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# Involutions on Generating Functions

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**Abstract**

We study a family of involutions on the space of sequences. Many arithmetically or combinatorially interesting sequences appear as eigensequences of the involutions. We develop new tools for studying sequences using these involutions.

## 1 Introduction

In his paper [8], Kaneko proved an interesting identity

$$\sum_{i=0}^{n+1} \binom{n+1}{i} (n+i+1) B_{n+i} = 0 \tag{1}$$

for Bernoulli numbers $B_n$ defined by the exponential generating function

$$\frac{x}{e^x - 1} = \sum_{n=0}^{\infty} B_n \frac{x^n}{n!}. \tag{2}$$

By using Kaneko's identity, we need about half the number of terms for computing $B_n$ compared with the usual identity

$$\sum_{i=0}^{n} \binom{n+1}{i} (-1)^i B_i = n+1.$$

Kaneko proved the identity by means of a continued fraction expansion. His paper also contains a sketch of another proof due to Zagier who uses an involutive linear action on sequences. Zagier's involution was further studied by various authors (see, for example, [6, 12, 13]).

The aim of this paper is to define an infinite family of involutions on the space of sequences, which generalize Zagier's involution, and to prove various properties of the involutions and their eigensequences. The definition of the involutions looks like that of modular forms and this resemblance enables us to show that generating functions of eigensequences enjoy properties analogous to modular forms. The importance of these involutions is not only to provide a tool to study general sequences but also to elicit a symmetry in classical and important sequences appearing as eigensequences. In fact, as an application, we show that these involutions yield many interesting identities involving the Bernoulli numbers, the Fibonacci numbers and so on.

This paper is organized as follows. In the next section, we define the involutions and show their basic properties. In Section 3, we give examples of eigensequences of the involutions. It will become apparent that many arithmetically and/or combinatorially interesting sequences (namely, many core sequences in OEIS [15]) are eigensequences. In Section 4, we construct differential operators on the generating functions of eigensequences. In particular, we prove the existence of analogues of the Cohen-Rankin brackets. These operators produce new eigensequences from given eigensequences. In Section 5, we consider the action of the involutions on the endomorphism ring of the sequences. By studying this action, we can produce infinitely many linear identities for eigensequences in Section 6 including a generalization of Kaneko's identity.

Throughout this paper, the binomial coefficient is a generalized one defined by

$$\binom{n}{k} = \frac{n(n-1)\cdots(n-k+1)}{k!} = \frac{n^{\underline{k}}}{k!}$$

for $k \geq 0$, where $n^{\underline{k}}$ is the falling factorial power defined by

$$n^{\underline{k}} = n(n-1)\cdots(n-k+1).$$

# 2 Definition and basic properties of involutions

Let $\mathscr{S}$ be the set of sequences in a field $F$ of characteristic 0. Though we are primarily interested in integer or rational sequences, we do not need such restriction to develop a general theory. Thus we start with a general field $F$.

Since we have to deal with many sequences and their generating functions, we need a more systematic notation than usual. We denote the geometric series $\{r^n\}$ of the ratio $r$ by

$\langle r \rangle$ and also by $\boldsymbol{\delta}_n$ the sequence whose terms are all 0 except for the $n$-th term which is 1. For any sequence $\boldsymbol{a} = \{a_n\} \in \mathscr{S}$, we associate formal power series

$$G_i(\boldsymbol{a}, x) = \sum_{n=0}^{\infty} a_n \frac{x^n}{(n!)^i} \in F[[x]] \quad (i = 0, 1).$$

The series $G_0(\boldsymbol{a}, x)$ is the ordinary generating function and $G_1(\boldsymbol{a}, x)$ is the exponential generating function of $\boldsymbol{a}$. Further, we use the notation

$$[n]\boldsymbol{a} = a_n \text{ and } \left[\frac{x^n}{(n!)^i}\right] G_i(\boldsymbol{a}, x) = a_n.$$

By the correspondence $\boldsymbol{a} \mapsto G_i(\boldsymbol{a}, x)$, we have isomorphisms of $F$-vector spaces:

$$\varphi_i \ : \ \mathscr{S} \longrightarrow F[[x]].$$

By these isomorphisms, we can endow $\mathscr{S}$ with $F$-algebra structures. Namely we can define products on $\mathscr{S}$ by

$$[n](\boldsymbol{a} \underset{i}{*} \boldsymbol{b}) = \sum_{j=0}^{n} \binom{n}{j}^i a_j b_{n-j}.$$

The operation $\underset{0}{*}$ is called the ordinary convolution and $\underset{1}{*}$ is the binomial convolution. We set $\mathscr{S}_i = \varphi_i(\mathscr{S})$.

In addition, we denote the term-wise product sequence simply by $\boldsymbol{ab}$, that is, $[n](\boldsymbol{ab}) = [n]\boldsymbol{a} \cdot [n]\boldsymbol{b}$.

**Definition 1.** We define an action of a lower triangular $2 \times 2$ regular matrix $A = \begin{bmatrix} a & 0 \\ c & d \end{bmatrix}$ over $F$ on $\mathscr{S}_0$ and $\mathscr{S}$ by

$$G_0|_{[A]_k}(\boldsymbol{a}, x) = G_0(\boldsymbol{a}|_{[A]_k}, x) = (cx + d)^{-k} G_0\left(\boldsymbol{a}, \frac{ax}{cx + d}\right),$$

where $k$ is an integer called the *weight* of the action.

It is plain to see that this operation satisfies

$$(G_0|_{[A]_k})|_{[B]_k} = G_0|_{[AB]_k} \tag{3}$$

for any such matrices $A$ and $B$. Thus this is a well-defined action on $\mathscr{S}$. The $n$-th term of the new sequence is explicitly given by

$$[n](\boldsymbol{a}|_{[A]_k}) = \frac{1}{d^{n+k}} \sum_{j=0}^{n} \binom{n+k-1}{j} a_{n-j}\ a^{n-j}\ (-c)^j. \tag{4}$$

If the order of $A$ in $\mathrm{GL}_2(\mathrm{F})$ is 2, then we have an involution on $\mathscr{S}$. Therefore we are particularly interested in the actions of the following matrices of order 2:

$$-I = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}, \ A_c = \begin{bmatrix} -1 & 0 \\ c & 1 \end{bmatrix}, \text{ and } -A_c = \begin{bmatrix} 1 & 0 \\ -c & -1 \end{bmatrix},$$

where $c$ is any element in $F$. (Later at some places we take $c$ from a ring containing $F$). For these matrices, the actions are given by

$$[n](\boldsymbol{a}|_{[-I]_k}) = (-1)^k a_n,$$

$$[n](\boldsymbol{a}|_{[A_c]_k}) = (-1)^n \sum_{j=0}^{n} \binom{n+k-1}{j} a_{n-j}\, c^j, \tag{5}$$

$$[n](\boldsymbol{a}|_{[-A_c]_k}) = (-1)^{n+k} \sum_{j=0}^{n} \binom{n+k-1}{j} a_{n-j}\, c^j. \tag{6}$$

In particular, a special case

$$[n](\boldsymbol{a}|_{[A_{-1}]_1}) = \sum_{j=0}^{n} \binom{n}{j} (-1)^j a_j$$

is the action which Zagier [8] used. Since the action of $-I$ is more or less trivial, we will not consider it further and we assume that $A$ is one of $\pm A_c$ unless otherwise specified. We write down the first few terms of $\boldsymbol{a}|_{[A_c]_k}$ for later convenience:

$$\begin{aligned}
&[0](\boldsymbol{a}|_{[A_c]_k}) = a_0;\\
&[1](\boldsymbol{a}|_{[A_c]_k}) = -ka_0c - a_1;\\
&[2](\boldsymbol{a}|_{[A_c]_k}) = \binom{1+k}{2} a_0c^2 + (1+k)a_1c + a_2;\\
&[3](\boldsymbol{a}|_{[A_c]_k}) = -\binom{2+k}{3}a_0c^3 - \binom{2+k}{2}a_1c^2 - (2+k)a_2c - a_3.
\end{aligned} \tag{7}$$

The action of $[A]_k$ on $\mathscr{S}$ induces an action on $\mathscr{S}_1$.

**Proposition 2.** *Let $\boldsymbol{a} \in \mathscr{S}$. For positive integers $k$, we have*

$$G_1(\boldsymbol{a}|_{[A]_k}, x) = \frac{1}{d^k}\frac{d^{k-1}}{dx^{k-1}}\left(\exp\left(-\frac{c}{d}x\right)\underbrace{\int\cdots\int}_{k-1} G_1\left(\left\langle\frac{a}{d}\right\rangle \boldsymbol{a}, x\right)\underbrace{dx\cdots dx}_{k-1}\right),$$

*where all the integral sign means the formal integration from 0 to $x$.*

*Proof.* We compute

$$\begin{aligned}
G_1(\boldsymbol{a}|_{[A]_k}, x) &= \frac{1}{d^k}\sum_{n\geq 0}\frac{1}{d^n}\left(\sum_{j=0}^{n}\binom{n+k-1}{j}(-c)^j a^{n-j}a_{n-j}\right)\frac{x^n}{n!}\\
&= \frac{1}{d^k}\sum_{n\geq 0}\frac{(n+k-1)!}{n!}\left(\sum_{j=0}^{n}\binom{n}{j}\left(-\frac{c}{d}\right)^j\left(\frac{a}{d}\right)^{n-j} a_{n-j}(n-j)!\frac{1}{(n-j+k-1)!}\right)\frac{x^n}{n!}.
\end{aligned}$$

The inner sum is the binomial convolution of

$$\left\langle -\frac{c}{d}\right\rangle \quad \text{and} \quad \left(\frac{a}{d}\right)^n a_n \frac{n!}{(n+k-1)!}.$$

The exponential generating function of the latter sequence is

$$x^{1-k}\underbrace{\int\cdots\int}_{k-1} G_1\left(\left\langle\frac{a}{d}\right\rangle \boldsymbol{a}, x\right)\underbrace{dx\cdots dx}_{k-1}.$$

This completes the proof of the proposition. □

The special case where $k=1$ will be important for us:

$$G_1(\boldsymbol{a}|_{[A]_1}, x) = \frac{1}{d}\exp\left(-\frac{c}{d}x\right) G_1\left(\left\langle\frac{a}{d}\right\rangle \boldsymbol{a}, x\right). \tag{8}$$

*Remark* 3. We can prove the following formula similarly for $k \le 0$:

$$\left[\frac{x^n}{n!}\right] G_1(\boldsymbol{a}|_{[A]_k}, x) = \left[\frac{x^n}{n!}\right]\frac{1}{d^k}\underbrace{\int\cdots\int}_{1-k}\left(\exp\left(-\frac{c}{d}x\right)\frac{d^{1-k}}{dx^{1-k}}G_1\left(\left\langle\frac{a}{d}\right\rangle \boldsymbol{a}, x\right)\right)\underbrace{dx\cdots dx}_{1-k}$$

provided $n \ge 1-k$.

Now we define the main object of our study.

