# Bivariate Generating Functions for a Class of Linear Recurrences. II. Applications

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**Abstract**

In a previous paper, we classified and obtained the exponential generating functions for the large class of bivariate linear recurrences proposed by Graham, Knuth, and Patashnik. In this paper, we obtain general Rodrigues–like formulas for the corresponding univariate row generating polynomials. We make special emphasis on two-parameter generalizations of the Eulerian numbers $\left\langle {n \atop k} \right\rangle$ and $\left\langle\!\left\langle {n \atop k} \right\rangle\!\right\rangle$. In these cases, starting from the exponential generating function, we achieve a complete solution to the problem: i.e., closed formulas for these numbers. Finally, we briefly discuss some other applications of combinatorial interest.

**Key Words:** Exponential generating functions, Polynomial row generating functions, Generalized Eulerian numbers, Generalized $\nu$-order Eulerian numbers, Rodrigues formula.

# 1 Introduction

In a previous paper [1] (we will hereafter refer to it as Paper I), we studied in a systematic way (using exponential generating functions) a problem originally posed by Graham, Knuth and Patashnik (GKP) [15, Problem 6.94, pp. 319 and 564] concerning the solution for a class of two-parameter linear recurrences. The statement of this problem is:

**Question 1.1** *Develop a general theory of the solutions to the two-parameter recurrence*

$$\begin{aligned}\left|\begin{matrix} n \\ k \end{matrix}\right| &= (\alpha n + \beta k + \gamma) \left|\begin{matrix} n-1 \\ k \end{matrix}\right| \\ &\quad + (\alpha' n + \beta' k + \gamma') \left|\begin{matrix} n-1 \\ k-1 \end{matrix}\right| + [n = k = 0]\,, \quad \text{for } n, k \geq 0, \end{aligned} \tag{1.1}$$

*assuming that* $\left|\begin{smallmatrix} n \\ k \end{smallmatrix}\right| = 0$ *when* $n < 0$ *or* $k < 0$.

This recurrence produces a triangular array of numbers: i.e., it takes non-trivial values only for $0 \leq k \leq n$ (for any $n \geq 0$, $\left|\begin{smallmatrix} n \\ k \end{smallmatrix}\right| = 0$ for any $k < 0$ and $k > n$). As shown in Paper I, many combinatorial numbers satisfy recurrences belonging to the class (1.1).

For consistency, we will follow the notation used in GKP [15]. In the second line of Eq. (1.1), we use Iverson's brackets [18]:

$$[A] = \begin{cases} 1 & \text{if } A \text{ is true,} \\ 0 & \text{if } A \text{ is false,} \end{cases} \tag{1.2}$$

where the argument $A$ is any statement that can be either true or false. We use the standard notation for the sets of integer, rational, real, and complex numbers: i.e., $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, and $\mathbb{C}$, respectively. Moreover, the set of positive integers (resp. non-negative integers) will be denoted as $\mathbb{N}$ (resp. $\mathbb{Z}_0 = \mathbb{N} \cup \{0\}$). There are two more "bracket" notations that should not be confused with (1.2) (although it will be clear which one we are using from the context): 1) $[n]$ for any $n \in \mathbb{N}$ denotes the set $\{1, 2, \ldots, n\}$ [7,27] (for $n = 0$, we define $[0] = \varnothing$ [27]), and 2) $[z^k] f(z)$ stands for the coefficient of $z^k$ in $f(z)$ [15]. The rising $x^{\overline{n}}$ and falling $x^{\underline{n}}$ factorials powers will be defined as in [15] for an arbitrary number $x$ and any $n \in \mathbb{Z}_0$. For $n > 0$, they are defined as follows:

$$x^{\overline{n}} = x(x+1)(x+2)\cdots(x+n-1)\,, \tag{1.3a}$$

$$x^{\underline{n}} = x(x-1)(x-2)\cdots(x-n+1)\,, \tag{1.3b}$$

while for $n = 0$, we define $x^{\overline{0}} = x^{\underline{0}} = 1$, as empty products are defined to take the value 1 (and empty sums take the value 0). Finally, to simplify the notation for partial derivatives, we will use a slightly different notation as in Paper I: given a function $f(x_1, x_2, \ldots, x_n)$, we use $f_i$ as a shorthand for $\frac{\partial f}{\partial x_i}$.

In addition to the thorough study of many particular cases of combinatorial interest of Question 1.1 in the literature [2,3,7,15,24, and references therein], there has been (to our knowledge) only a few recent attempts to study general sub-families of Question 1.1. In particular, Neuwirth [22] found the solution to the recursion (1.1) for the particular case $\alpha' = 0$ by using Galton arrays. A few years later, Spivey [25] found explicit solutions (using finite differences) for three particular cases: 1) $\alpha = -\beta$; 2) $\beta = \beta' = 0$; and 3) $\alpha/\beta = \alpha'/\beta' + 1$. Both works focused on finding closed expressions for $\left|{n \atop k}\right|$ in terms of simpler combinatorial numbers for the above particular cases.

In Paper I, we used exponential generating functions (EGF)

$$F(x,y) \;=\; \sum_{n,k\geq 0} \left|{n \atop k}\right| x^k \, \frac{y^n}{n!}\,, \tag{1.4}$$

to solve Question 1.1 in a more systematic way. We found that:

- The dependence of the solutions to (1.1) on the parameters $\gamma, \gamma'$ is rather trivial. Thus, we defined families of solutions using the other parameters $(\alpha, \beta; \alpha', \beta')$. For instance, the "binomial family" corresponds to $(0,0;0,0)$.
- The algebraic structure of the EGF for (1.1) depends strongly on the parameters $\beta, \beta'$. Actually, what really matters is whether these parameters are zero or non-zero. Therefore, we find four "types" of solutions to Question 1.1.

