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GENERATING IDENTITIES FOR FIBONACCI AND LUCAS TRIPLES

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Using the generating functions of

$$\{F_{n+m}\}_{n=0}^{\infty} \quad \text{and} \quad \{L_{n+m}\}_{n=0}^{\infty} ,$$

where F_{n+m} denotes the $(n+m)^{th}$ Fibonacci number and L_{n+m} denotes the $(n+m)^{th}$ Lucas number, many basic identities are easily deduced. From certain of these identities and the generating functions, we obtain identities for the triples $F_pF_qF_r$, $F_pF_qL_r$, $F_pL_qL_r$, and $L_pL_qL_r$, where p, q, and r are fixed integers.

To derive the desired generating functions we recall that

$$(0) \qquad F_{n+m} = \frac{\alpha^{n+m} - \beta^{n+m}}{\alpha - \beta} \quad \text{and} \quad L_{n+m} = \alpha^{n+m} + \beta^{n+m}$$

where

$$\alpha = \frac{1 - \sqrt{5}}{2} \quad \text{and} \quad \beta = \frac{1 + \sqrt{5}}{2} .$$

Note that α and β are the roots of the equation $x^2 - x - 1 = 0$, and hence $\alpha + \beta = 1$ and $\alpha\beta = -1$. The generating functions of

$$\{F_{n+m}\}_{n=0}^{\infty}$$

where m is any fixed integer is found using the given definition of F_{n+m}. We have

$$\sum_{n=0}^{\infty} F_{n+m}x^n = \sum_{n=0}^{\infty} \frac{\alpha^{n+m} - \beta^{n+m}}{\alpha - \beta} x^n$$

$$= \frac{1}{\alpha - \beta}\left[\alpha^m \sum_{n=0}^{\infty} \alpha^n x^n - \beta^m \sum_{n=0}^{\infty} \beta^n x^n\right]$$

$$(1) \qquad = \frac{1}{\alpha - \beta}\left[\alpha^m \frac{1}{1 - \alpha x} - \beta^m \frac{1}{1 - \beta x}\right]$$

$$= \frac{1}{\alpha - \beta}\left[\frac{(\alpha^m - \beta^m) - \alpha\beta(\alpha^{m-1} - \beta^{m-1})x}{(1 - \alpha x)(1 - \beta x)}\right]$$

$$= \frac{F_m + F_{m-1}x}{1 - x - x^2} .$$

In a similar fashion the generating function of $\{L_{n+m}\}_{n=0}^{\infty}$ is found to be

$$\sum_{n=0}^{\infty} L_{n+m} x^n = \frac{L_m + L_{m-1} x}{1 - x - x^2} \quad . \tag{2}$$

(Any reader who is unfamiliar with the general theory of generating functions will find references [1], [2], [3], and [4] enlightening.)

Before considering important special cases of the above results, two lemmas are given which are proved by appropriate substitution of formulas (0).

Lemma 1. $F_n L_n = F_{2n}$, $n \in Z$, the set of integers.

Lemma 2. $F_n L_{n-1} + F_{n-1} L_n = 2F_{2n-1}$, $n \in Z$.

In utilizing formulas (1) and (2) to generate basic identities, we must first evaluate the formulas at specific values of m. It is sufficient for our purposes to consider the cases $m = -2, -1, 0, 1, 2, 3, 4$.

SPECIAL CASES OF FORMULAS (1) AND (2)

(Let $1 - x - x^2 = \Delta$.)

$$\sum_{n=0}^{\infty} F_{n-2} x^n = \frac{F_{-2} + F_{-1}x}{\Delta} = \frac{-1 + 2x}{\Delta}, \quad \sum_{n=0}^{\infty} L_{n-2} x^n = \frac{L_{-2} + L_{-3}x}{\Delta} = \frac{3 - 4x}{\Delta}$$

$$\sum_{n=0}^{\infty} F_{n-1} x^n = \frac{F_{-1} + F_{-2}x}{\Delta} = \frac{1 - x}{\Delta}, \quad \sum_{n=0}^{\infty} L_{n-1} x^n = \frac{L_{-1} + L_{-2}x}{\Delta} = \frac{-1 + 3x}{\Delta}$$

$$\sum_{n=0}^{\infty} F_{n} x^n = \frac{F_{0} + F_{-1}x}{\Delta} = \frac{0 + x}{\Delta}, \quad \sum_{n=0}^{\infty} L_{n} x^n = \frac{L_{0} + L_{-1}x}{\Delta} = \frac{2 - x}{\Delta}$$

$$\sum_{n=0}^{\infty} F_{n+1} x^n = \frac{F_{1} + F_{0}x}{\Delta} = \frac{1 + 0x}{\Delta}, \quad \sum_{n=0}^{\infty} L_{n+1} x^n = \frac{L_{1} + L_{0}x}{\Delta} = \frac{1 + 2x}{\Delta}$$

