \sum_{n=0}^{\infty} \binom{2n}{n} x^n = \frac{1}{\sqrt{1-4x}},$$

the generating function for the central binomial numbers. So (26) implies, for  $m \in \mathbb{N}_0$ ,

$$G_B(m, x) = \left(\frac{x}{1-4x}\right)^m \frac{1}{\sqrt{1-4x}}. \tag{37}$$

For  $x \frac{d}{dx} G_B(m, x)$ , see (53). Therefore, we have proved the following proposition.

**Proposition 4 (column sequences of the  $B(n, m)$  triangle):** The sequence  $\{B(n, m)\}_{n=m}^{\infty}$ , defined for fixed  $m \in \mathbb{N}_0$  and  $n \in \mathbb{N}_0$  by (28), is the convolution of the central binomial sequence

$$\left\{ \binom{2k}{k} \right\}_{k \in \mathbb{N}_0}$$

and the  $m^{\text{th}}$  convolution of the (shifted) power sequence  $\{0, 1, 4^1, 4^2, \dots\}$ .

**Note 1:** The infinite dimensional matrix  $\mathbf{B}$  with elements  $B(n, m)$  given for  $n \geq m \geq 0$  by (28) and  $B(n, m) \equiv 0$  for  $n < m$  is an example of a Riordan matrix [7]. With the notation of this reference,

$$\mathbf{B} = \left( \frac{1}{\sqrt{1-4x}}, \frac{x}{1-4x} \right).$$

**Note 2:(Sheffer-type identities from Riordan matrices):** Triangular Riordan matrices

$$\mathbf{M} = (M_{i,j})_{i \geq j \geq 0} = (g(x), f(x)),$$

$M_{i,j} = 0$  for  $j > i$ , in the notation of [7], lead to polynomials that satisfy Sheffer-type identities (see [5] and its references, and also [1]),

$$S_n(x+y) = \sum_{k=0}^n S_k(y) P_{n-k}(x) = \sum_{k=0}^n P_k(y) S_{n-k}(x), \tag{38}$$

$$P_n(x+y) = \sum_{k=0}^n P_k(y) P_{n-k}(x) = \sum_{k=0}^n P_k(x) P_{n-k}(y), \tag{39}$$

where the polynomials  $S_n(x)$  and  $P_n(x)$  are defined by

$$S_n(x) = \sum_{m=0}^n M_{n,m} \frac{x^m}{m!}, \quad n \in \mathbb{N}_0, \quad P_n(x) = \sum_{m=1}^n P_{n,m} \frac{x^m}{m!}, \quad n \in \mathbb{N}, \quad P_0(x) \equiv 1, \tag{40}$$

with  $P_{n,m} := [z^n](f^m(z))$ ,  $n \geq m \geq 1$ . Here  $g(x)$  defines the first column of  $\mathbf{M}$ :  $M_{n,0} = [x^n]g(x)$ .

If one uses  $s_n(x) := n! S_n(x)$  and  $p_n(x) := n! P_n(x)$ , one obtains the Sheffer identities (also called binomial identities) treated in [5]. Then  $s_n(x)$  is Sheffer for  $(1/g(\bar{f}(t)), \bar{f}(t))$ , and  $p_n(x)$  is



associated to  $\bar{f}(t)$ —or Sheffer for  $(1, \bar{f}(t))$ —in the terminology of [5]. Here  $\bar{f}(t)$  stands for the inverse of  $f(t)$  as a function.

Let us give the relation between  $g_b(x; z)$  and  $G_B(m; x)$ .

**Proposition 5:** We have

$$g_b(x; z) = \sum_{m=0}^{\infty} (-1)^m G_B(m; xz) \left(\frac{1}{x}\right)^m. \tag{41}$$

**Proof:** One inserts the value of  $b_n(x)$  given in (20) into the definition (6) of  $g_b(x; z)$  and rewrites the Cauchy sum as two infinite sums which are then interchanged. Finally, the definition of  $G_B(m; x)$  in (36) is used.  $\square$

One can check (41) by using the explicit form of  $G_B(m; xz)$  given in (36) and comparing with (6).