For future reference, it is convenient to label these four types of solutions in the same way as in Paper I:

**Definition 1.2** *The solutions to Question 1.1 are classified in four different types according to the values of the parameters $\beta$ and $\beta'$:*

- **Type I**: $\beta\beta' \neq 0$.
- **Type II**: $\beta \neq 0$ *and* $\beta' = 0$.
- **Type III**: $\beta = 0$ *and* $\beta' \neq 0$.
- **Type IV**: $\beta = \beta' = 0$.

Let us summarize the main results and definitions of Paper I to make the present paper as self-contained as possible. We first proved that solving Question 1.1 was equivalent to solving[1]

**Question 1.3** *Find the EGF $F(x,y)$ (1.4) with the initial condition $F(x,0) = 1$ that satisfies the partial differential equation (PDE)*

$$a(x)\, F_1 + b(x,y)\, F_2 \;=\; c(x)\, F\,, \tag{1.5}$$

[1]Please, recall that we use a slightly different short-hand notation for the partial derivatives of a function.

*where the polynomials $a, b, c$ are given by*

$$a(x) = -(\beta + \beta' x)\, x\,, \tag{1.6a}$$
$$b(x,y) = 1 - \alpha\, y - \alpha' x\, y\,, \tag{1.6b}$$
$$c(x) = \alpha + \gamma + (\alpha' + \beta' + \gamma')\, x\,. \tag{1.6c}$$

For Type I solutions, it is convenient to make the change of variables $(x, y) \mapsto (X, Y)$ defined by:

$$X = \left|\frac{\beta'}{\beta}\right| x = \sigma\, \frac{\beta'}{\beta}\, x\,, \qquad \text{where } \sigma = \text{sgn}(\beta\beta'), \tag{1.7a}$$
$$Y = \beta\, y\,. \tag{1.7b}$$

Then, the function $F(x, y)$ (1.4) is given in terms of a new function $\mathcal{F}(X, Y)$ through the relation:

$$F(x,y) = \mathcal{F}(X,Y) = \mathcal{F}\left(\sigma\, \frac{\beta'}{\beta}\, x, \beta\, y\right)\,. \tag{1.8}$$

We then proved that solving Question 1.3 for a Type I problem was equivalent to solving

**Question 1.4** *Let us assume that $\beta\beta' \neq 0$. Find the solutions $\mathcal{F}(X, Y)$ (1.8) with the initial condition $\mathcal{F}(X, 0) = 1$ to the PDE*

$$A(X)\, \mathcal{F}_1 + B(X,Y)\, \mathcal{F}_2 = C(X)\, \mathcal{F}\,, \tag{1.9}$$

*where the polynomials $A, B, C$ are given by*

$$A(X) = -(1 + \sigma\, X)\, X\,, \tag{1.10a}$$
$$B(X,Y) = 1 - r\, Y - \sigma\, r'\, X\, Y\,, \tag{1.10b}$$
$$C(X) = s - \sigma\, s'\, X\,, \tag{1.10c}$$

*and the new parameters $r, r', s, s'$ are defined as:*

$$r = \frac{\alpha}{\beta}\,, \quad r' = \frac{\alpha'}{\beta'}\,, \quad s = \frac{\alpha + \gamma}{\beta}\,, \quad s' = -1 - \frac{\alpha' + \gamma'}{\beta'}\,. \tag{1.11}$$

The main result for solutions of Type I is given by the following[2]

**Theorem 1.5 (Theorem 3.3 of Paper I)** *The solution $\mathcal{F}$ to Question 1.4 is given by*

$$\mathcal{F}(X,Y) = \left(\frac{G(\xi(X,Y))}{X}\right)^s \left(\frac{1 + \sigma\, X}{1 + \sigma\, G(\xi(X,Y))}\right)^{s+s'}\,, \tag{1.12}$$

[2]Notice that in Theorem 3.3 of Paper I, we used a slightly different notation: $Q_{r,r',\sigma}(X)$ [resp. $Q^0_{r,r',\sigma}(X)$] was written as $Q(r, r'; \sigma; X)$ [resp. $Q_0(r, r'; \sigma; X)$]. This change of notation will make our formulas more compact and easier to understand. We will make similar changes of notations in Theorems 1.6 and 1.7.

*where*

$$\xi(X,Y) \;=\; Y\,X^{-r}\,(1+\sigma\,X)^{r-r'} + G^{-1}(X)\,, \tag{1.13}$$

*and*

$$G^{-1}(X) \;=\; \xi(X,0) \;=\; Q_{r,r',\sigma}(X)\,, \tag{1.14a}$$

$$Q_{r,r',\sigma}(X) \;=\; Q^0_{r,r',\sigma}(X) + [r\in\mathbb{Z}_0]\,\sigma^r \binom{-1-r'+r}{r} \log X\,, \tag{1.14b}$$

$$Q^0_{r,r',\sigma}(X) \;=\; \sum_{k\in\mathbb{Z}_0\setminus\{r\}} \sigma^k \binom{-1-r'+r}{k} \frac{X^{k-r}}{k-r}\,, \tag{1.14c}$$

*for* $0 < X < 1$.