$$\sum_{n=0}^{\infty} F_{n+2} x^n = \frac{F_{2} + F_{1}x}{\Delta} = \frac{1 + x}{\Delta}, \quad \sum_{n=0}^{\infty} L_{n+2} x^n = \frac{L_{2} + L_{1}x}{\Delta} = \frac{3 + x}{\Delta}$$

$$\sum_{n=0}^{\infty} F_{n+3} x^n = \frac{F_{3} + F_{2}x}{\Delta} = \frac{2 + x}{\Delta}, \quad \sum_{n=0}^{\infty} L_{n+3} x^n = \frac{L_{3} + L_{2}x}{\Delta} = \frac{4 + 3x}{\Delta}$$

$$\sum_{n=0}^{\infty} F_{n+4}\, x^n = \frac{F_4 + F_3 x}{\Delta} = \frac{3 + 2x}{\Delta}, \quad \sum_{n=0}^{\infty} L_{n+4}\, x^n = \frac{L_4 + L_3 x}{\Delta} = \frac{7 + 4x}{\Delta} .$$

Using the fact that two series are equal if and only if the corresponding coefficients are equal, we now find several elementary identities.

Since

$$\frac{2 - x}{\Delta} = \frac{1}{\Delta} + \frac{1 - x}{\Delta} ,$$

it follows that

$$\sum_{n=0}^{\infty} L_n\, x^n = \sum_{n=0}^{\infty} F_{n+1}\, x^n + \sum_{n=0}^{\infty} F_{n-1}\, x^n$$

$$= \sum_{n=0}^{\infty} (F_{n+1} + F_{n-1})\, x^n$$

and hence

Lemma 3. $L_n = F_{n+1} + F_{n-1}$, $n \in Z^+ \cup \{0\}$, the set of nonnegative integers.

Note from definition (0) that

$$F_{-n} = \frac{\alpha^{-n} - \beta^{-n}}{\alpha - \beta} = \frac{1}{\alpha - \beta}\left(\frac{1}{\alpha^n} - \frac{1}{\beta^n}\right)$$

$$(0') \qquad = \frac{1}{\alpha - \beta}\, \frac{\beta^n - \alpha^n}{(\alpha\beta)^n} = \frac{1}{\alpha - \beta}\, \frac{\beta^n - \alpha^n}{(-1)^n}$$

$$= (-1)^{n+1}\, \frac{\alpha^n - \beta^n}{\alpha - \beta} = (-1)^{n+1}\, F_n$$

and

$$L_{-n} = \alpha^{-n} + \beta^{-n} = (\alpha\beta)^{-n}(\alpha^n + \beta^n)$$

$$(0'') \qquad = (-1)^{-n} L_n = (-1)^n L_n$$

for any positive integer n.

Returning to Lemma 3, we now observe from this lemma and "definitions" (0') and (0'') that

$$\begin{aligned} F_{(-n)+1} + F_{(-n)-1} &= F_{-(n-1)} + F_{-(n+1)} \\ &= (-1)^{(n-1)+1} F_{n-1} + (-1)^{(n+1)+1} F_{n+1} \\ &= (-1)^n\, [F_{n-1} + F_{n+1}] \\ &= (-1)^n\, L_n = L_{-n} . \end{aligned}$$

Hence Lemma 3 holds for all integers n.

In a similar manner the additional lemmas are found.

Lemma 4. $5F_n = L_{n+1} + L_{n-1}, \quad n \in Z.$

Lemma 5. $2F_{n+1} = F_n + L_n = F_{n+2} + F_{n-1}, \quad n \in Z.$

Lemma 6. $2F_{n-1} = F_n - L_n, \quad n \in Z.$

Lemma 7. $F_{n+3} = F_{n+1} + L_{n-1}, \quad n \in Z.$

Lemma 8. $3F_n = L_{n+1} - F_{n-1}, \quad n \in Z.$

Lemma 9. $3L_{n+2} = L_n + L_{n+4}, \quad n \in Z.$

Lemma 10. $3F_{n+2} = F_n + F_{n+4}, \quad n \in Z.$

Lemma 11. $2F_{n+1} = L_{n+1} - F_{n-2}, \quad n \in Z.$

Lemma 12. $L_n + F_n = F_{n+2} + F_{n-1}, \quad n \in Z.$

Although these results are of interest in themselves, their principal use is as lemmas to more profound results. The reader is encouraged to consider additional special cases of formulas (0), and then generate additional Fibonacci and Lucas identities.

The next three results are also generated from formulas (1) and (2). These fundamental identities are essential to our development of Fibonacci and Lucas triples.

Theorem 1. $F_nL_m + F_{n-1}L_{m-1} = L_{n+m-1}$, for any $n, m \in Z$.

