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On Bilateral Generating Functions of Extended Jacobi Polynomials

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Abstract

In this note we have obtained a novel result on bilateral generating relations involving extended Jacobi polynomials by group theoretic method which in turn yields the corresponding results involving Hermite, Laguerre and Jacobi polynomials.

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1. Introduction

Generating functions play a large role in the study of special functions. In the investigation of generating functions, group theoretic method seems to be a potent one in comparison with analytic methods because of the fact that the unknown generating functions can only be obtained by group theoretic method whereas the known generating functions can be verified and the corresponding extension can be made by analytic method.

The aim at presenting this paper is to derive an unified representation of bilateral generating relations for certain special functions in terms of $F_n(\alpha,\ \beta + n;\ x)$-a modified form of $F_n(\alpha,\ \beta;\ x)$[3]:

$$F_n(\alpha,\beta;x) = \frac{(-1)^n}{n\,!}\left(\frac{\lambda}{b-a}\right)^n (x-a)^{-\alpha}(b-x)^{-\beta} \times D^n\left[(x-a)^{n+\alpha}(b-x)^{n+\beta}\right], \quad (1.1)$$

where $D \equiv \frac{d}{dx}$ and λ is a number such that $\frac{\lambda}{b-a} > 0$.

The main result of our investigation is stated in the form of the following theorem.

Theorem 1: If there exists a unilateral generating relation of the form:

$$G(x,w) = \sum_{n=0}^{\infty} a_n F_n(\alpha,\ \beta+n;\ x)\, w^n \quad (1.2)$$

then

$$\left\{1 - \frac{\lambda}{b-a}(x-b)w\right\}^{-1-\alpha-\beta} (1+\lambda w)^{\beta} \times G\left(\frac{x - \frac{a\lambda}{b-a}(x-b)w}{1-\frac{\lambda}{b-a}(x-b)w}, \frac{wz(1+\lambda w)}{\left\{1-\frac{\lambda}{b-a}(x-b)w\right\}^2}\right)$$

$$= \sum_{n=0}^{\infty} w^n\, \sigma_n(x,w), \quad (1.3)$$

where

$$\sigma_n(x,z) = \sum_{p=0}^{n} a_p \binom{n}{p} F_n(\alpha,\ \beta - n + 2p;\ x)\, z^p.$$

The importance of the above theorem lies in the fact that whenever one knows a generating relation of the form (1.2) then the corresponding bilateral generating relation can at once be written down from (1.3). So one can get a large number of bilateral generating relations by attributing different suitable values to a_n in (1.2).

2. Proof of the theorem

Using the differential recurrence relation:

$$
\begin{aligned}
&(x-a)(x-b)\frac{\partial}{\partial x}F_n(\alpha,\beta+n;x)\\
&\quad+\{n(2x-a-b)+\beta(x-a)+(1+\alpha)(x-b)\}\\
&\quad\times F_n(\alpha,\beta+n;x) \;=\frac{(b-a)(n+1)}{\lambda}F_{n+1}(\alpha,\beta+n-1;x)
\end{aligned}
\tag{2.1}
$$

we find the following partial differential operator R:

$$
\begin{aligned}
R=\frac{\lambda}{b-a}\Bigg[&\frac{(x-a)(x-b)z}{y^2}\frac{\partial}{\partial x}+\frac{(x-a)z}{y}\frac{\partial}{\partial y}+(2x-a-b)\frac{z^2}{y^2}\frac{\partial}{\partial z}\\
&+(1+\alpha)(x-b)\frac{z}{y^2}\Bigg]
\end{aligned}
\tag{2.2}
$$

such that

$$
R\big(F_n(\alpha,\beta+n;x)y^{\beta}z^n\big)=(n+1)F_{n+1}(\alpha,\beta+n-1;x)y^{\beta-2}z^{n+1}. \tag{2.3}
$$

The extended form of the group generated by R is given by

$$
\begin{aligned}
&e^{wR}\,f(x,\,y,\,z)=\left\{1-(x-b)\frac{\lambda\,w}{b-a}\;\frac{z}{y^2}\right\}^{-1-\alpha}\\
&\times f\left(\frac{x-(x-b)\frac{\lambda aw}{b-a}\;\frac{z}{y^2}}{1-(x-b)\frac{\lambda\,w}{b-a}\;\frac{z}{y^2}},\;\frac{y\left(1+\lambda w\frac{z}{y^2}\right)}{1-(x-b)\frac{\lambda\,w}{b-a}\;\frac{z}{y^2}},\;\frac{z\left(1+\lambda w\;\frac{z}{y^2}\right)}{\left\{1-(x-b)\frac{\lambda\,w}{b-a}\;\frac{z}{y^2}\right\}^2}\right).
\end{aligned}
\tag{2.4}
$$