In a similar vein, we can solve  $a_n(x)$  in (17) with  $b_n(x)$  given by (20) and (28). The coefficients  $a(n, k)$ , defined by (19), have to satisfy, after comparing coefficients of  $x^n$ ,  $x^0$ , and  $x^{n-k}$  for  $k = 1, 2, \dots, n-1$  and  $n \in \mathbb{N}_0$ :

$$x^n: a(n, 0) = 4a(n-1, 0) + C_n, \tag{42}$$

$$x^0: (n+1)a(n, n) = 1 + na(n-1, n-1), \tag{43}$$

$$x^{n-k}: (n+1)a(n, k) = ka(n-1, k-1) + 4(n+1+k)a(n-1, k) + B(n, k). \tag{44}$$

In (42) we have used (24), i.e.,  $B(n, 0) = (n+1)C_n$ ; in (43) we have used (25), i.e.,  $B(n, n) \equiv 1$ . From (42) one finds, with input  $a(0, 0) = 1$ ,

$$a(n, 0) = \sum_{k=0}^n C_k 4^{n-k}, \tag{45}$$

and from (43),

$$a(n, n) \equiv 1 \text{ or } a_n(0) = (-1)^n. \tag{46}$$

Note that  $a(n, 0) = s_n(4)$  of (31) with solution (35). It is convenient to define  $a(n-1, -1) := C_n$ ,  $n \in \mathbb{N}_0$ . Then the sequence  $\{a(n, 0)\}_{-1}^{\infty}$  is, with  $a(-1, 0) := 0$ , the convolution of the sequence  $\{a(k, -1)\}_{-1}^{\infty}$  and the shifted power sequence  $\{0, 1, 4^1, 4^2, \dots\}$ . Before solving (44), with  $B(n, k)$  from (28) inserted, we add to the triangular array of numbers  $a(n, m)$  the  $m = -1$  column and an extra row for  $n = -1$ , and define a new enlarged triangular array for  $n, m \in \mathbb{N}_0$  as

$$A(n, m) := a(n-1, m-1) \tag{47}$$

with  $A(n, 0) = a(n-1, -1) = C_n$  and  $A(0, m) = a(-1, m-1) = \delta_{0, m}$ . An inspection of the  $A(n, m)$  triangular array, partly depicted in Table 2, leads to the conjecture

$$A(n, m) = 4A(n-1, m) + A(n-1, m-1), \tag{48}$$

with  $A(n, 0) = C_n$  and  $A(n, m) \equiv 0$  for  $n < m$ . This recursion relation can be used to extend the array  $A(n, m)$  to negative integer values of  $m$ . This conjecture is correct for  $A(n+1, 1) = a(n, 0)$  found in (45), as well as for  $A(n+1, n+1) = a(n, n) \equiv 1$  known from (46). The generating function for the sequence appearing in the  $m^{\text{th}}$  column,

$$G_A(m, x) := \sum_{n=m}^{\infty} A(n, m)x^n, \tag{49}$$

satisfies, due to (48),  $G_A(m, x) = \frac{x}{1-4x}G_A(m-1, x)$ , remembering that  $A(m-1, m) \equiv 0$  and that  $G_A(0, x) = c(x)$ . Therefore,

$$G_A(m, x) = \left(\frac{x}{1-4x}\right)^m c(x). \tag{50}$$

**Note 3:** The infinite dimensional matrix  $A$  with elements  $A(n, m)$  given for  $n \geq m \geq 0$  by (48) and  $A(n, m) \equiv 0$  for  $n < m$  is another example of a Riordan matrix, written in the notation of [7] as  $(c(x), x/(1-4x))$ .