For problems of Types II–IV there is no need to perform a similar change of variables. The results of Paper I can be stated as follows:

**Theorem 1.6 (Theorem 3.5 of Paper I)** *When* $\beta \neq 0$ *and* $\beta' = 0$*, the solution* $F$ *to Question 1.3 is given by*

$$F(x,y) \;=\; \left(\frac{G(\xi(x,y))}{x}\right)^{(\alpha+\gamma)/\beta} \exp\left(-\frac{\alpha'+\gamma'}{\beta}\left(x - G(\xi(x,y))\right)\right)\,, \tag{1.15}$$

*where*

$$\xi(x,y) \;=\; y\,x^{-\alpha/\beta}\,e^{-\alpha' x/\beta} + G^{-1}(x)\,, \tag{1.16}$$

*and*

$$G^{-1}(x) \;=\; \xi(x,0) \;=\; \frac{1}{\beta}\,P_{\frac{\alpha}{\beta},\frac{\alpha'}{\beta}}(x)\,, \tag{1.17a}$$

$$P_{\mu,\nu}(x) \;=\; P^0_{\mu,\nu}(x) + [\mu\in\mathbb{Z}_0]\,\frac{(-\nu)^\mu}{\mu!}\,\log x\,, \tag{1.17b}$$

$$P^0_{\mu,\nu}(x) \;=\; \sum_{k\in\mathbb{Z}_0\setminus\{\mu\}} \frac{(-\nu)^k}{k!}\,\frac{x^{k-\mu}}{k-\mu}\,, \tag{1.17c}$$

*for any* $x > 0$.

**Theorem 1.7 (Theorem 3.6 of Paper I)** *When* $\beta = 0$ *and* $\beta' \neq 0$*, the solution* $F$ *to Question 1.3 is given by*

$$F(x,y) \;=\; \left(\frac{G(\xi(x,y))}{x}\right)^{1+(\alpha'+\gamma')/\beta'} \exp\left(\frac{\alpha+\gamma}{\beta'}\left(\frac{1}{x} - \frac{1}{G(\xi(x,y))}\right)\right)\,, \tag{1.18}$$

*where*

$$\xi(x,y) \;=\; y\,x^{-\alpha'/\beta'}\,e^{\alpha/(\beta' x)} + G^{-1}(x)\,, \tag{1.19}$$

*and*

$$G^{-1}(x) \;=\; \xi(x,0) \;=\; \frac{1}{\beta'}\, R_{\frac{\alpha}{\beta'},\frac{\alpha'}{\beta'}}(x)\,, \tag{1.20}$$

$$R_{\mu,\nu}(x) \;=\; R^0_{\mu,\nu}(x) + [-\nu\in\mathbb{N}]\,\frac{\mu^{-1-\nu}}{(-1-\nu)!}\,\log x\,, \tag{1.21}$$

$$R^0_{\mu,\nu}(x) \;=\; -\sum_{k\in\mathbb{Z}_0\setminus\{-1-\nu\}} \frac{\mu^k}{k!}\,\frac{1}{k+1+\nu}\,\frac{1}{x^{k+1+\nu}}\,, \tag{1.22}$$

*for* $x>0$*.*

**Theorem 1.8 (Theorem 3.7 of Paper I)** *When* $\beta=\beta'=0$ *the solution* $F$ *to Question 1.3 is*

$$F(x,y) \;=\; \begin{cases} \Big(1-(\alpha+\alpha'\,x)\,y\Big)^{-\frac{\alpha+\gamma+(\alpha'+\gamma')\,x}{\alpha+\alpha'\,x}} & \text{if } (\alpha,\alpha')\neq(0,0),\\ e^{(\gamma+\gamma'\,x)\,y} & \text{if } (\alpha,\alpha')=(0,0). \end{cases} \tag{1.23}$$

In the present paper we will deepen our understanding of the results of Paper I. In particular, we obtain the one-variable row generating functions associated with the EGF $F(x,y)$ (1.4):

$$P_n(x) \;=\; n!\,[y^n]F(x,y) \;=\; \sum_{k=0}^{n} \left|{n \atop k}\right| x^k\,. \tag{1.24}$$

These polynomials can be obtained for each of the four types discussed above.

The main contributions of this paper can be summarized as follows:

- We have obtained Rodrigues–like formulas for the row polynomials for all types of families of Question (1.1). In particular, when the functions $Q_{r,r',\sigma}$ (1.14b), $P_{\mu,\nu}$ (1.17b), and $R_{\mu,\nu}$ (1.21) are elementary functions, the row polynomials are given in terms of derivatives of elementary functions.

- For problems of Type IV, we have obtained closed formulas for the row polynomials and used them to derive the numbers $\left|{n \atop k}\right|$ obtained by Spivey [25, Theorem 9].

- For two-parameter generalizations of the Eulerian $\left\langle {n \atop k} \right\rangle$ and second-order Eulerian $\left\langle\!\left\langle {n \atop k} \right\rangle\!\right\rangle$ numbers (see e.g., [15] and [23, entries A173018 and A008517]), we have obtained closed formulas for both the row polynomials and the coefficients. These formulas are (to our knowledge) new.

The plan of the paper is as follows. In Section 2 we will compute the row polynomial generating functions (1.24) for each of the four types of solutions to Question 1.1. In Section 3, we will apply the preceding general results to some Eulerian families, which include the ordinary and second-order Eulerian numbers, and generalizations of them. In Section 4 we discuss a short selection of applications of our general method to some other families of combinatorial interest. Finally, in Appendix A we will prove some technical results stated in the main text.

# 2 Polynomial generating functions in one variable

The purpose of this section is to study the one-variable polynomials $P_n(x)$ (1.24) for the four types of solutions to Question 1.1 as defined in Definition 1.2, and obtain useful expressions of Rodrigues type for them. We will obtain these polynomials from the corresponding EGF $F(x,y)$ (1.4) (given by the theorems quoted in the Introduction) by employing complex-variable methods based on contour integration and the Hadamard formula (see e.g., [7, p. 85], [16]). We will start with the most general case (Type I), and will work out the proof of the main theorem with great detail. For the other cases, the corresponding proofs are very similar, so we will skip most of the common parts for the sake of brevity.