**Definition 4.** Let $A$ be a lower triangular regular matrix of order 2. A sequence $\boldsymbol{a}$ satisfying

$$\boldsymbol{a}|_{[A]_k} = s\boldsymbol{a}$$

with some $s \in \{\pm 1\}$ is called an *eigensequence* of $[A]_k$. We call $s$ the *sign* of the eigensequence. We denote the eigenspaces by

$$\mathscr{S}(A)_k^+ = \left\{\boldsymbol{a}\in\mathscr{S} \ : \ \boldsymbol{a}|_{[A]_k} = +\boldsymbol{a}\right\} \text{ and } \mathscr{S}(A)_k^- = \left\{\boldsymbol{a}\in\mathscr{S} \ : \ \boldsymbol{a}|_{[A]_k} = -\boldsymbol{a}\right\}.$$

Although a resemblance between this definition and that of modular forms is clear (see [11] for example), several differences will appear in the following. Here we only note that each space $\mathscr{S}(A)_k^\pm$ is an infinite-dimensional $F$-vector space for every integer $k$.

By (4) we have

$$\boldsymbol{a}\in\mathscr{S}(A)_k^\pm \Leftrightarrow \left(\pm 1 - \frac{a^n}{d^{n+k}}\right) a_n = \frac{1}{d^{n+k}}\sum_{j=1}^{n}\binom{n+k-1}{j} a_{n-j}a^{n-j}(-c)^j. \tag{9}$$

If $A = \pm A_c$, then $\left(\pm 1 - \frac{a^n}{d^{n+k}}\right)$ equals $\pm 2$ or 0 alternatively with $n$. This means that we can choose one of the two consecutive terms freely and the next term is determined automatically by the preceding terms. We also define

$$\mathscr{S}_i(A)_k^\pm = \varphi_i(\mathscr{S}(A)_k^\pm).$$

When $A = A_0$, the eigensequences are easy to describe.

**Proposition 5.** *The following equivalences hold for $i = 0, 1$:*

$$\boldsymbol{a} \in \mathscr{S}(A_0)_k^+ \iff G_i(\boldsymbol{a}, x) \text{ is an even power series};$$
$$\boldsymbol{a} \in \mathscr{S}(A_0)_k^- \iff G_i(\boldsymbol{a}, x) \text{ is an odd power series.}$$

*Proof.* If $A = A_0$, then Definition 1 implies that $\boldsymbol{a} \in \mathscr{S}(A_0)_k^{\pm}$ if and only if

$$G_0(\boldsymbol{a}, -x) = \pm G_0(\boldsymbol{a}, x).$$

This means that $G_0(\boldsymbol{a}, x)$ is an odd or even power series according to the sign of $\boldsymbol{a}$. Moreover, $G_0(\boldsymbol{a}, x)$ is an odd (resp. even) power series if and only if $G_1(\boldsymbol{a}, x)$ is an odd (resp. even) power series. This completes the proof. □

**Example 6.** The sequence of the Euler numbers $\boldsymbol{E} = \{E_n\}$ (see [7, p.559]) is an example in $\mathscr{S}(A_0)_k^+$, because it is defined by

$$G_1(\boldsymbol{E}, x) = \sec x = \sum_{n=0}^{\infty} E_{2n} \frac{x^{2n}}{(2n)!}. \tag{10}$$

An example from $\mathscr{S}(A_0)_k^-$ is the sequence of the tangent numbers $\boldsymbol{T} = \{T_n\}$ defined by

$$G_1(\boldsymbol{T}, x) = \tan x = \sum_{n=0}^{\infty} T_{2n+1} \frac{x^{2n+1}}{(2n+1)!},$$

which have a close relation to the Bernoulli numbers [7, p.287].

Also we can deduce the following identities readily from (5) and (6).

**Proposition 7.** *We have*

$$\mathscr{S}(A_c)_k^{\pm} = \begin{cases} \mathscr{S}(-A_c)_k^{\pm}, & \text{if } k \text{ is even;} \\ \mathscr{S}(-A_c)_k^{\mp}, & \text{if } k \text{ is odd.} \end{cases}$$

By these propositions, we shall mainly consider the eigenspaces $\mathscr{S}(A_c)_k^{\pm}$ with $c \neq 0$.

As in the case of modular forms, the space of eigensequences has a graded structure.

**Proposition 8.** *If $\boldsymbol{a} \in \mathscr{S}(A)_{k_1}^+$ and $\boldsymbol{b} \in \mathscr{S}(A)_{k_2}^+$, then $\boldsymbol{a} \underset{0}{*} \boldsymbol{b} \in \mathscr{S}(A)_{k_1+k_2}^+$. If $A \neq \pm A_0$, then $\mathscr{S}(A)^+ = \oplus_{k\in\mathbb{Z}} \mathscr{S}(A)_k^+$ is a graded ring under this product.*

*Proof.* Let $\boldsymbol{a} \in \mathscr{S}(A)_{k_1}^+$ and $\boldsymbol{b} \in \mathscr{S}(A)_{k_2}^+$. Then we have

$$\begin{aligned} G_0|_{[A]_{k_1+k_2}}(\boldsymbol{a} \underset{0}{*} \boldsymbol{b}, x) &= (G_0(\boldsymbol{a}, x) G_0(\boldsymbol{b}, x))|_{[A]_{k_1+k_2}} \\ &= (cx+d)^{-(k_1+k_2)} G_0\left(\boldsymbol{a}, \frac{ax}{cx+d}\right) G_0\left(\boldsymbol{b}, \frac{ax}{cx+d}\right) \\ &= G_0(\boldsymbol{a}|_{[A]_{k_1}}, x) G_0(\boldsymbol{b}|_{[A]_{k_2}}, x) \\ &= G_0(\boldsymbol{a}, x) G_0(\boldsymbol{b}, x) \\ &= G_0(\boldsymbol{a} \underset{0}{*} \boldsymbol{b}, x). \end{aligned}$$

This shows the first assertion. To show the second assertion, it suffices to prove that if $k_1 \neq k_2$, then $\mathscr{S}(A)^+_{k_1} \cap \mathscr{S}(A)^+_{k_2} = \{0\}$. Let $\boldsymbol{a} \in \mathscr{S}(A)^+_{k_1} \cap \mathscr{S}(A)^+_{k_2}$. Then

$$(cx+d)^{k_1} G_0\left(\boldsymbol{a}, \frac{ax}{cx+d}\right) = (cx+d)^{k_2} G_0\left(\boldsymbol{a}, \frac{ax}{cx+d}\right)$$

holds. Since $F[[x]]$ is an integral domain, we have $k_1 = k_2$ provided $c \neq 0$. □

In a similar manner, we can show the following proposition.

**Proposition 9.** *Let $s_1, s_2 \in \{\pm 1\}$. If $\boldsymbol{a} \in \mathscr{S}(A)^{s_1}_{k_1}$ and $\boldsymbol{b} \in \mathscr{S}(A)^{s_2}_{k_2}$, then $\boldsymbol{a} \underset{0}{*} \boldsymbol{b} \in \mathscr{S}(A)^{s_1 s_2}_{k_1+k_2}$.*

By these propositions, it is clear that considering these involutions with varying $k$ rather than a fixed $k$.

A special case of Proposition 9 implies the following.

**Proposition 10.** *There exists an isomorphism*

$$\mathscr{S}(A_c)^+_k \cong \mathscr{S}(A_c)^-_{k-1}, \quad \boldsymbol{a} \mapsto \boldsymbol{\delta}_1 \underset{0}{*} \boldsymbol{a}$$

*of vector spaces for every integer $k$.*

*Proof.* Since $G_0(\boldsymbol{\delta}_1, x) = x$, we can see $\boldsymbol{\delta}_1 \in \mathscr{S}(A_c)^-_{-1}$. It follows from Proposition 9 that the convolution $\boldsymbol{\delta}_1 \underset{0}{*} \boldsymbol{a}$ belongs to $\mathscr{S}(A_c)^-_{k-1}$. The given map is obviously an injective linear map. To show the surjectivity, we note that the sequence $\boldsymbol{\delta}_1 \underset{0}{*} \boldsymbol{a}$ is a shift of $\boldsymbol{a}$ to the right. Therefore it is enough to prove that every sequence in $\mathscr{S}(A_c)^-_{k-1}$ begins with 0. This assertion follows immediately from (7). □

In terms of generating functions, Proposition 10 gives the following isomorphism:

$$\mathscr{S}_0(A)^+_k \cong \mathscr{S}_0(A)^-_{k-1}, \quad G_0(\boldsymbol{a}, x) \mapsto x G_0(\boldsymbol{a}, x).$$

It is worth while to remark that the same correspondence gives an injective linear map from $\mathscr{S}(A_c)^-_k$ to $\mathscr{S}(A_c)^+_{k-1}$ but this is not surjective in general.

There is a variant of Proposition 9 for binomial convolution. Before we state it, we note the following equivalence. If $k = 1$, then by (8) we see

$$\boldsymbol{a} \in \mathscr{S}(A_c)^{\pm}_1 \iff G_1(\boldsymbol{a}, x) = \pm \exp(-cx) G_1(\langle -1 \rangle \boldsymbol{a}, x), \tag{11}$$

which is a generalization of [12, Theorem 3.2]. Observe here that we have an identity

$$G_1(\langle -1 \rangle \boldsymbol{a}, x) = G_1(\boldsymbol{a}, -x).$$

**Proposition 11.** *Let $s_1, s_2 \in \{\pm 1\}$. If $\boldsymbol{a} \in \mathscr{S}(A_{c_1})^{s_1}_1$ and $\boldsymbol{b} \in \mathscr{S}(A_{c_2})^{s_2}_1$, then*

$$\boldsymbol{a} \underset{1}{*} \boldsymbol{b} \in \mathscr{S}(A_{c_1+c_2})^{s_1 s_2}_1.$$

*Proof.* By (11), we have

$$G_1(\boldsymbol{a}, x) = s_1 \exp(-c_1 x) G_1(\langle -1 \rangle \boldsymbol{a}, x),$$
$$G_1(\boldsymbol{b}, x) = s_2 \exp(-c_2 x) G_1(\langle -1 \rangle \boldsymbol{b}, x).$$

Multiplying the both sides, we obtain

$$G_1(\boldsymbol{a}, x) G_1(\boldsymbol{b}, x) = s_1 s_2 \exp(-(c_1 + c_2)x) G_1(\langle -1 \rangle \boldsymbol{a}, x) G_1(\langle -1 \rangle \boldsymbol{a}, x).$$

The left hand side is equal to $G_1(\boldsymbol{a} \underset{1}{*} \boldsymbol{b}, x)$. From the right hand side, we get

$$G_1(\langle -1 \rangle \boldsymbol{a}, x) G_1(\langle -1 \rangle \boldsymbol{a}, x) = G_1(\boldsymbol{a}, -x) G_1(\boldsymbol{b}, -x) = G_1(\boldsymbol{a} \underset{1}{*} \boldsymbol{b}, -x) = G_1(\langle -1 \rangle (\boldsymbol{a} \underset{1}{*} \boldsymbol{b}), x).$$

Now the proposition follows from (11). □

Both Propositions 9 and 11 will be generalized in a broader context in Section 4 (see Theorems 31 and 33).

