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NOTE ON SOME GENERATING FUNCTIONS

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1. In a recent paper in this Quarterly, Bruckman [2] defined a sequence of positive integers B_k by means of

$$(1.1)\qquad (1-x)^{-1}(1+x)^{-\frac{1}{2}} = \sum_{k=0}^{\infty} B_k \frac{x^k}{2^k\cdot k!} .$$

This is equivalent to the recurrence

$$(1.2)\qquad B_k = B_{k-1} + (2k-1)(2k-2)B_{k-2} \qquad (k \geqslant 2), \qquad B_0 = B_1 = 1 .$$

Making use of (1.2) he showed that

$$(1.3)\qquad e^{x^2/2}\int_0^x e^{-u^2}du = \sum_{k=0}^{\infty} B_k \frac{x^{2k+1}}{(2k+1)!}$$

and

$$(1.4)\qquad (1-x^2)^{-1}\arctan x = \sum_{k=0}^{\infty} B_k^2 \frac{x^{2k+1}}{(2k+1)!} .$$

Bateman [1] has discussed the polynomial $g_n(y,z)$ defined by

$$(1.5)\qquad (1+x)^{y+z}(1-x)^{-y} = \sum_{n=0}^{\infty} x^n g_n(y,z);$$

see also [3]. On the other hand the Jacobi polynomial [6, Ch. 16]

$$(1.6)\qquad P_n^{(\alpha,\beta)}(x) = \sum_{k=0}^{n} \binom{\alpha+n}{n-k}\binom{\beta+n}{k}\left(\frac{x-1}{2}\right)^k\left(\frac{x+1}{2}\right)^{n-k}$$

satisfies

$$(1.7)\qquad \sum_{n=0}^{\infty} P_n^{(\alpha-n,\beta-n)}(x)z^n = \left(1+\frac{x+1}{2}z\right)^{\alpha}\left(1+\frac{x-1}{2}z\right)^{\beta}$$

and in particular, for $x = 0$,

$$(1.8)\qquad \sum_{n=0}^{\infty} P_n^{(\alpha-n,\beta-n)}(0)z^n = (1+\tfrac{1}{2}z)^{\alpha}(1-\tfrac{1}{2}z)^{\beta} .$$

It follows from (1.1) and (1.8) that

$$(1.9)\qquad \frac{1}{k!}B_k = 2^{2k}P_k^{(-\frac{1}{2}-k,-1-k)}(0) = (-1)^k 2^{2k} P_k^{(-1-k,-\frac{1}{2}-k)}(0) .$$

We shall show that both (1.3) and (1.4) can be generalized considerably. We also obtain the following congruence for B_n :

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$$(1.10)\qquad \sum_{s=0}^{r} (-1)^{r-s}\binom{r}{s} B_{n+sm}B_{(r-s)m} \equiv 0 \pmod{r!\,m^r},$$

where m and r are arbitrary positive integers.

It would be of interest to find a combinatorial interpretation of B_k.

2. The writer [4] has obtained the following bilinear generating function:

$$(2.1)\qquad \sum_{n=0}^{\infty} \frac{n!}{(\gamma)_n}(x-1)^n(y-1)^n w^n P_n^{(\alpha-n,-\alpha-\gamma-n)}\left(\frac{x+1}{x-1}\right) P_n^{(\beta-n,-\beta-\gamma-n)}\left(\frac{y+1}{y-1}\right)$$

$$= (1-w)^{-\alpha-\beta-\gamma}(1-xw)^{\alpha}(1-yw)^{\beta} F\left[-\alpha,-\beta;\gamma;\frac{(x-1)(y-1)w}{(1-xw)(1-yw)}\right],$$

where as usual

$$F(z,b;c;z) = \sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{n!\,(c)_n}z^n \qquad\text{and}\qquad (a)_n = a(a+1)\cdots(a+n-1),\quad (a)_0 = 1.$$

In particular, for $x = y = -1$ and $\gamma = -\alpha-\beta$, Eq. (2.1) reduces to

$$(2.2)\qquad \sum_{n=0}^{\infty}\frac{n!}{(-\alpha-\beta)_n}4^n w^n P_n^{(\alpha-n,\beta-n)}(0)P_n^{(\beta-n,\alpha-n)}(0) = (1+w)^{\alpha+\beta}F\left[-\alpha,-\beta;-\alpha-\beta;\frac{4w}{(1+w)^2}\right].$$

