SIAM J. DISC. MATH.
Vol. 5, No. 4, pp. 497–499, November 1992

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HOW TO GUESS A GENERATING FUNCTION*

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Abstract. In this note, a new method for taking the first few terms of a sequence and making an educated guess as to the generating function of the sequence is described. The method involves a matrix factorization into lower triangular, diagonal, and upper triangular matrices (the LDU decomposition), generating functions, and solving a first-order differential equation.

Key words. generating function, Hankel matrix, differential equations, preferential arrangement, Schröder numbers

AMS(MOS) subject classification. 05A15

Analyzing a sequence. Suppose that we have determined the first few terms of a sequence and would like to know more about it. The sequence 1, 3, 10, 37, 151, 674, 3263, 17007, 94824, $\cdots$ will be used to illustrate. From this limited information, we would like to guess either a generating function or a recursion relation. Suppose that looking at differences, Sloane's handbook [S], or looking for a recursion, have not helped in identifying the sequence. Here is a new technique that often provides some insight. Start by forming a Hankel matrix from the sequence. The illustrative sequence yields that

$$H=\begin{bmatrix} 1 & 3 & 10 & 37 & 151 & \\ 3 & 10 & 37 & 151 & 674 & \\ 10 & 37 & 151 & 674 & 3263 & \vdots \\ 37 & 151 & 674 & 3263 & 17007 & \\ 151 & 674 & 3263 & 17007 & 94824 & \\ & & \cdots & & & \end{bmatrix},$$

and Gauss elimination is used to find the LDU decomposition of H as follows:

$$L=\begin{bmatrix} 1 & & & & 0 \\ 3 & 1 & & & \vdots \\ 10 & 7 & 1 & & \\ 37 & 40 & 12 & 1 & \\ 151 & 221 & 103 & 18 & 1 \\ & & \cdots & & \end{bmatrix}, \quad D=\begin{bmatrix} 1 & & & & 0 & \\ & 1 & & & & \\ & & 2! & & & \\ & & & 3! & & \vdots \\ 0 & & & & 4! & \\ & & & & \cdots & \end{bmatrix},$$

and $U = L^T$, the transpose of L.

Obviously, L and the original sequence convey the same information, but often L is more tractable.

Let $C_0(x)$ be the generating function for the first column, $C_1(x)$ the generating function for the second, and find $f(x)$ such that $C_0(x)f(x) = C_1(x)$.

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* Received by the editors November 5, 1990; accepted for publication June 10, 1991. This research was supported by National Science Foundation grant R11-89-12667.

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In this example, exponential generating functions work, and we solve

$$\left(1+3x+10\frac{x^2}{2!}+37\frac{x^3}{3!}+\cdots\right)f(x)=x+7\frac{x^2}{2!}+40\frac{x^3}{3!}+221\frac{x^4}{4!}+\cdots$$

to obtain $f(x) = x + x^2/2! + x^3/3! + \cdots$, leading to a reasonable guess that $f(x) = e^x - 1$.

The next step is a comparison of the first two columns. Here it seems that $C_{n+1,0} = 3C_{n,0} + C_{n,1}$, where $L = (C_{n,k})_{n,k \geq 0}$. This leads to

$$C_0'(x)=3C_0(x)+C_1(x)=3C_0(x)+f(x)C_0(x)$$
$$=3C_0(x)+(e^x-1)C_0(x).$$

The solution of this elementary differential equation is

$$C_0(x)=e^{e^x+2x-1} \quad \text{since } C_0(0)=1.$$

We have guessed our generating function. If we define $C_0(x) = \sum_{n=0}^{\infty} P_n(x^n/n!)$, then differentiation yields