If the 0-th term of a sequence $\boldsymbol{a} \in \mathscr{S}$ is not 0, then $G_i(\boldsymbol{a}, x)$ $(i = 0, 1)$ are invertible in $F[[x]]$.

**Corollary 12.** *If $\boldsymbol{a} \in \mathscr{S}(A_c)_k^+$ and $[0]\boldsymbol{a} \neq 0$, then we have*

$$G_0(\boldsymbol{a}, x)^{-1} \in \mathscr{S}_0(A_c)_{-k}^+.$$

*Moreover, if $k = 1$, then we have*

$$G_1(\boldsymbol{a}, x)^{-1} \in \mathscr{S}_1(A_{-c})_1^+.$$

*Proof.* If the generating functions are invertible, then $G_i(\boldsymbol{a}, x) G_i(\boldsymbol{a}, x)^{-1} = 1 = G_i(\boldsymbol{\delta}_0, x)$. Since $\boldsymbol{\delta}_0$ belongs to $\mathscr{S}(A_c)_0^+ \cap \mathscr{S}(A_0)_k^+$ for all $c$ and $k$, the corollary follows from Propositions 9 and 11. □

The following propositions give easy ways to alter $c$.

**Proposition 13.** *If $\boldsymbol{a} \in \mathscr{S}(A_c)_k^\pm$, then $\langle r \rangle \boldsymbol{a} \in \mathscr{S}(A_{cr})_k^\pm$.*

We omit the proof since it is straightforward.

**Proposition 14.** *If $\boldsymbol{a} \in \mathscr{S}(A_{c_1})_k^\pm$, then $\boldsymbol{a}|_{[A_{c_2}]_k} \in \mathscr{S}(A_{2c_2 - c_1})_k^\pm$.*

*Proof.* Using (3) and the relation $A_{c_2} A_{2c_2 - c_1} = A_{c_1} A_{c_2}$, we compute

$$(\boldsymbol{a}|_{[A_{c_2}]_k})_{[A_{2c_2 - c_1}]_k} = \boldsymbol{a}|_{[A_{c_2} A_{2c_2 - c_1}]_k} = \boldsymbol{a}|_{[A_{c_1} A_{c_2}]_k} = (\boldsymbol{a}|_{[A_{c_1}]_k})_{[A_{c_2}]_k} = \pm \boldsymbol{a}|_{[A_{c_2}]_k}.$$

This proves the assertion. □

We close this section with the following remark. For each $A$ of order 2 and each $k$, we have projections

$$\pi(A)_k^+ = \frac{1+[A]_k}{2} \ : \ \mathscr{S} \longrightarrow \mathscr{S}(A)_k^+$$

and

$$\pi(A)_k^- = \frac{1-[A]_k}{2} \ : \ \mathscr{S} \longrightarrow \mathscr{S}(A)_k^-.$$

Also by Proposition 8, if $A \neq \pm A_0$, then there is a surjection

$$\pi(A) = \bigoplus_{k\in\mathbb{Z}} \pi(A)_k^+ \ : \ \mathscr{S} \longrightarrow \mathscr{S}(A)^+ = \bigoplus_{k\in\mathbb{Z}} \mathscr{S}(A)_k^+.$$

The kernel of $\pi(A)$ is $\cap_k \mathscr{S}(A)_k^-$. In a similar manner as in the proof of Proposition 8, it is shown that it is actually 0. Hence the surjection is an isomorphism.

# 3 Eigensequences

In this section, we give various examples of sequences in $\mathscr{S}(A_c)_k^\pm$ (see Propositions 5 and 7). Although it is easy to construct such sequences from arbitrary sequences using the projections $\pi(A)_k^\pm$, almost all sequences thus obtained are unnatural and useless. Therefore we systematically search interesting eigensequences in the following. In the course of searching, we can reveal the structure of $\mathscr{S}(A_c)_k^\pm$ (Theorem 18). Note that Z.-H. Sun [12, Section 2] found some examples in $\mathscr{S}(A_{-1})_1^+$ and some of the following examples are generalizations of his.

We begin with necessary conditions for a given sequence to be an eigensequence.

**Lemma 15.** *Let $\boldsymbol{a} \in \mathscr{S}$.*

1. *If $\boldsymbol{a} \in \mathscr{S}(A_c)_k^+$, then*

$$a_1 = -\frac{1}{2}kca_0, \qquad a_3 = \frac{1}{24}c(2+k)(-12a_2 + a_0c^2k + a_0c^2k^2). \tag{12}$$

   *Moreover $k$ is an integer solution of the quadratic equation*

$$(a_1^3 + 3a_0^2a_3 - 3a_0a_1a_2)k^2 + (3a_1^3 - 6a_0a_1a_2)k + 2a_1^3 = 0. \tag{13}$$

2. *If $\boldsymbol{a} \in \mathscr{S}(A_c)_k^-$, then*

$$a_0 = 0, \tag{14}$$

$$a_2 = -\frac{1}{2}(1+k)ca_1, \tag{15}$$

$$a_4 = \frac{1}{24}c(3+k)(-12a_3 + a_1c^2(k+1) + a_1c^2(k+1)^2). \tag{16}$$

*Moreover $k$ is an integer solution of the quadratic equation*

$$(a_2^3+3a_1^2a_4-3a_1a_2a_3)k^2+(6a_1^2a_4-12a_1a_2a_3+5a_2^3)k+3a_1^2a_4+6a_2^3-9a_1a_2a_3=0.$$

*Proof.* The conditions on each term is obtained directly from the identities following (7) (see also (9)). By eliminating $c$ from these equalities, we obtain the equations for $k$.

Note that the latter half 2 also follows from the isomorphism

$$\mathscr{S}(A_c)^+_{k+1}\cong\mathscr{S}(A_c)^-_k$$

in Proposition 10. In other words, the relations (14), (15) and (16) can be obtained by a right shift of the relations in 1 and a substitution of $k$ by $k+1$. □

Using this lemma, we can decide that a given sequence is not an eigensequence or determine candidates for $c$ and $k$.

We first give some examples of eigensequences with polynomial generating functions, namely finite eigensequences.

**Example 16** (Polynomial generating functions)**.** If the generating function $G_0(\boldsymbol{a},x)$ for $\boldsymbol{a}\in\mathscr{S}(A_c)^{\pm}_k$ $(c\neq 0)$ is a polynomial of degree $m\geq 1$, then it is easy to see that the weight $k$ should be negative and equal or less than $-m$. If $k$ is exactly equal to $-m$, then we can show that $[A_c]_{-m}$ defines an involution on the subspace

$$F[x]_m=\{G(x)\in F[x]\ :\ \deg G(x)\leq m\}.$$

By Lemma 15, we have an inequality

$$\dim_F(\mathscr{S}_0(A_c)^+_k\cap F[x]_m)\geq\dim_F(\mathscr{S}_0(A_c)^-_k\cap F[x]_m).$$

This implies

$$\dim_F(\mathscr{S}_0(A_c)^+_k\cap F[x]_m)=\left\lceil\frac{m+1}{2}\right\rceil,\quad\dim_F(\mathscr{S}_0(A_c)^-_k\cap F[x]_m)=\left\lfloor\frac{m+1}{2}\right\rfloor.$$

As $k$ gets smaller, we have additional constraints on the terms of eigensequences. Therefore the dimension gets smaller, too.

Suppose that we want to find generating functions of degree 1 for plus eigensequences. Setting $a_2=a_3=0$ in (13), we have

$$a_1^3k^2+3a_1^3k+2a_1^3=0.$$

This equation yields that $k=-1$ or $-2$. From (12) it follows that $a_1=ca_0/2$ or $ca_0$. We can easily show that these are indeed eigensequences. Similar calculation leads to the following polynomial generating functions:

$\deg G_0=1$

$$1+\frac{c}{2}x\in\mathscr{S}_0(A_c)^+_{-1},\qquad x\in\mathscr{S}_0(A_c)^-_{-1}\cap\mathscr{S}_0(A_0)^-_k,$$
$$1+cx\in\mathscr{S}_0(A_c)^+_{-2};$$

$\deg G_0 = 2$

$$t + ctx + sx^2 \in \mathscr{S}_0(A_c)^+_{-2}, \qquad x\left(1 + \frac{c}{2}x\right) \in \mathscr{S}_0(A_c)^-_{-2},$$
$$(1 + cx)\left(1 + \frac{c}{2}x\right) \in \mathscr{S}_0(A_c)^+_{-3}, \qquad x(1 + cx) \in \mathscr{S}_0(A_c)^-_{-3}$$
$$(1 + cx)^2 \in \mathscr{S}_0(A_c)^+_{-4};$$

where $s$ and $t$ are arbitrary parameters.

Since $\boldsymbol{\delta}_1 \in \mathscr{S}(A_c)^-_{-1}$, we can prove that

$$G_0(\boldsymbol{\delta}_m, x) = x^m = G_0(\boldsymbol{\delta}_1, x)^m \in \mathscr{S}_0(A_c)^{(-1)^m}_{-m}$$

for positive integer $m$ by using Proposition 9 repeatedly. Combining this with Proposition 5, we conclude

$$\boldsymbol{\delta}_m \in \mathscr{S}(A_c)^{(-1)^m}_{-m} \cap \mathscr{S}(A_0)^{(-1)^m}_{k} \tag{17}$$

for all $c$ and $k$.

These finite eigensequences themselves are not so interesting, but they provide primary ingredients of general eigensequences. We shall see this in the following rational generating functions.

**Example 17** (Rational generating functions)**.** Combining polynomial generating functions in various ways, we have rational generating functions of eigensequences, whose weights can be computed using Proposition 9 and Corollary 12. The simplest examples are

$$G_0(\langle c\rangle, x) = \frac{1}{1 - cx} \in \mathscr{S}_0(A_{-2c})^+_1 \cap \mathscr{S}_0(A_{-c})^+_2.$$

We also have

$$G_0(\boldsymbol{a}, x) = \frac{2 + cx}{t + tcx + sx^2} \in \mathscr{S}_0(A_c)^+_1, \quad G_0(\boldsymbol{b}, x) = \frac{x}{t + tcx + sx^2} \in \mathscr{S}_0(A_c)^-_1.$$

As a special case of $\boldsymbol{a}$, we have the Lucas sequence $\boldsymbol{L} = \{L_n\}$ defined by the recurrence

$$L_0 = 2,\ L_1 = 1,\ L_n = L_{n-1} + L_{n-2}.$$

In fact, since $G_0(\boldsymbol{L}, x) = \dfrac{2 - x}{1 - x - x^2}$, we have $\boldsymbol{L} \in \mathscr{S}(A_{-1})^+_1$. As for $\boldsymbol{b}$, we have the Fibonacci sequence $\boldsymbol{F} = \{F_n\}$ defined by

$$F_0 = 0,\ F_1 = 1,\ F_n = F_{n-1} + F_{n-2},$$

whose generating function is $G_0(\boldsymbol{F}, x) = \dfrac{x}{1 - x - x^2} \in \mathscr{S}_0(A_{-1})^-_1$. This means that $\boldsymbol{F}$ and $\boldsymbol{L}$ are not only a basis of the 2-dimensional vector space of sequences satisfying the recurrence $a_{n+1} = a_n + a_{n-1}$, but also a basis as a $\langle [A_{-1}]_1\rangle$-module. This fact plays an essential role in the researches of Fibonacci numbers although it is not noticed explicitly.