It is convenient to replace α,β by $-\alpha,-\beta$, so that (2.2) becomes

$$(2.3)\qquad \sum_{n=0}^{\infty}\frac{n!}{(\alpha+\beta)_n}4^n w^n P_n^{(-\alpha-n,-\beta-n)}(0)P_n^{(-\beta-n,-\alpha-n)}(0) = (1+w)^{-\alpha-\beta}F\left[\alpha,\beta;\alpha+\beta;\frac{4w}{(1+w)^2}\right].$$

Specializing further, we take $\beta = \alpha + \tfrac12$, so that

$$(2.4)\qquad \sum_{n=0}^{\infty}\frac{n!}{(2\alpha+\tfrac12)_n}4^n w^n P_n^{(-\alpha-n,-\alpha-\frac12-n)}(0)P_n^{(-\alpha-\frac12-n,-\alpha-n)}(0)$$

$$= (1+w)^{-2\alpha-\frac12}F\left[\alpha,\alpha+\tfrac12;2\alpha+\tfrac12,\frac{4w}{(1+w)^2}\right].$$

Next in formula (2) of [6, p. 66],

$$F\left[\tfrac12 a,\tfrac12 a+\tfrac12;a-b+1;\frac{4z}{(1+z)^2}\right] = (1+z)^a F[a,b;a-b+1;z]$$

take $a = 2\alpha$, $b = \tfrac12$. We get

$$(2.5)\qquad F\left[\alpha,\alpha+\tfrac12;2\alpha+\tfrac12;\frac{4z}{(1+z)^2}\right] = (1+z)^2F[2\alpha,\tfrac12;2\alpha+\tfrac12;z].$$

Hence (2.4) becomes

$$(2.6)\qquad \sum_{n=0}^{\infty}\frac{n!}{(2\alpha+\tfrac12)_n}4^n w^n P_n^{(-\alpha-n,-\alpha-\frac12-n)}(0)P_n^{(-\alpha-\frac12-n,-\alpha-n)}(0) = (1+w)^{-\frac12}F[2\alpha,\tfrac12;2\alpha+\tfrac12;w].$$

Since

$$P_n^{(\alpha,\beta)}(x) = (-1)^n P_n^{(\beta,\alpha)}(-x),$$

(2.6) may be written in the form

$$(2.7)\qquad \sum_{n=0}^{\infty}\frac{n!}{(2\alpha+\tfrac12)_n}4^n w^n \left\{P_n^{(-\alpha-n,-\alpha-\frac12-n)}(0)\right\}^2 = (1-w)^{-\frac12}F[2\alpha,\tfrac12;2\alpha+\tfrac12;-w].$$

In particular, for $\alpha = \tfrac12$, it follows from (2.7) and (1.9) that

$$\sum_{n=0}^{\infty}\frac{2^{-2n}w^n}{n!(3/2)_n}B_n^2 = (1-w)^{-\frac12}F[1,\tfrac12;3/2;-w].$$

Replacing w by z^2, this becomes

(2.8) $$\sum_{n=0}^{\infty} \frac{z^{2n+1}}{(2n+1)!} B_n^2 = z(1-z^2)^{-\frac{1}{2}} F[1, \tfrac{1}{2}; 3/2; -z^2].$$

Since

$$zF[1, \tfrac{1}{2}; 3/2; -z^2] = \sum_{n=0}^{\infty} (-1)^n \frac{(1/2)_n}{(3/2)_n} z^{2n+1} = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{2n+1} = \arctan z,$$

it is evident that (2.8) is the same as (1.4).

3. In (2.1) take $x = -1,\ y = 0,\ \gamma = -\alpha - \beta$. Since, by (1.6),

$$P_n^{(\beta-n,\alpha-n)}(-1) = \binom{\alpha}{n},$$

it is clear that (2.1) reduces to

$$\sum_{n=0}^{\infty} \frac{n!}{(-\alpha-\beta)_n} \binom{\alpha}{n} 2^n w^n P_n^{(\alpha-n,\beta-n)}(0) = (1+w)^{\alpha} F\left[-\alpha, -\beta; -\alpha-\beta; \frac{2w}{1+w}\right].$$

Replacing α,β by $-\alpha, -\beta$, this becomes

(3.1) $$\sum_{n=0}^{\infty} (-1)^n \frac{(\alpha)_n}{(\alpha+\beta)_n} 2^n w^n P_n^{(-\alpha-n,-\beta-n)}(0) = (1+w)^{-\alpha} F\left[\alpha, \beta; \alpha+\beta; \frac{2w}{1+w}\right].$$