$$C_0'(x)=C_0(x)(e^x+2), \quad \text{so}$$

$$p_{n+1}=2p_n+\sum_{l=0}^{n}\binom{n}{l}p_l.$$

We can even give a combinatorial interpretation for $e^{2x}e^{3^x-1}$. Since e^{e^x-1} generates the Bell numbers, we can take a set $[n]$, color some elements red and some others green, then partition the rest into disjoint nonempty blocks $B_1, B_2, \cdots, B_k$. Let G be the green elements, R the red elements, and let $B_1 \cup G, B_2 \cup G, \cdots, B_k \cup G$ be the atoms of a sublattice. The $\hat{O}$ element for this sublattice is G, while $\hat{1}$ is $[n] - R$. This process can be easily reversed. Thus the sequence 1, 3, 10, 37, 151, $\cdots$ could be the number of Boolean sublattices of the Boolean lattice of subsets of $[n]$.

We have seen one example where we started with the first nine terms of a sequence, formed the Hankel matrix, row-reduced to obtain the LDU decomposition, found a recurrence in L, guessed f, and solved a first-order differential equation. From this, we found the generating function, a recursion relation, and a combinatorial interpretation.

Here are two examples, set as exercises, to illustrate the technique.

(i) We use the sequence 1, 1, 3, 13, 75, 541, 4683, $\cdots$. We obtain that

$$L=\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & \\ 1 & 1 & 0 & 0 & 0 & \\ 3 & 5 & 1 & 0 & 0 & \vdots \\ 13 & 31 & 12 & 1 & 0 & \\ 75 & 233 & 133 & 22 & 1 & \\ & & \cdots & & & \end{bmatrix},$$

$$f(x)=x+3x^2/2!+13x^3/3!+75x^4/4!+\cdots.$$

A reasonable guess is that $C_0(x) - 1 = f(x)$. The differential equation then becomes

$$C_0'(x)=C_0(x)+2C_0(x)f(x),$$

so $C_0(x) = 1/(2 - e^x)$. These numbers can arise as the number of preferential arrangements.

(ii) We use the sequence 1, 2, 6, 22, 90, 394, 1806, $\cdots$. In this case, ordinary generating functions work better, as shown below:

$$L=\begin{bmatrix} 1 & 0 & 0 & 0 & \\ 2 & 1 & 0 & 0 & \vdots \\ 6 & 5 & 1 & 0 & \\ 22 & 23 & 8 & 1 & \\ & \cdots & & & \end{bmatrix},$$

$$f = x + 3x^2 + 11x^3 + 45x^4 + \cdots = \frac{C_0(x) - 1}{2},$$

$$C_{n+1,0} = 2C_{n,0} + 2C_{n,1} \quad \text{so}$$

$$C_0(x) = 1 + 2x[C_0(x) + C_1(x)] = 1 + 2x\left[C_0(x) + C_0(x)\left(\frac{C_0(x)-1}{2}\right)\right].$$

Now we solve a quadratic equation instead of a differential equation and obtain that

$$C_0(x) = \frac{1 - x - \sqrt{1 - 6x + x^2}}{2x},$$

which generates the (double) Schröder numbers.

If we have the stronger condition that the $C_n(x) = C_0(x)(f(x))^n$ or $C_n(x) = C_0(x)(f(x)^n)/n!$ for all n, then we obtain a group structure for the lower triangular matrices leading variously to the umbral calculus [R], paths and continued fraction expansions [F], [GJ], and combinatorics of orthogonal polynomials [V]. A brief introduction is given in [GSWW].

For a researcher discovering integer sequences, this technique can be very useful at the initial work stage. It is hard to judge how many sequences on which this would work, but the list does include factorials, derangement numbers, telephone numbers (the number of elements in S_n such that $x^2 = e$), Bernoulli numbers, number of permutations with no double descents, Bell numbers, both even and odd factorials, Euler numbers, numbers of preferential arrangements, secant numbers, Catalan numbers, Motzkin numbers, Schröder numbers (little and big), Delannoy numbers, central binomial coefficients, directed animals (single source), central trinomial coefficients, and some polynomial sequences such as Chebyshev Legendre and Hermite sequences.

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