The binomial coefficients $\left\{\binom{n+k-1}{n}\right\}_{n\geq 0}$ are also of this type, since the ordinary generating function is $(1-x)^{-k}$. Therefore we see

$$\left\{\binom{n+k-1}{n}\right\}_{n\geq 0} \in \mathscr{S}(A_{-1})^{-}_{2k}.$$

The generating function

$$\mathscr{E}_k(x) = \frac{1}{\left(1+\frac{c}{2}x\right)^k} = \sum_{n=0}^{\infty} \binom{n+k-1}{n} \left(-\frac{c}{2}\right)^n x^n \in \mathscr{S}_0(A_c)^{+}_{k}$$

plays a special role.

**Theorem 18.** *We have a direct sum decomposition*

$$\mathscr{S}_0(A_c)^{+}_{k} = F \cdot \mathscr{E}_k(x) \oplus x^2 \mathscr{S}_0(A_c)^{+}_{k+2}$$

*for any integer $k$. Moreover, any generating function $G_0(\boldsymbol{a}, x) \in \mathscr{S}_0(A_c)^{+}_{k}$ can be written uniquely as a sum of the form*

$$G_0(\boldsymbol{a}, x) = \sum_{i=0}^{\infty} \alpha_{2i} x^{2i} \mathscr{E}_{k+2i}(x) \quad (\alpha_{2i} \in F). \tag{18}$$

*If $G_0(\boldsymbol{a}, x) \in \mathscr{S}_0(A_c)^{-}_{k}$, then it can be written uniquely as*

$$G_0(\boldsymbol{a}, x) = \sum_{i=0}^{\infty} \alpha_{2i+1} x^{2i+1} \mathscr{E}_{k+2i+1}(x) \quad (\alpha_{2i+1} \in F). \tag{19}$$

*Proof.* Consider the map $g : \mathscr{S}_0(A_c)^{+}_{k} \longrightarrow F$ defined by $G_0(x) \mapsto G_0(0)$. The dimension of the image of $g$ is at most 1. Since $[x^0]\mathscr{E}_k(x) \neq 0$, the dimension is 1, indeed. Suppose that $G_0(x) \in \ker g$. Then $[x]G_0(x)$ is also 0 by (12). We obtain a factorization $G_0(x) = x^2 H(x)$. From (17), we have $H(x) \in \mathscr{S}_0(A_c)^{+}_{k+2}$. This proves the first assertion. Unless $H(x)$ is zero or a constant, we can continue the same argument for $H(x)$, we can expand $G_0(x)$ like (18). If $H(x)$ is zero or a constant, the process stops and obtain a finite sum expansion. The equation (19) follows from (18) by using the isomorphism in Proposition 10. □

An explicit formula for $\alpha_i$ in (18) and (19) will be given later (Proposition 28).

This theorem provides more precise test for eigensequences than Lemma 15. But we should note that it is still difficult to extract interesting sequences from this structure theorem. That being so, we continue our quest.

**Example 19.** In this example, we deal with irrational generating functions. Let $G_0(x) \in \mathscr{S}_0$ be an invertible power series. If we choose a square root $\sqrt{a_0}$ of $a_0 = G_0(0)$, then $\sqrt{G_0(x)}$ is uniquely determined and belongs to $\mathscr{S}_0 \otimes_F F(\sqrt{a_0})$. Further if $G_0(x) \in \mathscr{S}_0(A_c)^{+}_{2k}$, then it is easy to verify that $\sqrt{G_0(x)} \in \mathscr{S}_0(A_c)_k^+ \otimes_F F(\sqrt{a_0})$. As an example, since $\dfrac{1}{1-4x} \in \mathscr{S}_0(A_{-4})_2^+$, we have

$$\frac{1}{\sqrt{1-4x}} \in \mathscr{S}_0(A_{-4})_1^+,$$

whose $n$-th coefficient is $\binom{2n}{n}$. Let $\boldsymbol{C} = \{C_n\}$ be the Catalan numbers ([10, 6.2.1 Proposition]) whose generating function is known to be

$$G_0(\boldsymbol{C}, x) = \frac{1-\sqrt{1-4x}}{2x}.$$

We can show that $G_0(\boldsymbol{C}, x)^2 \in \mathscr{S}_0(A_{-4})_1^+$. Since it is well known that $G_0(\boldsymbol{C}, x)^2 = \dfrac{G_0(\boldsymbol{C}, x) - 1}{x}$, we have $\{C_{n+1}\}_{n \geq 0} \in \mathscr{S}(A_{-4})_1^+$.

The generating function of Motzkin numbers is another example:

$$\frac{1-x-\sqrt{1-2x-3x^2}}{2x} \in \mathscr{S}_0(A_{-2})_0^-.$$

**Example 20.** There are also examples of transcendental generating functions. Let $\boldsymbol{H}$ be the sequence of harmonic numbers ([7, Section 6.3]):

$$[0]\boldsymbol{H} = 0, \quad [n]\boldsymbol{H} = 1 + \frac{1}{2} + \cdots + \frac{1}{n}.$$

We have

$$G_0(\boldsymbol{H}, x) = \frac{1}{1-x} \log\left(\frac{1}{1-x}\right) \in \mathscr{S}_0(A_{-1})_2^-.$$

More generally, Pfaff's reflection law for the Gaussian hypergeometric function ([7, (5.101)])

$$\frac{1}{(-x+1)^\alpha} F\left(\begin{matrix}\alpha, \beta \\ \gamma\end{matrix} \,\middle|\, \frac{-x}{-x+1}\right) = F\left(\begin{matrix}\alpha, \gamma-\beta \\ \gamma\end{matrix} \,\middle|\, x\right)$$

implies

$$F\left(\begin{matrix}\alpha, \beta \\ 2\beta\end{matrix} \,\middle|\, x\right) \in \mathscr{S}_0(A_{-1})_\alpha^+.$$

We next proceed to find examples of exponential generating functions of eigensequences using (11).

**Example 21.** We begin with examples of polynomials of $\exp(x)$. It is possible to determine all such polynomials giving eigensequences. First of all, a monomial generating function $\exp(cx)$ gives

$$\langle c \rangle \in \mathscr{S}(A_{-2c})_1^+$$

as we saw in Example 17.

We say that a polynomial $P(t) \in F[t]$ is self-reciprocal if $u^{\deg P} P(t/u) \in F[t, u]$ is a symmetric polynomial: namely, setting $d = \deg P$, it satisfies

$$u^d P(t/u) = t^d P(u/t).$$

Also a polynomial $P(t)$ is called skew self-reciprocal if $u^{\deg P}P(t/u) \in F[t,u]$ is an alternating polynomial.

Let $P(t)$ be a self-reciprocal polynomial. Then, by setting $u = 1, t = \exp(x)$ in the above formula, we obtain $P(\exp(x)) = \exp(dx)P(\exp(-x))$. This means that $G_1(x) = P(\exp(x))$ satisfies (11) with $c = -\deg P$. Similarly using skew self-reciprocal polynomials, we are able to construct minus eigensequences.

The simplest examples of this construction are the binomials $P(t) = t^c \pm 1$, which give

$$G_1(\boldsymbol{a}^{\pm}, x) = \exp(cx) \pm 1 \in \mathscr{S}_1(A_{-c})_1^{\pm}.$$

**Example 22.** We can directly verify using (2) that the Bernoulli numbers $\boldsymbol{B} = \{B_n\}$ satisfies

$$\boldsymbol{B} \in \mathscr{S}(A_1)_1^{+}.$$

Similarly, the Genocchi numbers $\boldsymbol{G} = \{G_n\}$ ([10, Exercise 5.8 d]) defined by

$$G_1(\boldsymbol{G}, x) = \frac{2x}{\exp(x) + 1}$$

belongs to $\mathscr{S}_1(A_1)_1^{-}$.

**Example 23.** Kummer's first formula for confluent hypergeometric function (there is a typo in [2, (4.1.11)])

$$F\left(\begin{matrix}\alpha \\ \gamma\end{matrix}\middle|\, x\right) = \exp(x)\ F\left(\begin{matrix}\gamma-\alpha \\ \gamma\end{matrix}\middle|\, -x\right)$$

leads to

$$F\left(\begin{matrix}\alpha \\ 2\alpha\end{matrix}\middle|\, x\right) \in \mathscr{S}_1(A_{-1})_1^{+}.$$

We define polynomial sequences associated to a sequence.

**Definition 24.** For any $\boldsymbol{a} \in \mathscr{S}$, we define the associated polynomial sequence $\boldsymbol{P}\,(\boldsymbol{a}, t)_k$ of weight $k$ by

$$P_n(\boldsymbol{a}, t)_k = [n]\boldsymbol{P}\,(\boldsymbol{a}, t) = [n](\boldsymbol{a}|_{[A_t]_k}) = (-1)^n \sum_{j=0}^{n} \binom{n+k-1}{j} a_{n-j} t^j \in F[t].$$

Z.-W. Sun [13] and others adopted a slightly different definition. In our notation, their polynomial is $(\langle -1 \rangle \boldsymbol{a})\,|_{[A_t A_0]_1}$. Our definition seems to be more natural from our point of view.

The following proposition immediately follows from Proposition 14.

**Proposition 25.** *If* $\boldsymbol{a} \in \mathscr{S}(A_c)_k^s$*, then* $\boldsymbol{P}\,(\boldsymbol{a}, t)_k \in \mathscr{S}(A_{2t-c})_k^s$.

**Example 26.** A modified eigensequence $\langle -1 \rangle \boldsymbol{P}\,(\boldsymbol{B}, t)_1 \in \mathscr{S}(A_{1-2t})_1^{+}$ of polynomials associated to the Bernoulli sequence $\boldsymbol{B}$ is that of the Bernoulli polynomials.

The Euler polynomials $E_n(x)$ are defined by

$$\frac{2e^{tx}}{e^x + 1} = \sum_{n=0}^{\infty} E_n(t) \frac{x^n}{n!}.$$

The Euler numbers $\boldsymbol{E}$ (see Example 6) are related to the polynomials by $E_n\left(\frac{1}{2}\right) = \frac{E_n}{(-2)^n}$.

It is easy to see that $\boldsymbol{E}' = \left\{\frac{E_n}{(-2)^n}\right\} \in \mathscr{S}(A_0)_1^+$. The associated polynomials to $\boldsymbol{E}'$ are given by

$$P_n(\boldsymbol{E}', t)_1 = (-1)^n E_n\left(t + \frac{1}{2}\right)$$

and it belongs to $\mathscr{S}(A_{2t})_1^+$.

The coefficients $\alpha_{2i}$ in (18) are closely related to associated polynomials. To show this relation, we first prove the following proposition, which generalizes well-known symmetry formulas for Bernoulli and Euler polynomials.