Proof. Let m be any fixed integer. Then

$$\begin{aligned}\sum_{n=0}^{\infty}(F_nL_m + F_{n-1}L_{m-1})x^n &= L_m\sum_{n=0}^{\infty}F_n x^n + L_{m-1}\sum_{n=0}^{\infty}F_{n-1}x^n \\ &= L_m\frac{x}{\Delta} + L_{m-1}\frac{(1-x)}{\Delta} \\ &= \frac{L_{m-1} + (L_m - L_{m-1})x}{\Delta} \\ &= \frac{L_{m-1} + L_{m-2}x}{\Delta} \\ &= \sum_{n=0}^{\infty}L_{n+m-1}x^n \quad ,\end{aligned}$$

by formula (2). Results (0') and (0'') complete the proof.

From a development similar to the above proof, we find a companion result to Theorem 1.

Theorem 2. $F_nF_m + F_{n-1}F_{m-1} = F_{n+m-1}$, for any $n, m \in Z$.

Theorem 3. $L_nL_m + L_{n-1}L_{m-1} = L_{n+m} + L_{n+m-2} = 5F_{n+m-1}$, for any $n, m \in Z$.

Proof. Since $L_{n+m} + L_{n+m-2} = 5F_{n+m-1}$ by Lemma 4, we need only consider the first part of the identity. Let m be any fixed integer. Now

$$\sum_{n=0}^{\infty} (L_nL_m + L_{n-1}L_{m-1})x^n = L_m \sum_{n=0}^{\infty} L_nx^n + L_{m-1} \sum_{n=0}^{\infty} L_{n-1}x^n$$

$$= L_m\left(\frac{2 - x}{\Delta}\right) + L_{m-1}\left(\frac{-1 + 3x}{\Delta}\right)$$

$$= \frac{[L_m + (L_m - L_{m-1})] + [2L_{m-1} + (L_{m-1} - L_m)]x}{\Delta}$$

$$= \frac{[L_m + L_{m-2}] + [L_{m-1} + (L_{m-1} - L_{m-2})]x}{\Delta}$$

$$= \frac{L_m + L_{m-1}x}{\Delta} + \frac{L_{m-2} + L_{m-3}x}{\Delta}$$

$$= \sum_{n=0}^{\infty} (L_{n+m} + L_{n+m-2})x^n \quad .$$

Aided by the partial fractions technique we find the final result needed to generate the specified Fibonacci and Lucas triples. It is the following:

$$(3) \quad \frac{(p + qx)}{\Delta}\,\frac{(r + tx)}{\Delta} = \frac{pr + (pt + qr)x + qtx^2}{\Delta^2} = \frac{-qt}{\Delta} + \frac{(pr + qt) + (pt + qr - qt)x}{\Delta^2} \quad .$$

The identities are now found by convoluting series (generating functions) of the forms (1) and (2). We begin by specifying m and s as fixed integers. Now

$$(4) \quad \frac{F_m + F_{m-1}x}{\Delta} \cdot \frac{L_s + L_{s-1}x}{\Delta} = \sum_{n=0}^{\infty} F_{n+m}x^n \sum_{n=0}^{\infty} L_{n+s}x^n = \sum_{n=0}^{\infty} \sum_{k=0}^{n} F_{k+m}L_{n-k+s}x^n \quad ,$$

and by Eq. (3) this product also equals

$$\frac{-F_{m-1}L_{s-1}}{\Delta} + \frac{(F_mL_s + F_{m-1}L_{s-1}) + (F_mL_{s-1} + F_{m-1}L_s - F_{m-1}L_{s-1})x}{\Delta^2}$$

$$= \frac{-F_{m-1}L_{s-1}}{\Delta} + \frac{L_{m+s-1} + (F_{m-1}L_s + F_{m-2}L_{s-1})x}{\Delta^2}$$

by Theorem 1 and substitution of F_{m-2} for $F_m - F_{m-1}$.

$$= -F_{m-1}L_{s-1}\frac{1}{\Delta} + \frac{L_{m+s-1} + L_{m+s-2}x}{\Delta}\frac{1}{\Delta}$$

by Theorem 1

$$= -F_{m-1}L_{s-1}\sum_{n=0}^{\infty} F_{n+1}x^n + \sum_{n=0}^{\infty} L_{n+m+s-1}x^n \cdot \sum_{n=0}^{\infty} F_{n+1}x^n$$

by definition of generating functions (1) and (2)

$$= \sum_{n=0}^{\infty} [-F_{m-1}L_{s-1}F_{n+1}]x^n + \sum_{n=0}^{\infty}\sum_{k=0}^{n} F_{k+1}L_{n-k+m+s-1}x^n$$