## 2.1 Type I case

When $\beta\beta' \neq 0$, it is convenient to work with the variables $X, Y$ introduced in (1.7), and define the auxiliary polynomials

$$\mathcal{P}_n(X) \;=\; n!\,[Y^n]\mathcal{F}(X,Y)\,, \tag{2.1}$$

so that

$$P_n(x) \;=\; \beta^n\,\mathcal{P}_n\left(\sigma\frac{\beta'}{\beta}x\right)\,. \tag{2.2}$$

An important idea that we will use in this section is to write $\mathcal{P}_n(X)$ as a contour integral by using Cauchy's theorem, rewrite it in a convenient form by an appropriate change of variables, and finally compute the resulting integral with the help of residues. As we will show, in many occasions this procedure provides simple expressions for $\mathcal{P}_n(X)$. The possibility of employing this procedure depends crucially on the analyticity properties of the generating functions $\mathcal{F}(X,Y)$ (1.8) that, in turn, hinge upon those of the function $G(Z)$ [cf., (1.14)]. These can be studied by using the complex implicit-function theorem [19], and taking advantage of the explicit form of the function $\xi(X,Y)$ (1.13). Our result can be summarized in the following

**Theorem 2.1** *The polynomials $\mathcal{P}_n(X)$ (2.1) corresponding to the Type-I EGF $\mathcal{F}(X,Y)$ (1.12) are given by*

$$\begin{aligned}\mathcal{P}_n(X) \;&=\; \frac{(1+\sigma X)^{n(r-r')+s+s'}}{X^{s+rn}}\\ &\qquad\times\lim_{Z\to X}\frac{\partial^n}{\partial Z^n}\left[\frac{Z^{s-r-1}}{(1+\sigma Z)^\eta}\left(\frac{Z-X}{Q_{r,r',\sigma}(Z)-Q_{r,r',\sigma}(X)}\right)^{n+1}\right]\,,\end{aligned} \tag{2.3}$$

*where $\eta = s+s'+1+r'-r$, or in the following alternative form if $r\in\mathbb{Z}_0$:*

$$\begin{aligned}\mathcal{P}_n(X) \;&=\; \frac{(1+\sigma X)^{n(r-r')+s+s'}}{X^{s+rn}}\,\sigma^{(n+1)r}\binom{-1-r'+r}{r}^{-n-1)}\\ &\qquad\times\lim_{Z\to X}\frac{\partial^n}{\partial Z^n}\left[\frac{Z^{s-r-1}(Z-X)^{n+1}}{(1+\sigma Z)^\eta}\left[\log\frac{Z\,\widehat{Q}^0(Z)}{X\,\widehat{Q}^0(X)}\right]^{-n-1}\right]\,.\end{aligned} \tag{2.4}$$

*The function* $\widehat{Q}^0(X)$ *is a short-hand for*

$$\widehat{Q}^0(X) \; = \; \exp\left(\frac{Q^0_{r,r',\sigma}(X)}{\sigma^r\binom{-1-r'+r}{r}}\right) , \tag{2.5}$$

*where* $Q^0_{r,r',\sigma}(X)$ *is given in (1.14c).*

PROOF. Let us pick $X \in \mathbb{C}\backslash\{0\}$ contained within the convergence disk of $Q^0$ ($|X| < 1$; cf., (1.14c)). Because $Q^0$ contains a term $X^{-r}$, when $r$ is a non-integer number, we should treat the origin with care, as it can be a singular point (either a pole or a branch point). We now consider the function

$$A\colon U \subset \mathbb{C}^3 \; \to \; \mathbb{C}\colon (X_1, X_2, X_3) \; \mapsto \; A(X_1, X_2, X_3) \; = \; \xi(X_1, X_2) - \xi(X_3, 0) \, , \tag{2.6}$$

where $U \subset \mathbb{C}^3$ is an open neighborhood of $(X, 0, X)$ and $\xi(X_1, X_2)$ is given by (1.13). Now, as $A(X, 0, X) = 0$, and its partial derivative w.r.t. $X_3$ at $(X, 0, X)$ is given by

$$A_3(X, 0, X) \; = \; -\xi_1(X, 0) \; = \; -\frac{(1 + \sigma X)^{-1-r'+r}}{X^{1+r}} \; \neq \; 0 \, , \tag{2.7}$$

for all $X \in \mathbb{C}$ such that $0 < |X| < 1$, there exist open neighborhoods $U_1 \subset \mathbb{C}^2$ and $U_2 \subset \mathbb{C}$ of $(X, 0)$ and $X$, respectively, with $U_1 \times U_2 \subset U$, and a unique holomorphic function $\theta\colon U_1 \to U_2$ such that

$$A^{-1}(0) \cap (U_1 \times U_2) \; = \; \{((X, Y), \theta(X, Y))\colon (X, Y) \in U_1\} \; . \tag{2.8}$$

An important consequence of this result and the definition of $G$ [cf., (1.14)] is that

$$G(\xi(X, Y)) \; = \; G(\xi(\theta(X, Y), 0)) \; = \; \theta(X, Y) \, . \tag{2.9}$$

The results just proved for the analyticity of $\theta$ imply that there exists an open neighborhood $\Omega$ of the origin of the complex $Y$–plane such that, for every $X \in \mathbb{C}$ satisfying $0 < |X| < 1$, the function $Y \mapsto \mathcal{F}(X, Y)$ given by (1.12) is analytic in $\Omega$. By using Cauchy's theorem we can then write

$$[Y^n]\mathcal{F}(X, Y) \; = \; \frac{1}{2\pi i} \int_\Gamma \left(\frac{G(\xi(X, Y))}{X}\right)^s \left(\frac{1 + \sigma X}{1 + \sigma G(\xi(X, Y))}\right)^{s+s'} \frac{dY}{Y^{n+1}} \, , \tag{2.10}$$

where $\Gamma$ is a simple closed curve of index $+1$, contained in $\Omega$, that surrounds the origin $Y = 0$ and no other singularity of the integrand.