**Proposition 27.** *If $\boldsymbol{a} \in \mathscr{S}(A_c)_k^s$, then we have*

$$(-1)^n P_n(\boldsymbol{a}, c-t)_k = sP_n(\boldsymbol{a}, t)_k.$$

*Proof.* By Proposition 25, the sequence $\boldsymbol{P}(\boldsymbol{a}, t)_k$ belongs to $\mathscr{S}(A_{2t-c})_k^s$. Therefore we have $[n]\boldsymbol{P}(\boldsymbol{a}, t)_k|_{[A_{2t-c}]_k} = sP_n(\boldsymbol{a}, t)_k$. On the other hand, using the definition of the associated polynomials, we compute

$$\begin{aligned}[n]\boldsymbol{P}(\boldsymbol{a}, t)_k|_{[A_{2t-c}]_k} &= [n]\left(\boldsymbol{a}|_{[A_t]_k}\right)|_{[A_{2t-c}]_k} = [n]\left(\boldsymbol{a}|_{[A_t A_{2t-c}]_k}\right) \\ &= [n]\left(\boldsymbol{a}|_{[A_{c-t}A_0]_k}\right) = [n]\left(\boldsymbol{P}(\boldsymbol{a}, c-t)\right)_k|_{[A_0]_k} = (-1)^n P_n(\boldsymbol{a}, c-t)_k.\end{aligned}$$

Here we used a similar method as in Proposition 14. This completes the proof. □

Now we can show the following proposition.

**Proposition 28.** *Let $\boldsymbol{a} \in \mathscr{S}(A_c)_k^{\pm}$. We have an explicit formula for (18) and (19):*

$$G_0(\boldsymbol{a}, x) = \sum_{i=0}^{\infty} P_i\left(\boldsymbol{a}, \frac{c}{2}\right)_k x^i \mathscr{E}_{k+i}(x).$$

*Proof.* First note that the central value $t = c/2$ is a root of $P_n(\boldsymbol{a}, t)$ for odd $n$ if $\boldsymbol{a} \in \mathscr{S}(A_c)_k^+$ and for even $n$ if $\boldsymbol{a} \in \mathscr{S}(A_c)_k^-$. This fact readily follows from Proposition 27. We calculate the coefficient of $x^m$ in the right hand side of the equality using (4):

$$\sum_{j=0}^{m} \binom{m+k-1}{j} P_{m-j}\left(\boldsymbol{a}, \frac{c}{2}\right)_k \left(-\frac{c}{2}\right)^j = [m]\left((\boldsymbol{a}|_{[A_{c/2}]_k})|_{[A_{c/2}]_k}\right).$$

Since $[A_{c/2}]_k$ is an involution, this is equal to $[m]\boldsymbol{a}$. This proves the proposition. □

**Example 29.** As an example, we compute the explicit expansion of the generating function $G_0(\boldsymbol{F}, x) \in \mathscr{S}_0(A_{-1})_1^-$ of Fibonacci numbers. It is easy to compute

$$\sum_{n\geq 0} P_n\left(\boldsymbol{F}, -\frac{1}{2}\right)_1 x^n = G_0|_{[A_{-1/2}]_1}(\boldsymbol{F}, x) = \frac{-x}{1 - \frac{5}{4}x^2}.$$

Therefore we get

$$G_0(\boldsymbol{F}, x) = \sum_{i=0}^{\infty} \left(\frac{5}{4}\right)^i x^{2i+1} \mathscr{E}_{2i+2}(x).$$

Taking $[x^n]$ of the both sides, we obtain a classical formula due to Catalan:

$$F_n = \frac{1}{2^{n-1}} \sum_{i=0}^{\lfloor \frac{n-1}{2} \rfloor} 5^i \binom{n}{n-2i-1}.$$

**Example 30.** There is another important family of polynomial eigensequences. They are orthogonal polynomials. The Jacobi polynomial $P_n^{(\alpha,\beta)}(t)$ is defined in terms of Gaussian hypergeometric function ([2, Definition 2.5.1]) and its exponential generating function is

$$\frac{2^{\alpha+\beta}}{R(1-x+R)^{\alpha}(1+x+R)^{\beta}}$$

(see [2, Theorem 6.4.2]) where $R$ is defined by

$$R = \sqrt{1-2tx+x^2},$$

which belongs to $\mathscr{S}_0(A_{-2t})^+_{-1}$ by Examples 16 and 19. The product $(1-x+R)(1+x-R) = 2(R+(1-xt))$ is also in $\mathscr{S}_0(A_{-2t})^+_{-1}$. Thus if $\alpha = \beta$, then the exponential generating function belongs to $\mathscr{S}_0(A_{-2t})^+_{\alpha+1}$. We conclude

$$\left\{ \frac{P_n^{(\alpha,\alpha)}(t)}{n!} \right\}_{n \geq 0} \in \mathscr{S}(A_{-2t})^+_{\alpha+1}.$$

When $\alpha = \beta = 0$, the polynomial $P_n(t) = P_n^{(0,0)}(t)$ is called the Legendre polynomial.

The following orthogonal polynomials also form polynomial eigensequences:

- The Gegenbauer polynomial $C_n^{\lambda}(t)$ ([2, p.302]).

  $$G_0(\{C_n^{\lambda}(t)\}, x) = \frac{1}{R^{2\lambda}} \in \mathscr{S}_0(A_{-2t})^+_{2\lambda}.$$

  The special case $U_n(t) = C_n^1(t)$ is called Chebyshev polynomial of second kind.

- The Chebyshev polynomial of first kind $T_n(t)$ ([2, Remark 2.5.3]). The ordinary generating function is

  $$\frac{1-tx}{R^2} \in \mathscr{S}_0(A_{-2t})^+_1.$$

- The Hermite polynomial $H_n(t)$ ([2, Section 6.1]). The exponential generating function is $\exp(2tx - t^2) \in \mathscr{S}_1(A_{-4t})^+_1$.

# 4 Differential operations on eigensequences

The main result in this section is the following analogue of the Rankin-Cohen differential operator. Since the sign differs from the original formula (see [14, Section 1]), we state and prove the following theorem for a general lower triangular matrix $A = \begin{bmatrix} a & 0 \\ c & d \end{bmatrix}$.

**Theorem 31.** *Let $n$ be a non-negative integer and $k$ and $\ell$ integers such that the following conditions are not satisfied:*

$$-n < k \leq 0 \quad or \quad -n < \ell \leq 0. \tag{20}$$

*For $f \in \mathscr{S}_0(A)_k^{s_1}$ and $g \in \mathscr{S}_0(A)_\ell^{s_2}$, we define the $n$-th Rankin-Cohen bracket of $f$ and $g$ by the formula*

$$[f,g]_n(x) = \sum_{r+s=n} (-1)^r \binom{n+k-1}{s} \binom{n+\ell-1}{r} f^{(r)}(x) g^{(s)}(x).$$

*Then we have*

$$[f,g]_n \in \mathscr{S}_0(A)_{k+\ell+2n}^{\frac{s_1 s_2}{(ad)^n}}.$$

*Proof.* Our formulation and proof are based on Zagier's article [14]. Therefore we only give a sketch of the proof. However we have to take care of negative $k$ and $\ell$. We first associate a formal power series $\tilde{f}(x,t)$ to $f(x)$

$$\tilde{f}(x,t) = \sum_{i=0}^{n} \frac{f^{(i)}(x)}{(i+k-1)^{\underline{i}}} \frac{t^i}{i!}.$$

The formal power series $\tilde{g}(x,t)$ for $g(x)$ is similarly defined. This definition makes sense if (20) are not satisfied.

Then we can prove

$$[t^n]\tilde{f}(x,-t)\tilde{g}(x,t) = \frac{[f,g]_n(x)}{(n+k-1)^{\underline{n}}(n+\ell-1)^{\underline{n}}}$$

On the other hand, we can show the following formulas for the higher derivatives by induction:

$$\frac{(ad)^n f^{(n)}\left(\frac{ax}{cx+d}\right)}{n!(n+k-1)^{\underline{n}}} = s_1 \sum_{m=0}^{n} \frac{c^{n-m}(cx+d)^{k+n+m}}{(n-m)!} \frac{f^{(m)}(x)}{m!(m+k-1)^{\underline{m}}}.$$

It follows from these formulas that $\tilde{f}$ satisfies the transformation law

$$\tilde{f}\left(\frac{ax}{cx+d}, \frac{(ad)t}{(cx+d)^2}\right) \equiv s_1 (cx+d)^k \exp\left(\frac{ct}{cx+d}\right) \tilde{f}(x,t) \pmod{t^{n+1}}.$$

A similar formula holds also for $\tilde{g}$:

$$\tilde{g}\left(\frac{ax}{cx+d}, \frac{(ad)t}{(cx+d)^2}\right) \equiv s_2 (cx+d)^\ell \exp\left(\frac{ct}{cx+d}\right) \tilde{g}(x,t) \pmod{t^{n+1}}.$$

Multiplying the both sides, we have

$$\tilde{f}\left(\frac{ax}{cx+d}, -\frac{(ad)t}{(cx+d)^2}\right)\tilde{g}\left(\frac{ax}{cx+d}, \frac{(ad)t}{(cx+d)^2}\right) \equiv s_1 s_2 (cx+d)^{k+\ell}\tilde{f}(x,-t)\tilde{g}(x,t) \pmod{t^{n+1}}.$$

Comparing the coefficients of $t^n$, we obtain

$$[f,g]_n\left(\frac{ax}{cx+d}\right) = \frac{s_1 s_2}{(ad)^n}(cx+d)^{k+\ell+2n}[f,g]_n(x).$$

This means that $[f,g]_n \in \mathscr{S}_0(A)_{k+\ell+2n}^{\frac{s_1 s_2}{(ad)^n}}$. This completes the proof of the theorem. □

As in the paper [14], the following identities involving the brackets hold for $f \in \mathscr{S}_0(A)_k^{s_1}$, $g \in \mathscr{S}_0(A)_\ell^{s_2}$ and $h \in \mathscr{S}_0(A)_m^{s_3}$:

$$\begin{gathered}
[f,g]_n = (-1)^n [g,f]_n, \\
[[f,g]_1,h]_1 + [[g,h]_1,f]_1 + [[h,f]_1,g]_1 = 0 \quad \text{(the Jacobi identity)}, \\
[[f,g]_0,h]_1 + [[g,h]_0,f]_1 + [[h,f]_0,g]_1 = 0, \\
m[[f,g]_1,h]_0 + k[[g,h]_1,f]_0 + \ell[[h,f]_1,g]_0 = 0, \\
[[f,g]_0,h]_1 = [[g,h]_1,f]_0 + [[h,f]_1,g]_0, \\
(k+m+\ell)[[f,g]_1,h]_0 = k[[h,f]_0,g]_1 - \ell[[g,h]_0,f]_1.
\end{gathered}$$

Here we write down the brackets of low degree in the case of $A = A_c$:

$$\begin{aligned}
&[f,g]_0 = fg \in \mathscr{S}_0(A_c)_{k+\ell}^{s_1 s_2}, \\
&[f,g]_1 = kfg' - \ell f'g \in \mathscr{S}_0(A_c)_{k+\ell+2}^{-s_1 s_2}, \\
&[f,g]_2 = \frac{\ell(\ell+1)}{2}f''g - (k+1)(\ell+1)f'g' + \frac{k(k+1)}{2}fg'' \in \mathscr{S}_0(A_c)_{k+\ell+4}^{s_1 s_2}.
\end{aligned}$$