Because of (37) and  $\sqrt{1-4x}c(x) = 2 - c(x)$ , these generating functions of the conjectured  $A(n, m)$  column sequences obey

$$G_A(m, x) = (2 - c(x))G_B(m, x). \tag{51}$$

If we use the conjecture (48) in (44), which is written with (47) in the form

$$(n+1)A(n+1, m+1) = mA(n, m) + 4(n+m+1)A(n, m+1) + B(n, m)$$

for  $n \in \mathbb{N}_0, m \in \{1, 2, \dots, n-1\}$ , we have

$$mA(n+1, m+1) - (n+1)A(n, m) + B(n, m) = 0. \tag{52}$$

This recursion relation can be written with the help of the generating functions (36) and (49) as

$$\left(x\frac{d}{dx} + 1\right)G_A(m, x) - \frac{m}{x}G_A(m+1, x) = G_B(m, x), \tag{53}$$

or with (50) (i.e., the conjecture) as

$$\left(x\frac{d}{dx} + 1 - \frac{m}{1-4x}\right)G_A(m, x) = G_B(m, x). \tag{54}$$

Together with (51), this means

$$x\frac{d}{dx}((2 - c(x))G_B(m, x)) = \left[\left(\frac{m}{1-4x} - 1\right)(2 - c(x)) + 1\right]G_B(m, x). \tag{55}$$

If we can prove this equation with  $G_B(x)$  given by (37), we have shown that (44) is equivalent to the conjecture (48). In order to prove (55), we first compute from (37) for  $m \in \mathbb{N}_0$ ,

$$x\frac{d}{dx}G_B(m, x) = \left(2 + \frac{m}{x}\right)G_B(m+1, x) = \frac{2x+m}{1-4x}G_B(m, x). \tag{56}$$

With this result, (55) reduces to

$$\left(-xc'(x) + (2 - c(x))\frac{1-2x}{1-4x} - 1\right)G_B(m, x) = 0, \tag{57}$$

and with (1), the factor in front of  $G_B(m, x)$  vanishes identically for  $x \neq 1/4$ . Therefore, we have proved the following two propositions concerning the column sequences of the  $A(n, m)$  triangular array and the triangular  $A(n, m)$  array, respectively.

**Proposition 6:** The triangular array of numbers  $A(n, m)$ , defined for  $n, m \in \mathbb{N}_0$  by equation (48),  $A(n, 0) = C_n$ ,  $A(n, m) \equiv 0$  for  $n < m$  has as its  $m^{\text{th}}$  column sequence  $\{A(n, m)\}_{n=m}^{\infty}$  the convolution of the Catalan sequence and the  $m^{\text{th}}$  convolution of the shifted power sequence  $\{0, 1, 4^1, 4^2, \dots\}$ .

**Proof:** Use (50) with (49).  $\square$

**Proposition 7:** The triangular array  $A(n, m)$  of Proposition 6 coincides with the one defined by (47) and (42), (43) and (44) with  $B(n, m)$  given by (28).

**Proof:** On one hand,  $a(n, 0) = A(n+1, 1)$  and  $a(n, n) = A(n+1, n+1) \equiv 1$  of (42) and (43), i.e., (45) and (46), respectively, satisfy (48). On the other hand, (44) is rewritten with the aid of (47) as (52), and (52) has been proved by (53)-(57).  $\square$

Alternatively, one can use the now proven conjecture (48), together with (47), in (44) and derive for  $n \in \mathbb{N}_0, m \in \mathbb{N}_0$ ,

$$4ma(n-1, m) = (n+1-m)a(n-1, m-1) - B(n, m). \tag{58}$$

This is written in terms of the polynomials  $a_{n-1}(x)$  of (19) and  $b_n(x)$  of (20) as

$$x(1-4x)a'_{n-1}(x) + (1-4x+4nx)a_{n-1}(x) - \binom{2n}{n}x^n + b_n(x) = 0. \tag{59}$$

With this result, (17) becomes an inhomogeneous recursion relation for  $a_n(x)$ :

$$a_n(x) = (4x-1)a_{n-1}(x) + C_n x^n, \quad a_0(x) \equiv 1. \tag{60}$$

Moreover, (59) can also be considered as an inhomogeneous linear differential equation for  $a_{n-1}(x)$  with given  $b_n(x)$ . To find the solution this way is, however, a bit tedious. Let us give an alternative form for  $a_n(x)$  in the following proposition.