A natural change of variables, suggested by the form of (2.10), is to put [cf., (2.9)]

$$Z \; = \; G(\xi(X, Y)) \; = \; \theta(X, Y) \, , \tag{2.11}$$

so let us consider the one-parameter family of holomorphic maps ($\pi_2$ denotes the projection onto the second argument)

$$Z_X\colon \pi_2((\{X\} \times \mathbb{C}) \cap U_1) \subset \mathbb{C} \; \to \; \mathbb{C}\colon Y \; \mapsto \; Z_X(Y) \; = \; G(\xi(X, Y)) \, . \tag{2.12}$$

They satisfy

$$Z'_X(0) \;=\; \frac{dZ_X}{dY}(0) \;=\; \theta_2(X,0) \;=\; X\,(1+\sigma\,X) \;\neq\; 0\,, \tag{2.13}$$

for all $X \in \mathbb{C}$ such that $0 < |X| < 1$. This equation is proven by differentiating $\xi(X_1,Y) - \xi(\theta(X_1,Y),0) = 0$ with respect to $Y$, and evaluating the result at $(X,0)$ to get the following condition:

$$\xi_2(X,0) - \xi_1(X,0)\,\theta_2(X,0) \;=\; 0 \tag{2.14}$$

(remember that $\theta(X,0) = X$). Now using the fact that for $0 < |X| < 1$,

$$\xi_1(X,0) \;=\; X^{-1-r}\,(1+\sigma\,X)^{-1-r'+r} \;\neq\; 0\,, \tag{2.15a}$$
$$\xi_2(X,0) \;=\; X^{-r}\,(1+\sigma\,X)^{r-r'} \;\neq\; 0\,, \tag{2.15b}$$

we immediately obtain (2.13). Notice that $Z'_X(0) \neq 0$ implies that $Z_X(\Gamma)$, the image of the original integration contour $\Gamma$, will be also a closed, simple curve of index $+1$, contained in $Z_X(\Omega)$ and surrounding the point $Z = X$ in the complex $Z$–plane. In Figure 1 we show the neighborhood $Z_X(\Omega)$ of the complex $Z$–plane, and the contour $Z_X(\Gamma) \subset Z_X(\Omega)$ surrounding the point $Z = X$. Notice also that, given any open neighborhood $V_X$ of $Z = X$, it is possible to choose the original integration contour $\Gamma$ in such a way that $Z_X(\Gamma) \subset V_X$.

It is now straightforward to rewrite the integral in (2.10) as a contour integral on $Z_X(\Gamma)$. In practice one needs to compute [cf., (1.14a)]

$$\xi(X,Y) \;=\; G^{-1}(Z) \;=\; \xi(Z,0) \;=\; Q_{r,r',\sigma}(Z)\,, \tag{2.16}$$

that can be written as

$$Y\,X^{-r}\,(1+\sigma X)^{r-r'} + Q_{r,r',\sigma}(X) \;=\; Q_{r,r',\sigma}(Z) \tag{2.17}$$

and leads to the following expression for $Y$ in terms of $Z$ ($X$ should be considered as a parameter here):

$$Y \;=\; X^r\,(1+\sigma X)^{r'-r}\,\left(Q_{r,r',\sigma}(Z) - Q_{r,r',\sigma}(X)\right)\,. \tag{2.18}$$

By using (1.13)/(1.14b)/(1.14c), we also find

$$\xi_2(X,0) \;=\; X^{-r}\,(1+\sigma\,X)^{r-r'}\,, \tag{2.19a}$$
$$Q'_{r,r',\sigma}(Z) \;=\; Z^{-1-r}\,(1+\sigma\,Z)^{-1-r'+r}\,, \tag{2.19b}$$

so that

$$dY \;=\; X^r\,(1+\sigma\,X)^{r'-r}\,Q'_{r,r',\sigma}(Z)\,dZ \;=\; \frac{X^r\,(1+\sigma\,X)^{r'-r}}{Z^{1+r}\,(1+\sigma\,Z)^{1+r'-r}}\,dZ\,. \tag{2.20}$$

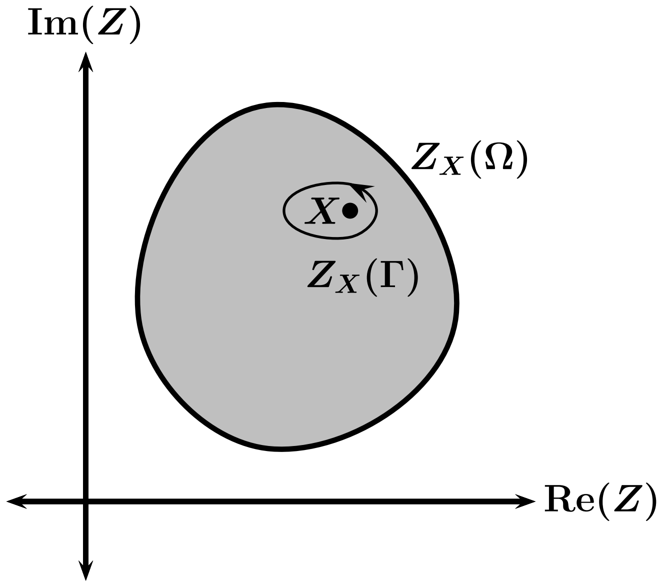


Figure 1: Integration contour in the complex $Z$-plane. The gray area is the set $Z_X(\Omega)$, i.e., the image under $Z_X$ of the neighborhood $\Omega$ of the origin of the complex $Y$–plane. $\mathcal{F}(X,Y)$ is analytic in $\Omega$ as a function of $Y$ (for any fixed value of $X$ satisfying $0<|X|<1$). The contour $Z_X(\Gamma)$ is the image under $Z_X$ of the contour $\Gamma$ appearing in the contour integral (2.10). While $\Gamma$ encircles the point $Y=0$, $Z_X(\Gamma)$ encircles the point $Z=X$, where the integrand of Eq. (2.21) has a *single* pole of order $n+1$.