**Example 32.** Let $\boldsymbol{a} = \{a_i\} \in \mathscr{S}(A_c)_k^s$ $(k > 0)$. Since $x^\ell \in \mathscr{S}(A_c)_{-\ell}^{(-1)^\ell}$ (see (17)), we have

$$\begin{aligned}
[G_0(\boldsymbol{a},x), x^\ell]_n &= \sum_{s=0}^{n}(-1)^{n-s}\binom{n+k-1}{s}\binom{n-\ell-1}{n-s}\left(\sum_{i=0}^{\infty}(i+n-s)^{\underline{n-s}}a_{i+n-s}x^i\right)\ \ell^{\underline{s}}\ x^{\ell-s} \\
&= \sum_{i=0}^{\infty}\left(\sum_{s=0}^{n}(-1)^{n-s}\binom{n+k-1}{s}\binom{n-\ell-1}{n-s}\ell^{\underline{s}}(i+n-\ell)^{\underline{n-s}}\right)a_{i+n-\ell}x^i \\
&= \sum_{i=0}^{\infty}\left(\sum_{s=0}^{n}(-1)^{n-s}\binom{n+k-1}{s}(-1)^{n-s}\binom{\ell-s}{n-s}s!\binom{\ell}{s}(n-s)!\binom{i-\ell+n}{n-s}\right)a_{i+n-\ell}x^i \\
&= n!\binom{\ell}{n}\sum_{i=0}^{\infty}\left(\sum_{s=0}^{n}\binom{n+k-1}{s}\binom{i-\ell+n}{n-s}a_{i+n-\ell}\right)x^i \\
&= n!\binom{\ell}{n}\sum_{i=0}^{\infty}\binom{2n+k+i-\ell-1}{n}a_{i+n-\ell}x^i \in \mathscr{S}_0(A_c)_{k-\ell+2n}^{s(-1)^{n+\ell}}.
\end{aligned}$$

for $n \le \ell$. Here the last equality follows from Vandermonde's convolution. An interesting case is $\ell = 2n$. Then $[*, x^{2n}]_n$ preserves the weight and we have

$$[G_0(\boldsymbol{a}, x), x^{2n}]_n = (2n)^{\underline{n}} \sum_{i=n}^{\infty} \binom{i+k-1}{n} a_{i-n} x^i \in \mathscr{S}_0(A_c)_k^{(-1)^n s}.$$

We conclude that

$$\left\{ \binom{i+k-1}{n} a_{i-n} \right\}_{i \ge 0} \in \mathscr{S}(A_c)_k^{(-1)^n s}$$

and

$$\{(i+k-1)^{\underline{n}}\, a_{i-n}\}_{i \ge 0} \in \mathscr{S}(A_c)_k^{(-1)^n s}.$$

There is also an analogue of Theorem 31 for exponential generating functions.

**Theorem 33.** *Let $n$ be a non-negative integer. If $f \in \mathscr{S}_1(A_{c_1})_1^{s_1}$ and $g \in \mathscr{S}_1(A_{c_2})_1^{s_2}$, then*

$$\{f, g\}_n = \sum_{j=0}^{n} \binom{n}{j} c_2^j (-c_1)^{n-j} f^{(j)}(x) g^{(n-j)}(x) \in \mathscr{S}_1(A_{c_1+c_2})_1^{(-1)^n s_1 s_2}.$$

*Proof.* Using Leibniz's rule, we calculate

$$\begin{aligned}
&\exp(-(c_1+c_2)x) \sum_{j=0}^{n} \binom{n}{j} c_2^j (-c_1)^{n-j} f^{(j)}(-x) \otimes g^{(n-j)}(-x) \\
&= \exp(-(c_1+c_2)x) \sum_{j=0}^{n} \binom{n}{j} c_2^j (-c_1)^{n-j} \\
&\times \left( s_1 \exp(c_1 x) \left(-c_1 - \frac{d}{dx}\right)^j f(x) \right) \otimes \left( s_2 \exp(c_2 x) \left(-c_2 - \frac{d}{dx}\right)^{n-j} g(x) \right) \\
&= s_1 s_2 \sum_{j=0}^{n} \binom{n}{j} c_2^j (-c_1)^{n-j} \left( \left(-c_1 - \frac{d}{dx}\right)^j f(x) \otimes \left(-c_2 - \frac{d}{dx}\right)^{n-j} g(x) \right) \\
&= s_1 s_2 \left( c_2 \left(-c_1 - \frac{d}{dx}\right) + (-c_1) \left(-c_2 - \frac{d}{dx}\right) \right)^n f(x) \otimes g(x) \\
&= (-1)^n s_1 s_2 \left( c_2 \frac{d}{dx} - c_1 \frac{d}{dx} \right)^n f(x) \otimes g(x) \\
&= (-1)^n s_1 s_2 \sum_{j=0}^{n} \binom{n}{j} c_2^j (-c_1)^{n-j} f^{(j)}(x) g^{(n-j)}(x).
\end{aligned}$$

This implies that

$$\exp(-(c_1+c_2)x)\{f, g\}_n(-x) = (-1)^n s_1 s_2 \{f, g\}_n(x).$$

The result follows from (11). □

If $f(x) = G_1(\boldsymbol{a}, x)$ and $g(x) = G_1(\boldsymbol{b}, x)$, then we compute

$$\left[\frac{x^m}{m!}\right] \{G_1(\boldsymbol{a}, x), G_1(\boldsymbol{b}, x)\}_n = \sum_{j=0}^{n} \binom{n}{j} c_2^j (-c_1)^{n-j} \sum_{i=0}^{m} \binom{m}{i} a_{i+j} b_{m+n-i-j}. \tag{21}$$

In particular, if $c_2 = -c_1$, then

$$\left[\frac{x^m}{m!}\right] \{G_1(\boldsymbol{a}, x), G_1(\boldsymbol{b}, x)\}_n = (-c_1)^n \left[\frac{x^{m+n}}{(m+n)!}\right] \{G_1(\boldsymbol{a}, x), G_1(\boldsymbol{b}, x)\}_0.$$

We again write down the brackets of low degree:

$$\begin{aligned} \{f, g\}_0 &= fg \in \mathscr{S}_1(A_{c_1+c_2})_1^{s_1 s_2}, \\ \{f, g\}_1 &= -c_1 f g' + c_2 f' g \in \mathscr{S}_1(A_{c_1+c_2})_1^{-s_1 s_2}, \\ \{f, g\}_2 &= c_1^2 f g'' - 2c_1 c_2 f' g' + c_2^2 f'' g \in \mathscr{S}_1(A_{c_1+c_2})_1^{s_1 s_2}. \end{aligned}$$

These brackets also satisfy the analogous properties of the Rankin-Cohen brackets. We omit to write down these formulas.

We conclude this section with a few remarks.

We can apply Theorem 33 only to eigensequences of weight 1. Thus it is worth considering a method to transform a general eigensequence to a weight-1 sequence.

The simplest method is shifting. Namely by shifting an eigensequence of positive weight to the right, it eventually becomes a sequence of weight 1. The right shifts correspond to iterated formal integration of the exponential generating function (cf. Proposition 2).

There is another interesting method. Let $\boldsymbol{a} \in \mathscr{S}(A_c)_k^{\pm}$ ($c \neq 0$). We choose integer parameters $\alpha$ and $\beta$ and define a new sequence $\boldsymbol{b}$ by

$$G_1(\boldsymbol{b}, x) = \exp(\beta x) G_0\left(\boldsymbol{a}, \frac{1}{c}(\exp(\alpha x) - 1)\right). \tag{22}$$

An easy calculation shows that

$$G_1(\boldsymbol{b}, x) = \pm \exp((2\beta - k\alpha)x) G_1(\boldsymbol{b}, -x) \tag{23}$$

holds. By (11) we conclude $\boldsymbol{b} \in \mathscr{S}(A_{k\alpha-2\beta})_1^{\pm}$.

The transformation (22) is closely related to a generalization of the Akiyama-Tanigawa algorithm (see [1] and [9]). We explain this connection briefly. Starting from the initial sequence $\boldsymbol{a} = \{a_{0,m}\}_{m\geq 0}$, we define new sequences $\{a_{n,m}\}_{m\geq 0}$ for $n = 1, 2, \ldots$ by

$$a_{n,m} = (\alpha m + \beta) a_{n-1,m} + (\gamma m + \delta) a_{n-1,m+1} \tag{24}$$

with given parameters $\alpha, \beta$ , $\gamma$ and $\delta$. Then we collect the 0-th terms and form a sequence $\boldsymbol{b} = \{a_{n,0}\}_{n\geq 0}$. If $\delta = \gamma$, then by a similar method as [4], we can prove

$$a_{n,0} = \sum_{j=0}^{n} \binom{n}{j} \beta^{n-j} \left( \sum_{m=0}^{n} \left\{ {n \atop m} \right\} \alpha^{n-m} \gamma^m m!\, a_{0,m} \right),$$

where $\begin{Bmatrix} n \\ m \end{Bmatrix}$ is the Stirling number of the second kind and

$$G_1(\boldsymbol{b}, x) = \exp(\beta x) G_0\left(\boldsymbol{a}, \frac{\gamma}{\alpha}(\exp(\alpha x) - 1)\right).$$

Chen [4] proved these formulas for the special cases where $(\alpha, \beta, \gamma) = (1, 1, -1)$ and $(1, 0, -1)$.

We have chosen $\gamma = \alpha/c$ in (24).

Incidentally, if $\alpha = \gamma = 0$, then the sequences $\{a_{n,m}\}$ form a matrix called generalized Seidel matrix. In this case, assuming $\delta \neq 0$, we have

$$G_1(\boldsymbol{a}, x) = \exp\left(-\frac{\beta x}{\delta}\right) G_1\left(\boldsymbol{b}, \frac{x}{\delta}\right)$$

(see [5, Theorem 2.1]). Hence this transformation can be written by our action (but it is not a convolution). Moreover, if $\boldsymbol{a} \in \mathscr{S}(A_c)_1^s$, then we can prove $\boldsymbol{b} \in \mathscr{S}(A_{c\delta - 2\beta})_1^s$.

# 5 Involutive action on endomorphisms

In this section, we study the action of the involutions on the endomorphism algebra $\mathrm{End}(\mathscr{S})$ of the vector space $\mathscr{S}$.

**Definition 34.** Let $A \in \mathrm{GL}_2(F)$ be a lower triangular regular matrix of order 2. We define a left action of $A$ of weight $k$ on $\mathrm{End}(\mathscr{S})$ by

$$([A]_k f)(\boldsymbol{a}) = (f(\boldsymbol{a}|_{[A]_k}))|_{[A]_k} \tag{25}$$

for $f \in \mathrm{End}(\mathscr{S})$ and $\boldsymbol{a} \in \mathscr{S}$.