**Proposition 8:** The solution of the recursion relation (60) is given by (8).

**Proof:** Iteration of (60).  $\square$

Next, we give a corollary.

**Corollary 2:** The generating function  $g_a(x, z) := \sum_{n=0}^{\infty} a_n(x) z^n$  is given by (9).

**Proof:** Equation (8) above shows that  $a_n(x)$  is a convolution of the functional sequences  $\{C_k x^k\}_{k \in \mathbb{N}_0}$  and  $\{(4x-1)^k\}_{k \in \mathbb{N}_0}$  with generating functions  $c(xz)$  and  $1/(1+(1-4x)z)$ . Therefore,  $g_a(x, z)$  is the product of these generating functions.  $\square$

We now have a relation between  $g_a(x, z)$  and  $G_A(m, x)$ .

**Proposition 9:**

$$g_a(x, z) = \frac{1}{1-4xz} \sum_{m=0}^{\infty} (-1)^m G_A(m, xz) \left(\frac{1}{x}\right)^m. \tag{61}$$

**Proof:** Analogous to the proof of Proposition 5.  $\square$

One can check (61) by putting in the explicit form (50) of  $G_A(m, x)$  and compare with (9). Let us state the relation between  $b_n(x)$  and  $a_{n-1}(x)$  as Proposition 10.

**Proposition 10:** For  $n \in \mathbb{N}_0$  and  $a_{-1}(x) \equiv 0$ , the relation between  $b_n(x)$  and  $a_{n-1}(x)$  is given by (10).

**Proof:** The alternative expressions (5) and (8) for these two families of polynomials are used. One splits off the  $k = 0$  term in (5) with  $C_{-1} = -1/2$  from the sum and shifts the summation variable.  $\square$

**Corollary 3:** The coefficients of the triangular arrays  $A(n, m)$  and  $B(n, m)$  are related as given by (11).

**Proof:** The relation (10) between the polynomials is, with the help of (19) and (20), written for the coefficients  $a(n-1, m)$ , or by (47) for  $A(n, m+1)$  and  $B(n, m)$ .  $\square$

It remains to compute the explicit expression for the coefficients  $a(n, k)$  of  $a_n(x)$  defined by (19). Because of (47), it suffices to determine  $A(n, m)$ .

**Corollary 4:** The triangular array numbers  $A(n, m)$  are given explicitly by formula (7).

**Proof:** The formula (4) written for  $B(n, m-1)$  is used in relation (11).  $\square$

**Note 4:** This formula for  $A(n, m)$  satisfies indeed the recursion relation (48) with the given input. The first term,

$$\frac{1}{2} 4^{n-m+1} \binom{n}{m-1}$$

satisfies it because of the binomial identity

$$\binom{n}{m-1} = \binom{n-1}{m-1} + \binom{n-1}{m-2}.$$

For the second term of  $A(n, m)$  in (7) one has to prove

$$\binom{n}{m-1} \binom{2n}{n} = 4 \binom{n-1}{m-1} \binom{2(n-1)}{n-1} + \binom{n-1}{m-2} \binom{2(n-1)}{n-1} \frac{2(2m-3)}{m-1},$$

or after division by  $\binom{2(n-1)}{n-1}$ ,

$$\frac{2n-1}{n} \binom{n}{m-1} = 2 \binom{n-1}{m-1} + \binom{n-1}{m-2} \frac{2m-3}{m-1},$$

which reduces to the trivial identity  $2n-1 = 2(n-m+1) + 2m-3$ . Both terms together, i.e., (7), satisfy the input  $A(n, n) \equiv 1$ .