Let us assume the following unilateral generating relation of the form:

$$
G(x,w)=\sum_{n=0}^{\infty}a_nF_n(\alpha,\beta+n;x)\,w^n. \tag{2.5}
$$

Replacing w by wz and multiplying both sides of (2.5) by y^{β} and finally operating e^{wR} on both sides, we get

$$
e^{wR}\left(y^{\beta}\,G(x,wz)\right)=e^{wR}\left(\sum_{n=0}^{\infty}a_n\big(F_n(\alpha,\beta+n;x)y^{\beta}z^n\big)w^n\right). \tag{2.6}
$$

Now the left member of (2.6), with the help of (2.4), reduces to

$$
\left\{1-(x-b)\frac{\lambda\,w}{b-a}\;\frac{z}{y^2}\right\}^{-1-\alpha-\beta}y^{\beta}\left(1+\lambda w\frac{z}{y^2}\right)^{\beta}
$$

$$\times G\left(\frac{x-(x-b)\frac{\lambda a w}{b-a}\ \frac{z}{y^2}}{1-(x-b)\frac{\lambda\, w}{b-a}\ \frac{z}{y^2}},\qquad \frac{wz\left(1+\lambda w\ \frac{z}{y^2}\right)}{\left\{1-(x-b)\frac{\lambda\, w}{b-a}\ \frac{z}{y^2}\right\}^2}\right). \qquad (2.7)$$

The right member of (2.6), with the help of (2.3), becomes

$$=\sum_{n=0}^{\infty}\sum_{p=0}^{n} a_{n-p}(wz)^n \binom{n}{p} F_n(\alpha,\beta+n-2p;x)y^{\beta-2p}. \qquad (2.8)$$

Now equating (2.7) and (2.8) and then substituting $\frac{z}{y^2}=1$, we get

$$\left\{1-\frac{\lambda}{b-a}(x-b)w\right\}^{-1-\alpha-\beta}(1+\lambda w)^{\beta}\, G\left(\frac{x-\ \frac{a\lambda}{b-a}(x-b)w}{1-\frac{\lambda}{b-a}(x-b)w},\frac{wz(1+\lambda w)}{\left\{1-\frac{\lambda}{b-a}(x-b)w\right\}^2}\right)$$
$$=\sum_{n=0}^{\infty} w^n\,\sigma_n(x,z),$$

where

$$\sigma_n(x,z)=\sum_{p=0}^{n} a_p\binom{n}{p}F_n(\alpha,\ \beta-n+2p;\ x)\,z^p.$$

This completes the proof of the theorem.

3. Special cases:

We now discuss some special cases of the above theorem.

Special cases 1 (On Laguerre polynomials):

Putting $a=0, \lambda=1$ and $b=\beta$ in Theorem 1 and then simplifying and finally taking limit as $\beta\to\infty$, we get the following results on bilateral generating functions involving Laguerre polynomials:

Theorem 2: If there exists a unilateral generating relation of the form:

$$G(x,w)=\sum_{n=0}^{\infty} a_n\, L_n^{(\alpha)}(x)\ w^n$$

then

$$(1-w)^{-1-\alpha}\exp\left(\frac{-xw}{1-w}\right)G\left(\frac{x}{1-w},\ \frac{wz}{1-w}\right)=\sum_{n=0}^{\infty} w^n\,\sigma_n(z), \qquad (1.3)$$

where

$$\sigma_n(z) = \sum_{p=0}^{n} a_p \binom{n}{p} z^p,$$

which is found derived in [1, 5].

Special cases 2(On Hermite polynomials):

Similarly, putting $\alpha = \beta, -a = b = \sqrt{\alpha}$ and $\lambda = \frac{2}{\sqrt{\alpha}}$ in Theorem 1, then simplifying and finally taking limit as $\alpha \to \infty$, we get the corresponding bilateral generating relations involving Hermite polynomials which is found derived in [1, 4].

Special cases 3 (On Jacobi polynomials):

Again, if we Put $-a = b = 1$, $\lambda = 1$ and interchanging α, β in Theorem 1, we get the corresponding bilateral generating relations involving Jacobi polynomials found derived in [2].

References

[1] A.K. Chongdar, and S.K. Chatterjea, On a class of trilateral generating relations with Tchebychev polynomials from the view point of one parameter group of continuous transformations, Bull. Cal. Math. Soc.73(1981), 127-140.

[2] C.S. Bera and A.K. Chongdar, On some generating functions of modified Jacobi polynomials, Ultra Scientist , 24(1)A(2012), 41-46.

[3] I. Fujiwara, A unified presentation of classical orthogonal polynomials, Math. Japan 11(1966), 133-148.

[4] S.K. Chatterjea, Une nouvelle fonction generatrice bilaterale pour les polynomes d'Hermite, C.R. Acad. Sci., Paris, 266A(1968), 979-981.

[5] W.A. Al-Salam, Operational representations for the Laguerre and other polynomials, Duke Math. Jour., 31(1964), 127-142.

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