Since $A$ acts as an involution on $\mathrm{End}(\mathscr{S})$, we can decompose $\mathrm{End}(\mathscr{S})$ as an $F[A]_k$-module:

$$\mathrm{End}(\mathscr{S}) = \mathrm{End}(\mathscr{S})^+ \oplus \mathrm{End}(\mathscr{S})^-,$$

where the eigenspaces are given by

$$\mathrm{End}(\mathscr{S})^+ = \{f \in \mathrm{End}(\mathscr{S}) \mid [A]_k f = f\}, \quad \mathrm{End}(\mathscr{S})^- = \{f \in \mathrm{End}(\mathscr{S}) \mid [A]_k f = -f\}.$$

It is easy to see that $\mathrm{End}(\mathscr{S})^+$ coincides with the submodule $\mathrm{End}_{F[A]_k}(\mathscr{S})$ of $F[A]_k$-endomorphisms. From this fact, we can show the following proposition easily.

**Proposition 35.** *If $\boldsymbol{a}^+ \in \mathscr{S}(A)_k^+, \boldsymbol{a}^- \in \mathscr{S}(A)_k^-$, $f^+ \in \mathrm{End}(\mathscr{S})^+ and f^- \in \mathrm{End}(\mathscr{S})^-$, then we have*

$$f^+(\boldsymbol{a}^+) \in \mathscr{S}(A)_k^+, \quad f^+(\boldsymbol{a}^-) \in \mathscr{S}(A)_k^-, \quad f^-(\boldsymbol{a}^+) \in \mathscr{S}(A)_k^-, \quad f^-(\boldsymbol{a}^-) \in \mathscr{S}(A)_k^+.$$

This proposition gives another method to generate new eigensequences.

We shall compute the action of $[A_c]_k$ on $f \in \mathrm{End}(\mathscr{S})$ explicitly for several elementary endomorphisms $f$.

In the following, until the end of this section, we restrict ourselves to the case where the weight $k = 1$, for simplicity.

**Proposition 36.** *Let $\partial^m$ be the differentiation on exponential generating function:*

$$\partial^m : G_1(\boldsymbol{a}, x) \mapsto \frac{d^m}{dx^m} G_1(\boldsymbol{a}, x).$$

*Then we have*

$$[A_c]_1 \partial^m (G_1(\boldsymbol{a}, x)) = (-c - \partial)^m \, G_1(\boldsymbol{a}, x). \tag{26}$$

*Proof.* Applying Leibniz's rule to (8), we have

$$\begin{aligned}
\partial^m G_1(\boldsymbol{a}|_{[A_c]_1}, x) &= \sum_{i=0}^{m} \binom{m}{i} (-c)^{m-i} \exp(-cx) \; (-1)^i \partial^i \, G_1(\boldsymbol{a}, -x) \\
&= (-1)^m \sum_{i=0}^{m} \binom{m}{i} \; c^{m-i} \exp(-cx) \; \partial^i \, G_1(\boldsymbol{a}, -x)
\end{aligned}$$

Therefore we get

$$\begin{aligned}
[A_c]_1 \partial^m (G_1(\boldsymbol{a}, x)) &= \exp(-cx)(-1)^m \sum_{i=0}^{m} \binom{m}{i} c^{m-i} \exp(cx) \; \partial^i \, G_1(\boldsymbol{a}, x) \\
&= (-1)^m \sum_{i=0}^{m} \binom{m}{i} c^{m-i} \partial^i \; G_1(\boldsymbol{a}, x) \\
&= (-c - \partial)^m G_1(\boldsymbol{a}, x).
\end{aligned}$$

□

We rewrite this result in terms of sequence.

**Proposition 37.** *Let $\boldsymbol{a} \in \mathscr{S}$ and $f = L^m$ the map shifting $m$ $(m \geq 1)$ terms to the left:*

$$[n](L^m(\boldsymbol{a})) = a_{m+n}, \quad [0](L^m(\boldsymbol{a})) = a_m.$$

*Then we have*

$$\begin{aligned}
[n](([A_c]_1 L^m)(\boldsymbol{a})) &= (-1)^m \sum_{i=0}^{m} \binom{m}{i} c^{m-i} a_{n+i} = [m](L^n(\boldsymbol{a})|_{[A_c]_1}), \\
[0](([A_c]_1 L^m)(\boldsymbol{a})) &= (-1)^m \sum_{i=0}^{m} \binom{m}{i} c^{m-i} a_i = [m](\boldsymbol{a}|_{[A_c]_1}).
\end{aligned}$$

*Proof.* We obviously have $\partial^m G_1(\boldsymbol{a}, x) = G_1(L^m \boldsymbol{a}, x)$. Hence the result follows by taking the $n$-th terms of (26). □

**Proposition 38.** *Let $f = \Delta$ be the difference operator defined by $a_n \mapsto a_{n+1} - a_n$. Then we have*

$$\begin{aligned}
[n](([A_c]_1 \Delta^m)(\boldsymbol{a})) &= (-1)^m \sum_{j=0}^{m} \binom{m}{j} (1+c)^{m-j} a_{n+j} = [m](L^n(\boldsymbol{a})|_{[A_{1+c}]_1}), \\
[0](([A_c]_1 \Delta^m)(\boldsymbol{a})) &= (-1)^m \sum_{j=0}^{m} \binom{m}{j} (1+c)^{m-j} a_j = [m](\boldsymbol{a}|_{[A_{1+c}]_1}).
\end{aligned}$$

*Proof.* It is easy to show that

$$[n](\Delta^m(\boldsymbol{a})) = \sum_{i=0}^{m} \binom{m}{i}(-1)^{m+i} a_{n+i}.$$

Hence we have

$$\Delta^m(G_0(\boldsymbol{a}|_{[A_c]_1}, x)) = \sum_{i=0}^{m} \binom{m}{i}(-1)^{m+i} L^i G_0(\boldsymbol{a}|_{[A_c]_1}, x)).$$

From the definition (25) it follows that

$$([A_c]_1\Delta^m)(G_0(\boldsymbol{a}, x)) = \sum_{i=0}^{m} \binom{m}{i}(-1)^{m+i}([A_c]_1 L^i) G_0(\boldsymbol{a}, x).$$

Therefore, by Proposition 37, the $n$-th term is given by

$$\begin{aligned}[n](([A_c]_1\Delta^m)(\boldsymbol{a})) &= (-1)^m \sum_{i=0}^{m}\sum_{t=0}^{i} \binom{m}{i}\binom{i}{t} c^t a_{n+i-t} \\ &= (-1)^m \sum_{j=0}^{m}\sum_{i=j}^{m} \binom{m}{i}\binom{i}{j} c^{i-j} a_{n+j}.\end{aligned}$$

The binomial identity $\binom{m}{i}\binom{i}{j} = \binom{m}{j}\binom{m-j}{m-i}$ yields

$$[n](([A_c]_1\Delta^m)(\boldsymbol{a})) = (-1)^m \sum_{j=0}^{m} \binom{m}{j} \sum_{i=0}^{m-j} \binom{m-j}{i} c^i a_{n+j} = (-1)^m \sum_{j=0}^{m} \binom{m}{j}(1+c)^{m-j} a_{n+j}$$

as desired. □

Another differential operators producing new eigensequences are given in the following corollary, whose proof readily follows from Propositions 35 and 36.

**Corollary 39.** *Let $G(x) \in \mathscr{S}_1(A_c)_1^t$. For any polynomial $p(\partial) \in F[\partial]$, we define*

$$p(\partial)G(x) = p\left(\frac{d}{dx}\right) G(x).$$

*If $p(\partial) \in F[\partial]$ satisfies*

$$p(\partial) = s \cdot p(-c - \partial) \tag{27}$$

*with some $s \in \{\pm 1\}$, then we have*

$$p(\partial)G(x) \in \mathscr{S}(A_c)_1^{st}.$$

The following lemma tells us how to find polynomials satisfying (27).

**Lemma 40.** *Let $u(\partial)$ be an even (resp. odd) polynomial in $\partial$. Then $p(\partial) = u\left(\partial + \frac{c}{2}\right)$ satisfies (27) with $s = 1$ (resp. $s = -1$). Conversely every polynomial satisfying (27) is obtained in this way.*

*Proof.* The first half follows from an easy calculation. We shall prove the latter half. Notice that the map $\partial \mapsto -c - \partial$ is an involution on $F[\partial]$. Therefore any polynomial $p$ satisfying (27) is obtained by $p(\partial) = h(\partial) + s\, h(-c - \partial)$ for some $h(\partial) \in F[\partial]$. Then we have

$$p\left(\partial - \frac{c}{2}\right) = h\left(\partial - \frac{c}{2}\right) + s\, h\left(-\partial - \frac{c}{2}\right).$$

This shows that $p\left(\partial - \frac{c}{2}\right)$ is an even (resp. odd) polynomial if $s = 1$ (resp. $s = -1$). □

**Example 41.** We give some examples of polynomials satisfying (27). First we use $u(\partial)$ in Lemma 40 to obtain

$$\frac{c}{2} + \partial \quad (s = -1), \tag{28}$$

$$c\partial + \partial^2 \quad (s = 1). \tag{29}$$

Although we have $\left(\partial + \frac{c}{2}\right)^2 = \partial^2 + c\partial + \frac{c^2}{4}$, we can drop the constant term since $1 \in F[\partial]$ defines an operator with $s = 1$. These two operators are found in [12, Corollary 3.1] as operators related to $[A_{-1}]_1$.

We have another type of operators like

$$\partial^n \pm (-c - \partial)^n, \tag{30}$$

$$\partial^n (-c - \partial)^m \pm (-c - \partial)^n \partial^m, \tag{31}$$

$$(\partial^2 - c^2)^n \pm (\partial^2 + 2c\partial)^n \tag{32}$$

with $s = \pm 1$ and $m, n \in \mathbb{Z}_{>0}$.

These operators will be used to produce identities of eigensequences in the next section.

# 6 Identities for eigensequences

In this section, we shall deduce some identities involving eigensequences as an application of the theory we have developed so far in the preceding sections.

First we show that we can construct eigensequences from arbitrary sequences easily by applying 'twisted' endomorphisms.

**Proposition 42.** *Let $f \in \mathrm{End}(\mathscr{S})$ and $\boldsymbol{a} \in \mathscr{S}$. Let $A \in \mathrm{GL}_2(F)$ be a lower triangular matrix of order 2. Then we have*

1. $f(\boldsymbol{a}) + ([A]_k f)(\boldsymbol{a}|_{[A]_k}) \in \mathscr{S}(A)_k^+$;
2. $f(\boldsymbol{a}) - ([A]_k f)(\boldsymbol{a}|_{[A]_k}) \in \mathscr{S}(A)_k^-$;

3. $([A]_k f)(\boldsymbol{a}) + f(\boldsymbol{a}|_{[A]_k}) \in \mathscr{S}(A)_k^+$;

4. $([A]_k f)(\boldsymbol{a}) - f(\boldsymbol{a}|_{[A]_k}) \in \mathscr{S}(A)_k^-$.