In particular, for $\beta = \frac{1}{2}$, we get

(3.2) $$\sum_{n=0}^{\infty} \frac{(\alpha)_n}{(\alpha+\frac{1}{2})_n} 2^n z^n P_n^{(-\alpha-n,-\frac{1}{2}-n)}(0) = (1-z)^{-\alpha} F\left[\alpha, \tfrac{1}{2}; \alpha+\tfrac{1}{2}; \frac{-2z}{1-z}\right].$$

For $\alpha = 1$, Eq. (3.2) becomes

(3.3) $$\sum_{n=0}^{\infty} \frac{z^n}{2^n (3/2)_n} B_k = (1-z)^{-1} F\left[1, \tfrac{1}{2}; 3/2; -\frac{2z}{1-z}\right].$$

This is not the same as (1.3).

The right-hand side of (3.2) is equal to

$$\sum_{r=0}^{\infty} \frac{(\alpha)_r (\frac{1}{2})_r}{r!(\alpha+\frac{1}{2})_r} (-2z)^r (1-z)^{-\alpha-r} = \sum_{r=0}^{\infty} \frac{(\alpha)_r (\frac{1}{2})_r}{r!(\alpha+\frac{1}{2})_r} (-2z)^r \sum_{s=0}^{\infty} \frac{(\alpha+r)_s}{s!} z^s$$

$$= \sum_{n=0}^{\infty} (\alpha)_n z^n \sum_{r=0}^{n} (-2)^r \frac{(\frac{1}{2})_r}{r!(n-r)!(\alpha+\frac{1}{2})_r}.$$

Hence (3.2) implies

$$\sum_{n=0}^{\infty} \frac{2^n z^n}{(+\frac{1}{2})_n} P_n^{(-\alpha-n,-\frac{1}{2}-n)}(0) = \sum_{n=0}^{\infty} z^n \sum_{r=0}^{n} (-2)^r \frac{(\frac{1}{2})_r}{r!(n-r)!(\alpha+\frac{1}{2})_r} = \sum_{r=0}^{\infty} \frac{(\frac{1}{2})_r (-2z)^r}{r!(\alpha+\frac{1}{2})_r} \sum_{n=r}^{\infty} \frac{z^{n-r}}{(n-r)!},$$

so that

(3.4) $$\sum_{n=0}^{\infty} \frac{z^n}{(\alpha+\frac{1}{2})_n} P_n^{(-\alpha-n,-\frac{1}{2}-n)}(0) = e^{\frac{1}{2}z} \sum_{r=0}^{\infty} \frac{(\frac{1}{2})_r (-z)^r}{r!(\alpha+\frac{1}{2})_r}.$$

For $\alpha = 1$, Eq. (3.4) reduces to (1.3).

4. Put

(4.1) $$(1-x)^{\alpha}(1+x)^{\beta} = \sum_{n=0}^{\infty} c_n(\alpha,\beta) x^n.$$

Then

$$\sum_{m,n=0}^{\infty} c_m(\alpha,\beta)c_n(\alpha,\beta)x^m y^n = (1-x)^\alpha(1-y)^\alpha(1+x)^\beta(1+y)^\beta = (1+xy-x-y)^\alpha(1+xy+x+y)^\beta$$

$$= (1+xy)^{\alpha+\beta}\left(1-\frac{x+y}{1+xy}\right)^\alpha\left(1+\frac{x+y}{1+xy}\right)^\beta$$

$$= (1+xy)^{\alpha+\beta}\sum_{k=0}^{\infty} c_k(\alpha,\beta)\left(\frac{x+y}{1+xy}\right)^k$$

$$= \sum_{k=0}^{\infty} c_k(\alpha,\beta)\sum_{s=0}^{k}\binom{k}{s}x^s y^{k-s}\sum_{r=0}^{\infty}\binom{\alpha+\beta-k}{t}x^t y^t$$

$$= \sum_{m,n=0}^{\infty} x^m y^n \sum_{\substack{s+t=m\\ k-s+t=n}}\binom{k}{s}\binom{\alpha+\beta-k}{t}c_k(\alpha,\beta).$$

It follows that

$$(4.2)\qquad c_m(\alpha,\beta)c_n(\alpha,\beta) = \sum_{t=0}^{\min(m,n)}\binom{m+n-2t}{m-t}\binom{\alpha+\beta-m-n+2t}{t}c_{m+n-2t}(\alpha,\beta).$$

The proof follows Kaluza [6]; see also [3].