We can then rewrite the integral (2.10) in terms of $Z$ to obtain the following expression for the polynomials $\mathcal{P}_n(X)$:

$$\begin{aligned}\mathcal{P}_n(X) &= n!\,\frac{(1+\sigma\,X)^{n(r-r')+s+s'}}{X^{s+rn}}\\ &\quad\times\frac{1}{2\pi i}\int_{Z_X(\Gamma)}\frac{Z^{s-r-1}}{(1+\sigma\,Z)^{\eta}}\frac{dZ}{(Q_{r,r',\sigma}(Z)-Q_{r,r',\sigma}(X))^{n+1}}\,,\end{aligned}\tag{2.21}$$

where $\eta=s+s'+1+r'-r$.

The analytic structure of the integrand of (2.21) shows that it is possible to choose $\Gamma$ in such a way that 1) we can avoid the singularity that may appear due to the term $Z^{s-r-1}$, and 2) the only singularity surrounded by $Z_X(\Gamma)$ is $Z=X$. We can, hence, compute the integral by simply obtaining the residue of the integrand at this point, that can be immediately seen to be a pole of order $n+1$ because $Q'_{r,r',\sigma}(X)\neq 0$ if $X$ satisfies $0<|X|<1$. (See Figure 1.) By doing this we get

$$\begin{aligned}\mathcal{P}_n(X) &= n!\,\frac{(1+\,\sigma X)^{n(r-r')+s+s'}}{X^{s+rn}}\\ &\quad\times\mathrm{Res}\left(\frac{Z^{s-r-1}}{(1+\sigma\,Z)^{\eta}}\,\frac{1}{(Q_{r,r',\sigma}(Z)-Q_{r,r',\sigma}(X))^{n+1}}\,;Z=X\right),\end{aligned}\tag{2.22}$$

where $\mathrm{Res}(f(z); z = x)$ denotes the residue of the function $f$ at the point $x$. This last equation leads immediately to (2.3), as claimed.

When $r \in \mathbb{Z}_0$, it is convenient to explicitly take into account the logarithmic terms appearing in $Q_{r,r',\sigma}(Z)$ and $Q_{r,r',\sigma}(X)$ and rewrite (2.22) in the form

$$\begin{aligned} \mathcal{P}_n(X) &= n!\, \frac{(1+\sigma X)^{n(r-r')+s+s'}}{X^{s+rn}}\, \sigma^{(n+1)r} \binom{-1-r'+r}{r}^{-n-1} \\ &\quad \times \mathrm{Res}\left( \frac{Z^{s-r-1}}{(1+\sigma Z)^{\eta}} \left[ \log \frac{Z\,\widehat{Q}^0(Z)}{X\,\widehat{Q}^0(X)} \right]^{-n-1} ; Z = X \right), \qquad (2.23) \end{aligned}$$

where we have used the short-hand notation (2.5). Then, from (2.23), we easily obtain the claimed result (2.4). ∎

**Remarks.** 1. Notice that the procedure that we have followed above allows us to partially sidestep the difficulties associated with the impossibility to obtain closed form expressions for the function $G(Z)$ (1.14); we only need the functions $Q_{r,r',\sigma}(X)$ or $Q^0_{r,r',\sigma}(X)$ which, in the worst case scenario, can be written explicitly in terms of hypergeometric functions. Notice that these formulas are valid for every value of the parameters $r, r', s, s'$.

2. As discussed above, the integrand of (2.21) has a pole of order $n+1$ at $Z = X$. The expressions written above are based on the computation of the residue at this point and take advantage of the fact that the integrand is a meromorphic function in an open neighborhood of it. It is possible, however, to consider analytic extensions of the integrand and move the integration contour to rewrite the integral in more convenient ways. For instance, if for a particular choice of the parameters $(r, r'; s, s')$ [or the parameters $(\alpha, \beta, \gamma; \alpha', \beta'\gamma')$], the singularities of the integrand (other than the one at $Z = X$) are poles, it may happen that their order is *independent* of $n$ (though it will generically depend on the above parameters) in which case it may be possible to obtain closed form formulas for the $\mathcal{P}_n(X)$ by explicit computing the corresponding residues (including possibly the one at infinity) because this will only require the computation of derivatives up to a fixed (i.e., independent of $n$) order.

3. Formula (2.23) suggest the change of variables: $e^U = Z\widehat{Q}^0(Z)$ and $e^V = X\widehat{Q}^0(X)$. This change leads to simple Rodrigues-like formulas for the row polynomials. We will show some applications of this idea in the following sections.

## 2.2 Type II case

When $\beta \neq 0$ and $\beta' = 0$, the EGF $F(x, y)$ have the general form given by Theorem 1.6. In this case we can work with the original variables $x, y$. Our goal is to express the one-variable polynomials $P_n(x)$ (1.24) as a contour integral by following the same steps that led to Theorem 2.1.

Using the complex implicit function theorem, we can show (as in the proof of Theorem 2.1) that there is an open neighborhood $\Omega$ of the origin of the complex

$y$–plane where $F(x,y)$ (1.15) is analytic as a function of $y$ (for every $x \in \mathbb{C}$ satisfying $0 < |x| < 1$). Then, $P_n(x)$ can be expressed as a contour integral by using Cauchy's theorem:

$$P_n(x) \;=\; \frac{n!}{2\pi i} \int_{\Gamma_x} \left( \frac{G(\xi(x,y))}{x} \right)^{\frac{\alpha+\gamma}{\beta}} \exp\left( -\frac{\alpha'+\gamma'}{\beta} \Big( x - G(\xi(x,y)) \Big) \right) \frac{dy}{y^{n+1}}\,, \quad (2.24)$$

where $\Gamma_x$ is a closed, simple curve of index $+1$, that surrounds the origin $y=0$ and no other singularity of the integrand, $\xi(x,y)$ is given by (1.16), and $G$ is given by (1.17).