*Proof.* The assertions 3 and 4 follow from 1 and 2 respectively taking $[A]_k f$ as $f$. Therefore we have only to prove the first two assertions. Recall that we have projections $\pi(A)_k^+$ (resp. $\pi(A)_k^-$) from $\mathscr{S}$ to $\mathscr{S}(A)_k^+$ (resp. $\mathscr{S}(A)_k^+$) (see the remark after Proposition 14). By Definition 34, we have

$$f(\boldsymbol{a}) + ([A]_k f)(\boldsymbol{a}|_{[A]_k}) = f(\boldsymbol{a}) + f(\boldsymbol{a})|_{[A]_k} = 2\pi(A)_k^+(f(\boldsymbol{a})) \in \mathscr{S}(A)_k^+.$$

Similarly, we get

$$f(\boldsymbol{a}) - ([A]_k f)(\boldsymbol{a}|_{[A]_k}) = 2\pi(A)_k^-(f(\boldsymbol{a})) \in \mathscr{S}(A)_k^-.$$

This completes the proof. □

We have the following corollary.

**Corollary 43.** *Let $f, g \in \mathrm{End}(\mathscr{S})$ and $\boldsymbol{a} \in \mathscr{S}$. Then we have*

1. $([A]_k f \circ g)(\boldsymbol{a}) + (f \circ [A]_k g)(\boldsymbol{a}|_{[A]_k}) \in \mathscr{S}(A)_k^+$;

2. $([A]_k f \circ g)(\boldsymbol{a}) - (f \circ [A]_k g)(\boldsymbol{a}|_{[A]_k}) \in \mathscr{S}(A)_k^-$.

*Proof.* We first compute the action of $[A_c]_k$ in $f \circ g$:

$$([A_c]_k(f \circ g))(\boldsymbol{a}) = f(g(\boldsymbol{a}|_{[A_c]_k}))|_{[A_c]_k} = ([A_c]_k f)(g(\boldsymbol{a}|_{[A_c]_k})|_{[A_c]_k}) = ([A_c]_k f) \circ ([A_c]_k g)(\boldsymbol{a}).$$

Consequently we have $[A_c]_k([A_c]_k f \circ g) = f \circ [A_c]_k g$. Hence the results follow from Proposition 42. □

Among various properties of eigensequences, one of the easiest is (14):

$$[0]\boldsymbol{a} = 0 \text{ if } \boldsymbol{a} \in \mathscr{S}^-.$$

Thus by Proposition 42 2, 4 and Corollary 43 the following equalities hold for *any* sequence $\boldsymbol{a} \in \mathscr{S}$ and any $f, g \in \mathrm{End}(\mathscr{S})$:

$$[0]f(\boldsymbol{a}) = [0]([A_c]_k f)(\boldsymbol{a}|_{[A_c]_k}), \tag{33}$$
$$[0]([A_c]_k f)(\boldsymbol{a}) = [0]f(\boldsymbol{a}|_{[A_c]_k}), \tag{34}$$
$$[0]([A_c]_k f \circ g)(\boldsymbol{a}) = [0](f \circ [A_c]_k g)(\boldsymbol{a}|_{[A_c]_k}). \tag{35}$$

For every $f$ and $g$, we obtain an identity involving $\boldsymbol{a}$. We give several explicit examples of such formulas in the following proposition. We restrict ourselves again to the case $k = 1$ for simplicity.

**Proposition 44.** *Let $\boldsymbol{a} = \{a_n\}$ be any sequence and $a_n^* = [n](\boldsymbol{a}|_{[A_c]_1})$. Then the following identities hold.*

*1.* $\displaystyle\sum_{i=0}^{m}\binom{m}{i}(-1)^i a_i = \sum_{i=0}^{m}\binom{m}{i}(1+c)^{m-i}a_i^*.$

*2.* $\displaystyle(-1)^n\sum_{i=0}^{n}\binom{n}{i}c^{n-i}a_{m+i} = (-1)^m\sum_{i=0}^{m}\binom{m}{i}c^{m-i}a_{n+i}^*.$

*3.* $\displaystyle\sum_{i=0}^{m}\binom{m}{i}(2c)^{m-i}a_{m+i} = \sum_{i=0}^{m}\binom{m}{i}(-c^2)^{m-i}a_{2i}^*.$

*Note that the roles of $a_n$ and $a_n^*$ are interchangeable.*

*Proof.* Let $\boldsymbol{a}^* = \boldsymbol{a}|_{[A_c]_1}$. The following identities are used to obtain the first two formulas:

1. $[0](\Delta^m(\boldsymbol{a})) = [0]([A_c]_1\Delta^m(\boldsymbol{a}^*))$ (we take $f = \Delta^m$ in (33));

2. $[0](L^m \circ [A_c]_1 L^n(\boldsymbol{a})) = [0]([A_c]_1 L^m \circ L^n(\boldsymbol{a}^*))$ (we take $f = L^m$ and $g = L^n$ in (35)).

Here we have already computed the explicit formulas for $[A_c]_1\Delta^m$ and $[A_c]_1 L^m$ in Propositions 37 and 38. The identity 3 is obtained by taking $f = (L^2 - c^2)^m$ and using (34) since we have $[n](L^2 - c^2)(\boldsymbol{a}) = a_{n+2} - c^2 a_n$ and $[n]([A_c]_1 L^2 - c^2)(\boldsymbol{a}) = 2ca_{n+1} + a_{n+2}$. □

Note that 2 is a generalization of the formula due to Chen [5, Theorem 2.1] and Gessel [6, Theorem 7.4]. Kaneko's recursion formula (2) is an easy consequence of 2 in the preceding proposition as is noted by Gessel [6, Lemma 7.2]. The third formula in the proposition seems to be not known before.

These identities will be simple and particularly interesting if $\boldsymbol{a}|_{[A_c]_k}$ is simple. This is the case if $\boldsymbol{a}$ is an eigensequence in $\mathscr{S}(A_c)_k^{\pm}$.

In addition to eigensequences, there are some interesting pairs of $\boldsymbol{a}$ and $\boldsymbol{a}|_{[A_c]_k}$:

1. $\boldsymbol{a} = \{n!\}$ and derangement numbers $D(n)$ with $[A_{-1}]_1$ ([6, Section 7]);

2. The special values $\xi_k(-n)$ of negative integers of Arakawa-Kaneko zeta function and the poly-Bernoulli number $B_n^{(k)}$ with $[A_{-1}]_1$ ([3, Theorem 6]).

As we have seen in Proposition 5 and Lemma 15, the terms in eigensequences satisfy simple relations. These relations can be used to obtain identities for eigensequences.

**Proposition 45.** *Assume that $c \neq 0$. Let $\boldsymbol{a} \in \mathscr{S}(A_c)_1^{s_1}$ and $\boldsymbol{b} \in \mathscr{S}(A_{-c})_1^{s_2}$. Then we have*

$$[n](\boldsymbol{a} \underset{1}{*} \boldsymbol{b}) = \sum_{i=0}^{n}\binom{n}{i}\, a_i b_{n-i} = 0$$

*either $s_1 s_2 = 1$ and $n$ is odd or $s_1 s_2 = -1$ and $i$ is even.*

*Proof.* By Proposition 11 we have $\boldsymbol{a} \underset{1}{*} \boldsymbol{b} \in \mathscr{S}(A_0)_1^{s_1 s_2}$. By Proposition 5, the exponential generating function of $\boldsymbol{a} \underset{1}{*} \boldsymbol{b}$ is an even (resp. odd) power series if $s_1 s_2 = 1$( resp. $s_1 s_2 = -1$). The proposition is now clear from this. □

The identities

$$\begin{cases} \sum_{i=0}^{n} \binom{n}{i} B_i L_{n-i} = 0, & \text{if } n \text{ is odd;} \\ \sum_{i=0}^{n} \binom{n}{i} B_i F_{n-i} = 0, & \text{if } n \text{ is even} \end{cases}$$

involving the Lucas, Bernoulli and Fibonacci numbers proved in [12, (4.1) and (4.2)] follow readily from Proposition 45

Along with various differential operators, we can obtain more identities.

**Corollary 46.** *If $\boldsymbol{a} \in \mathscr{S}(A_c)_1^+$ (resp. $\mathscr{S}(A_c)_1^-$), then we have*

$$c\sum_{j=0}^{n} \binom{n}{j} (-1)^{n-j} a_j a_{n-j} + 2\sum_{j=0}^{n} \binom{n}{j} (-1)^{n-j} a_j a_{n-j+1} = 0$$

*for even $n$ (resp. odd $n$).*

*Proof.* Let $\boldsymbol{a} \in \mathscr{S}(A_c)_1^+$. Then it follows from (28) that

$$\left(2 + c\frac{d}{dx}\right) G_1(\boldsymbol{a}, x) \in \mathscr{S}_1(A_c)_1^-$$

and by Proposition 13 we have

$$\exp(-x) \cdot \left(2 + c\frac{d}{dx}\right) G_1(\boldsymbol{a}, x) \in \mathscr{S}_1(A_{-c})_1^-.$$

Therefore we obtain

$$\left\{G_1(\boldsymbol{a}, x), \exp(-x) \cdot \left(\frac{c}{2} + \frac{d}{dx}\right) G_1(\boldsymbol{a}, x)\right\}_0 \in \mathscr{S}_1(A_0)_1^-,$$

which is an odd power series by Proposition 5. The left hand side of the identity in the statement is the $n$-th term of this sequence (see (21)). This proves the assertion. The proof for $\boldsymbol{a} \in \mathscr{S}(A_c)_1^-$ is similar. □

**Proposition 47.** *Let $\boldsymbol{a} \in \mathscr{S}(A_c)_1^s$. Choose a positive integer $j$ such that $(-1)^j s = -1$. Then we have*

$$\sum_{i=0}^{n} \binom{n}{i} c^{n-i} (n+i)^{\underline{j}} \, a_{n-j+i} = 0. \tag{36}$$

*Proof.* By Example 32, we have $\{i^{\underline{j}} \, a_{i-j}\}_{i \geq 0} \in \mathscr{S}(A_c)_1^-$. By applying $c\partial + \partial^2$ in (29) $n$ times, the $m$-th term of the new sequence is

$$\sum_{i=0}^{n} \binom{n}{i} c^{n-i} (n+i+m)^{\underline{j}} \, a_{n+i+m-j}.$$

Since this sequence belongs to $\mathscr{S}(A_c)_1^-$, the 0-th term is 0 by Lemma 15:

$$\sum_{i=0}^{n} \binom{n}{i} c^{n-i} (n+i)^{\underline{j}} \, a_{n+i-j} = 0.$$

□

The following corollary readily follows from (36) by setting $j = 1$.

**Corollary 48** (Generalized Kaneko's identity)**.** *If* $\boldsymbol{a} \in \mathscr{S}(A_c)_1^+$*, then*

$$\sum_{i=0}^{n} \binom{n}{i} c^{n-i}(n+i)a_{n+i-1} = 0.$$

By noting that the Bernoulli numbers $\boldsymbol{B} \in \mathscr{S}(A_1)_1^+$ (see Example 22), the original formula (4) follows by changing $n$ by $n+1$.

As we have seen, many identities for eigensequences can be obtained from our study on the involutions instead of long and complicate computations of iterated sums.

We hope that our method using the involutions and the differential operators shed new light on the study of sequences.

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