Comparing (4.1) with (1.1), we have

$$(4.3)\qquad B_k = 2^k\cdot k!c_k(-1,-\tfrac{1}{2}).$$

Thus (4.2) implies

$$(4.4)\qquad B_mB_n = \sum_{t=0}^{\min(m,n)}(-1)^t2^t\binom{m}{t}\binom{n}{t}t!\prod_{j=0}^{t-1}(2m+2n-2t-2j+1)B_{m+n-2t}\,.$$

For $m = 1$, Eq. (4.4) reduces to (1.2). It is not difficult to prove (4.4) by induction.

The writer has proved the following result [5].

Let $f(n)$, $g(n)$ denote polynomials in n with integral coefficients. Define u_n by means of

$$(4.5)\qquad u_{n+1} = f(n)u_n + g(n)u_{n-1}\qquad (n \geqslant 1),$$

where

$$(4.6)\qquad u_0 = 1,\qquad u_1 = f(0),\qquad g(0) = 0.$$

Then u_n satisfies the following congruence:

$$(4.7)\qquad \Delta^{2r}u_n \equiv \Delta^{2r-1}u_n \equiv 0 \pmod{m^r},$$

for all $m \geqslant 1,\ n \geqslant 0,\ r \geqslant 1,$ **where**

$$\Delta^r u_n = \sum_{s=0}^{r}(-1)^{r-s}\binom{r}{s}u_{n+sm}u_{(r-s)m}\,.$$

Comparing (4.5) with

$$B_{n+1} = B_n + 2n(2n+1)B_{n-1}\,,$$

it is clear that (4.6) holds. We have therefore

$$(4.8)\qquad \sum_{s=0}^{r}(-1)^{r-s}\binom{r}{s}B_{n+sm}B_{(r-s)m} \equiv 0 \pmod{m^{[(r+1)/2]}}.$$

However a better result can be obtained. By (4.4) we have

$$\sum_{s=0}^{r} (-1)^{r-s} \binom{r}{s} B_{n+sm}B_{(r-s)m} = \sum_{s=0}^{r} (-1)^{r-s} \binom{r}{s} \sum_{t} (-1)^t 2^t \binom{n+sm}{t}\binom{(r-s)m}{t} t! \cdot \prod_{j=0}^{t-1}$$

$$\cdot (2n+2rm-2t-2j+1) \cdot B_{n+rm-2t} = \sum_{t} (-1)^t \frac{2^t}{t!} B_{n+rm-2t} \prod_{j=0}^{t-1} (2n+2rm-2t-2j+1)$$

$$\cdot \sum_{s=0}^{r} (-1)^{r-s} \binom{r}{s} f(s),$$

where

$$f(s) = (n+sm-t+1)_t((r-s)m-t+1)_t\,.$$

Clearly

$$f(s) = a_0 + a_1 sm + \cdots + a_{2t}(sm)^{2t}\,,$$

where the a_i are integers. Then

$$\sum_{s=0}^{r} (-1)^{r-s} \binom{r}{s} f(s) = \sum_{i=r}^{2t} a_i m^{2i} \Delta^r 0^i \equiv 0 \pmod{r!m^r}.$$

Since

$$\frac{2^t}{t!} \prod_{j=0}^{t-1} (2n+2rm-2t-2j+1)$$

is integral, it follows at once that

$$\sum_{s=0}^{r} (-1)^{r-s} \binom{r}{s} B_{n+sm}B_{(r-s)m} \equiv 0 \pmod{r!m^r}.$$

REFERENCES

1. H. Bateman, "The Polynomial of Mittag-Leffler," *Proc. Nat. Acad. Sci.,* USA, 26 (1946), pp. 491–496.
2. P.S. Bruckman, "An Interesting Sequence of Numbers Derived from Various Generating Functions," *The Fibonacci Quarterly,* Vol. 10, No. 2 (April 1972), pp. 169–181.
3. L. Carlitz, "A Special Functional Equation," *Riv. Mat.,* Univ. Parma, 7 (1956), pp. 211–233.
4. L. Carlitz, "A Bilinear Generating Function for the Jacobi Polynomials," *Bolletino Unione Matematica Italiana* (3), 18 (1963), pp. 87–89.
5. L. Carlitz, "Congruence Properties of the Polynomials of Hermite, Laguerre and Legendre," *Math. Zeitschrift,* 59 (1954), pp. 474–483.
6. Th. Kaluza, "Elementarer Beweis einer Vermutung von K. Friedrichs," *Mathematische Zeitschrift,* 37 (1933), pp. 689–697.
7. E.D. Rainville, *Special Functions,* Macmillan, New York, 1960.

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