As we did in the preceding section, it is convenient now to perform the change of variables $z = G(\xi(x,y))$. The same algebra used in the proof of Theorem 2.1 leads to the expressions:

$$y \;=\; \frac{1}{\beta}\, x^{\alpha/\beta}\, e^{\alpha' x/\beta} \left( P_{\frac{\alpha}{\beta},\frac{\alpha'}{\beta}}(z) - P_{\frac{\alpha}{\beta},\frac{\alpha'}{\beta}}(x) \right)\,, \quad (2.25)$$

$$dy \;=\; \frac{1}{\beta}\, \frac{x^{\alpha/\beta}}{z^{1+\alpha/\beta}}\, e^{\alpha'(x-z)/\beta}\, dz\,, \quad (2.26)$$

so that

$$P_n(x) \;=\; \frac{n!\,\beta^n}{2\pi i}\, \frac{e^{-((n+1)\alpha'+\gamma')x/\beta}}{x^{((n+1)\alpha+\gamma)/\beta}} \int_\Gamma \frac{z^{\gamma/\beta-1}\, e^{\gamma' z/\beta}}{\left( P_{\frac{\alpha}{\beta},\frac{\alpha'}{\beta}}(z) - P_{\frac{\alpha}{\beta},\frac{\alpha'}{\beta}}(x) \right)^{n+1}}\, dz\,, \quad (2.27)$$

where the integration contour in (2.27) is a closed, simple curve of index $+1$, that surrounds the point $z = x$ and no other singularity of the integrand. Again, as

$$P'_{\frac{\alpha}{\beta},\frac{\alpha'}{\beta}}(x) \;=\; x^{-\alpha/\beta-1}\, e^{-\alpha' x/\beta} \;\neq\; 0\,, \quad (2.28)$$

for any $x \in \mathbb{C}$ such that $0 < |x| < 1$, then the integrand in (2.27) has a pole of order $n+1$ at $z = x$. Then, we can compute this integral using residues:

$$P_n(x) \;=\; n!\,\beta^n\, \frac{e^{-((n+1)\alpha'+\gamma')x/\beta}}{x^{((n+1)\alpha+\gamma)/\beta}}\, \mathrm{Res}\left( \frac{z^{\gamma/\beta-1}\, e^{\gamma' z/\beta}}{\left( P_{\frac{\alpha}{\beta},\frac{\alpha'}{\beta}}(z) - P_{\frac{\alpha}{\beta},\frac{\alpha'}{\beta}}(x) \right)^{n+1}}\,; z = x \right)\,. \quad (2.29)$$

The computation of the residue in (2.29) is straightforward. The above discussion can be summarized in the following

**Theorem 2.2** *The polynomials $P_n(x)$ (1.24) corresponding to the Type-II EGF (1.15) are given by*

$$P_n(x) \;=\; \beta^n\, \frac{e^{-((n+1)\alpha'+\gamma')x/\beta}}{x^{((n+1)\alpha+\gamma)/\beta}}\, \lim_{z\to x} \frac{\partial^n}{\partial z^n} \frac{z^{\gamma/\beta-1}\, e^{\gamma' z/\beta}\, (z-x)^{n+1}}{\left( P_{\frac{\alpha}{\beta},\frac{\alpha'}{\beta}}(z) - P_{\frac{\alpha}{\beta},\frac{\alpha'}{\beta}}(x) \right)^{n+1}}\,. \quad (2.30)$$

## 2.3 Type III case

When $\beta = 0$ and $\beta' \neq 0$, the EGF $F(x,y)$ have the general form given by Theorem 1.7. Again, we can work with the original variables $x, y$, and our aim is to write the polynomials $P_n(x)$ (1.24) as a contour integral.

Using the complex implicit function theorem, we can show (as in the proof of Theorem 2.1) that there is an open neighborhood $\Omega$ of the origin of the complex $y$–plane where $F(x,y)$ (1.18) is analytic as a function of $y$ (for every $x \in \mathbb{C}$ satisfying $0 < |x| < 1$). Then, $P_n(x)$ can be expressed as a contour integral by using Cauchy's theorem:

$$P_n(x) \;=\; \frac{n!}{2\pi i} \int_{\Gamma_x} \left( \frac{G(\xi(x,y))}{x} \right)^{1+\frac{\alpha'+\gamma'}{\beta'}} \exp\left( \frac{\alpha+\gamma}{\beta'} \left( \frac{1}{x} - \frac{1}{G(\xi(x,y))} \right) \right) \frac{dy}{y^{n+1}}\,, \quad (2.31)$$

where $\Gamma_x$ is a closed, simple curve of index $+1$, that surrounds the origin $y = 0$ and no other singularity of the integrand, $\xi(x,y)$ is given by (1.19), and $G$ is given by (1.22).