**Note 5:**  $A(n, m)$  was found originally after iteration in the form (with  $n \geq m > 0$  and  $(-1)!! := 1$ )

$$A(n, m) = 2 \cdot 4^{n-m} \binom{n}{m-1} - \frac{\prod_{k=1}^m (2(n-m) + 2k - 1)}{(2m-3)!!} C_{n-m}. \quad (62)$$

$A(n, 0) = C_n$ . It is easy to establish the equivalence with (7).

In the original derivation of the formula (7) for  $A(n, m)$ , it turned out to be convenient to introduce a rectangular array of integers  $\hat{A}(n, m)$  for  $n, m \in \mathbb{N}_0$  as follows:  $\hat{A}(0, m) \equiv 1$ ,  $\hat{A}(n, 0) := -C_n$  for  $n \in \mathbb{N}$ , and for  $m \in \mathbb{N}$  and  $n \in \mathbb{N}_0$ ,  $\hat{A}(n, m)$  is defined by (12) or, equivalently, by (13). The  $A(n, m)$  recursion (48) translates (with the help of the Pascal-triangle identity) into

$$\hat{A}(n, m) = 4\hat{A}(n-1, m) + \hat{A}(n, m-1). \tag{63}$$

This leads, after iteration and use of  $\hat{A}(0, m) \equiv 1$  from (12) with  $A(n, n) \equiv 1$ , to

$$\hat{A}(n, m) = 4^n \sum_{k=0}^n \hat{A}(k, m-1) / 4^k. \tag{64}$$

Thus, the following proposition describes column sequences of the  $\hat{A}(n, m) \equiv C4(n, m)$  array.

**Proposition 11:** The  $m^{\text{th}}$  column sequence of the  $\hat{A}(n, m)$  array,  $\{\hat{A}(n, m)\}_{n \in \mathbb{N}_0}$ , is the convolution of the sequence  $\{\hat{A}(n, 0)\}_{n \in \mathbb{N}_0} = \{1, -1, -2, -5, \dots\}$ , generated by  $2 - c(x)$ , and the  $m^{\text{th}}$  convolution of the power sequence  $\{4^k\}_{k \in \mathbb{N}_0}$ .

*Proof:* Iteration of (64) with the  $\hat{A}(n, 0)$  input.  $\square$

**Corollary 5:** The ordinary generating function of the  $m^{\text{th}}$  column sequence of the  $\hat{A}(n, m)$  array (13) is given by

$$G_{\hat{A}}(m, x) := \sum_{n=0}^{\infty} \hat{A}(n, m)x^n = (2 - c(x)) \left( \frac{1}{1-4x} \right)^m \tag{65}$$

for  $m \in \mathbb{N}_0$ .

*Proof:* Use Proposition 11 written for generating functions.  $\square$

Because of the convolution of the (negative) Catalan sequence with powers of 4, we shall call this  $\hat{A}(n, m)$  array also  $C4(n, m)$ . A part of it is shown in Table 3 above. The second column sequence is given by

$$\hat{A}(n, 1) \equiv C4(n, 1) = \binom{2n+1}{n}$$

and appears as nr. 2848 in the book [8], or as A001700 in the web encyclopedia [8]. The sequence of the third column  $\{\hat{A}(n, 2) \equiv C4(n, 2)\}_{n \in \mathbb{N}_0} = \{1, 7, 38, 187, \dots\}$  is, from (64) and (62) with (12), determined by

$$4^n \sum_{k=0}^n \binom{2k+1}{k} / 4^k = (2n+3)(2n+1)C_n - 2^{2n+1},$$

and is listed as A000531 in the web encyclopedia [8]. There the fourth column sequence is now listed as A029887.

**Note 6:** The infinite dimensional lower triangular matrix  $\tilde{\mathbf{A}}$  related to the array  $\hat{A}(n, m) \equiv C4(n, m)$  by  $\tilde{A}(n, m) := \hat{A}(n-m, m+1)$  for  $n \geq m \geq 0$  and  $\tilde{A}(n, m) := 0$  for  $n < m$  is again an example of a Riordan matrix [7]. In the notation of [7],  $\tilde{\mathbf{A}} = (c(x) / \sqrt{1-4x}, x / \sqrt{1-4x})$ .