(5) $$= \sum_{n=0}^{\infty}\left[-F_{m-1}L_{s-1}F_{n+1} + \sum_{k=0}^{n} F_{k+1}L_{n-k+m+s-1}\right]x^n .$$

By equating the coefficients of series (4) and (5), the first identity is deduced. It may be expressed as

$$\sum_{k=0}^{n} F_{k+m}L_{n-k+s} = -F_{m-1}L_{s-1}F_{n+1} + \sum_{k=0}^{n} F_{k+1}L_{n-k+m+s-1}$$

or

$$F_{m-1}L_{s-1}F_{n+1} = \sum_{k=0}^{n} (F_{k+1}L_{n-k+m+s-1} + F_{k+m}L_{n-k+s}) .$$

Letting $p = m - 1$, $q = n + 1$, and $r = s - 1$, the identity becomes

Theorem 4.

$$F_pF_qL_r = \sum_{k=0}^{q-1} (F_{k+1}L_{p+q+r-k-1} + F_{p+k+1}L_{q+r-k}) ,$$

for any integers p, q, and r.

One notes the need of definitions (0') and (0'') if any of the above integers is negative.

Following the procedure given above, aided by the given lemmas, Theorems 1-3, and definitions, two additional identities are found. The first is a result of the convolution of

$$\frac{F_m + F_{m-1}x}{\Delta}$$

with

$$\frac{F_t + F_{t-1}x}{\Delta},$$

and the second is determined by the convolution of

$$\frac{L_m + L_{m-1}x}{\Delta}$$

with

$$\frac{L_t + L_{t-1}x}{\Delta}.$$

Theorem 5.

$$F_pF_qF_r = \sum_{k=0}^{r-1}(F_{p+q+k+1}F_{r-k} - F_{p+k+1}F_{r+q-k}),$$

for any $p, q, r \in Z$.

Theorem 6.

$$F_pL_qL_r = \sum_{k=0}^{p-1}(5F_{p-k}F_{q+r+k+1} - L_{q+k+1}L_{p+r-k}),$$

for any $p, q, r \in Z$.

Theorem 7.

$$L_pL_qL_r = 5\left[\sum_{k=0}^{p-2}(F_{q+r+k+1}L_{p-k} - F_{p+r-k}L_{q+k+1}) - F_{p+q+r}\right] - L_{p+q}L_{r+1},$$

for any $p, q, r \in Z$.

Proof. From Lemma 3, we obtain

$$\begin{aligned} L_pL_qL_r &= (F_{p+1} + F_{p-1})L_qL_r \\ &= F_{p+1}L_qL_r + F_{p-1}L_qL_r . \end{aligned}$$

Now from Theorem 6, it follows that

$$L_pL_qL_r = \sum_{k=0}^{p} (5F_{p-k+1}F_{q+r+k+1} - L_{q+k+1}L_{p+r-k+1})$$

$$+ \sum_{k=0}^{p-2} (5F_{p-k-1}F_{q+r+k+1} - L_{q+k+1}L_{p+r-k-1})$$

$$= \sum_{k=0}^{p-2} [5F_{q+r+k+1}(F_{p-k+1} + F_{p-k-1}) - L_{q+k+1}(L_{p+r-k+1} + L_{p+r-k-1})]$$

$$+ (5F_2F_{p+q+r} - L_{p+q}L_{r+2}) + (5F_1F_{p+q+r+1} - L_{p+q+1}L_{r+1})$$

$$= 5\sum_{k=0}^{p-2} (F_{q+r+k+1}L_{p-k} - F_{p+r+k}L_{q+k+1})$$

$$+ 5F_{p+q+r+2} - (5F_{p+q+r+1} + L_{p+q}L_{r+1})$$

by Lemmas 2 and 4 and Theorem 4

$$= 5\left[\sum_{k=0}^{p-2} (F_{q+r+k+1}L_{p-k} - F_{p+r-k}L_{q+k+1}) - F_{p+q+r}\right] - L_{p+q}L_{r+1} .$$

Many corollaries to the last three theorems are immediate by making substitution(s) for p, q, and r, respectively, in the given identities. The formulation and derivation of these results we leave to the reader.

REFERENCES

1. Nathan J. Fine, "Generating Functions," Enrichment Mathematics for High School (NCTM Twenty-eighth yearbook), 1963, Washington 6, D.C., pp. 355-367.
2. V. E. Hoggatt and D. A. Lind, "A Primer for the Fibonacci Numbers: Part VI," The Fibonacci Quarterly, Vol. 5, 1967, pp. 445-460.
3. H. W. Gould, "Generating Functions for Products of Powers of Fibonacci Numbers," The Fibonacci Quarterly, Vol. 1, 1963, pp. 1-16.
4. John Riordan, "Generating Functions for Powers of Fibonacci Numbers," Duke Math. J., Vol. 29, 1962, pp. 5-12.