Finally, we derive identities by using, for  $n \in \mathbb{N}_0$ , equation (14) for the left-hand side of (3) and the results for  $a_{n-1}(x)$  and  $b_n(x)$  for the right-hand side. Because there are no negative powers of  $x$  on the left-hand side of (3), such powers have to vanish on the right-hand side. This leads to the first family of identities. Because

$$(1-4x)^{-n} = \sum_{k=0}^{\infty} \frac{\binom{n}{k}}{k!} 4^k x^k,$$

with Pochhammer's symbol defined after (28), this means that  $x^p](a_{n-1}(x) + b_n(x)c(x))$ , the coefficient proportional to  $x^p$ , has to vanish for  $p = 0, 1, \dots, n-1, n \in \mathbb{N}$ . This requirement reads

$$(-1)^{n-1-p} a(n-1, n-1-p) + \sum_{k=0}^p (-1)^{n-k} B(n, n-k) C_{p-k} \equiv 0. \quad (66)$$

The sum is restricted to  $k \leq p$  ( $< n$ ) because no number  $C_l$  with negative index is found in  $c(x)$ . Inserting the known coefficients produces (15).

**Proposition 12:** For  $n \in \mathbb{N}$  and  $p \in \{0, 1, \dots, n-1\}$  identity (D1), given by (15), holds.

**Proof:** With (47), (66) becomes

$$\sum_{k=0}^p (-1)^{p-k} C_{p-k} B(n, n-k) = A(n, n-p), \quad (67)$$

which is (D1) of (15) if the summation index  $k$  is changed into  $p-k$ , and the symmetry of the binomial coefficients is used.  $\square$

**Example 4:** Take  $p = n-1 \in \mathbb{N}_0$ :

$$\sum_{k=0}^{n-1} (-1)^k \binom{n}{k+1} \frac{1}{2k+1} = 4^n / \binom{2n}{n} - 1 = 2A(n, 1) / \binom{2n}{n}. \quad (68)$$

With this identity we have found a sum representation for the convolution of the Catalan sequence and powers of 4:

$$s_{n-1}(4) := 4^{n-1} \sum_{k=0}^{n-1} C_k / 4^k = \frac{1}{2} \binom{2n}{n} \sum_{k=0}^{n-1} (-1)^k \binom{n}{k+1} \frac{1}{2k+1}$$

[cf. (35) with (31)].

The second family of identities, (D2) of (16), results from comparing powers  $x^k$  with  $k \in \mathbb{N}_0$  on both sides of (3) after expansion of  $(1-4x)^{-n}$  as given above in the text before (66). Only the second term  $b_n(x)c(x)$  contributes because  $a_{n-1}(x)/x^n$  has only negative powers of  $x$ . Thus, with definition (14), one finds, for  $k \in \mathbb{N}_0$  and  $n \in \mathbb{N}$ ,

$$C(n, k) = \sum_{l=0}^k \frac{(n)_l 4^l}{l!} \sum_{j=0}^n (-1)^{n-j} B(n, n-j) C_{n-j+k-l}, \quad (69)$$

which is, after interchange of the summations and insertion of  $B(n, n-j)$  from (4), the desired identity (D2) if also the summation index  $j$  is changed to  $n-q$ .

Thus, we have shown

**Proposition 13:** For  $k \in \mathbb{N}_0$  and  $n \in \mathbb{N}$ , identity (D2) of (16) with  $C(n, k)$  defined by (14) holds true.

**Example 5:** Take  $k = 0, n \in \mathbb{N}$ . Then we have

$$\sum_{j=0}^n (-1)^j \binom{n+1}{j+1} \equiv 1, \quad (70)$$

which is elementary.

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