As we did in the Section 2.1, we perform the change of variables $z = G(\xi(x,y))$. After some algebra, we arrive at the expressions:

$$y \;=\; \frac{1}{\beta'}\, x^{\alpha'/\beta'}\, e^{-\alpha/(\beta' x)} \left( R_{\frac{\alpha}{\beta'},\frac{\alpha'}{\beta'}}(z) - R_{\frac{\alpha}{\beta'},\frac{\alpha'}{\beta'}}(x) \right), \quad (2.32)$$

$$dy \;=\; \frac{1}{\beta'} \frac{x^{\alpha'/\beta'}}{z^{2+\alpha'/\beta'}} \exp\left( \frac{\alpha}{\beta'} \left( \frac{1}{z} - \frac{1}{x} \right) \right) dz\,, \quad (2.33)$$

so that

$$P_n(x) \;=\; \frac{n!\,\beta'^n}{2\pi i} \frac{e^{((n+1)\alpha+\gamma)/(\beta' x)}}{x^{1+((n+1)\alpha'+\gamma')/\beta'}} \int_\Gamma \frac{z^{\gamma'/\beta'-1}\, e^{-\gamma/(\beta' z)}}{\left( R_{\frac{\alpha}{\beta'},\frac{\alpha'}{\beta'}}(z) - R_{\frac{\alpha}{\beta'},\frac{\alpha'}{\beta'}}(x) \right)^{n+1}}\, dz\,, \quad (2.34)$$

where the integration contour in (2.34) is a closed, simple curve of index $+1$, that surrounds the point $z = x$ and no other singularity of the integrand. Again, as

$$R'_{\frac{\alpha}{\beta'},\frac{\alpha'}{\beta'}}(x) \;=\; x^{-\alpha'/\beta'-2}\, e^{\alpha/(\beta' x)} \;\neq\; 0\,, \quad (2.35)$$

for any $x \in \mathbb{C}$ such that $0 < |x| < 1$, then the integrand in (2.34) has a pole of order $n+1$ at $z = x$. Then, we can compute this integral using residues:

$$P_n(x) \;=\; n!\,\beta'^n \frac{e^{((n+1)\alpha+\gamma)/(\beta' x)}}{x^{1+((n+1)\alpha'+\gamma')/\beta'}} \operatorname{Res}\left( \frac{z^{\gamma'/\beta'-1}\, e^{\gamma/(\beta' z)}}{\left( R_{\frac{\alpha}{\beta'},\frac{\alpha'}{\beta'}}(z) - R_{\frac{\alpha}{\beta'},\frac{\alpha'}{\beta'}}(x) \right)^{n+1}} ; z = x \right). \quad (2.36)$$

The computation of the residue in (2.36) is straightforward. The above discussion can be summarized in the following

**Theorem 2.3** *The polynomials $P_n(x)$ (1.24) corresponding to the Type-III EGF (1.18) are given by*

$$P_n(x) \;=\; \beta'^n \frac{e^{((n+1)\alpha+\gamma)/(\beta' x)}}{x^{1+((n+1)\alpha'+\gamma')/\beta'}} \lim_{z\to x} \frac{\partial^n}{\partial z^n} \frac{z^{\gamma'/\beta'-1}\, e^{\gamma/(\beta' z)}\, (z-x)^{n+1}}{\left( R_{\frac{\alpha}{\beta'},\frac{\alpha'}{\beta'}}(z) - R_{\frac{\alpha}{\beta'},\frac{\alpha'}{\beta'}}(x) \right)^{n+1}}\,. \quad (2.37)$$

## 2.4 Type IV case

This corresponds to Spivey's case 2 [25]. The EGF $F(x,y)$ for the families $(\alpha,0;\alpha',0)$ are given in closed form by (1.23), so it is not necessary to provide the integral representation used above. The result is easy to obtain, so we simply quote it here:

**Theorem 2.4** *The polynomials $P_n(x)$ (1.24) corresponding to the Type-IV EGF (1.23) are given by*

$$P_n(x) \;=\; \prod_{k=1}^{n} \Big(k\,\alpha + \gamma + (k\,\alpha' + \gamma')\,x\Big)\,. \tag{2.38}$$

Actually, the form of the coefficients for this case is also easy to obtain after some little algebra:

**Corollary 2.5** *The coefficients $\left|{n \atop k}\right|$ for $n \ge 0$ and $0 \le k \le n$ corresponding to solutions to Question 1.1 of Type-IV are given by*

$$\left|{n \atop k}\right| \;=\; \sum_{t=0}^{n}\sum_{s=0}^{k} \left[{n \atop t}\right] \binom{t}{s} \binom{n-t}{k-s} (\alpha+\gamma)^{t-s}\,(\alpha'+\gamma')^{s}\alpha^{n-t+s-k}(\alpha')^{k-s}\,. \tag{2.39}$$

**Remarks.** 1. A similar formula has been obtained by Spivey [25, Theorem 9].

2. In proving (2.39) for $(\alpha,\alpha') \neq (0,0)$, we have used the formula [15, Eq. (6.11)]

$$x^{\overline{n}} \;=\; \sum_{k=0}^{n} \left[{n \atop k}\right] x^k\,, \qquad \text{for any } n \in \mathbb{Z}_0, \tag{2.40}$$

where $\left[{n \atop k}\right]$ are the Stirling cycle numbers [15].

3. Notice that when $\alpha = \alpha' = 0$, the formulas (2.38)/(2.39) reduce easily to those for the binomial family:

$$P_n(x) \;=\; (\gamma + x\gamma')^n \quad \Rightarrow \quad \left|{n \atop k}\right| \;=\; \binom{n}{k} \gamma^{n-k}\,(\gamma')^k\,. \tag{2.41}$$

4. For the Stirling cycle numbers [i.e., the family $(1,0;0,0)$ with $(\gamma,\gamma') = (-1,1)$], (2.38)/(2.39) reduce to

$$P_n(x) \;=\; \prod_{k=0}^{n-1}(x+k) \;=\; x^{\overline{n}} \quad \Rightarrow \quad \left|{n \atop k}\right| \;=\; \left[{n \atop k}\right], \tag{2.42}$$

as expected from (2.40).

5. For the Stirling numbers of first kind [i.e., the family $(-1,0;0,0)$ with $(\gamma,\gamma') = (1,1)$], (2.38)/(2.39) reduce as expected to [cf., [15, Eq. (6.13)]]:

$$P_n(x) \;=\; \prod_{k=0}^{n-1}(x-k) \;=\; x^{\underline{n}} \quad \Rightarrow \quad \left|{n \atop k}\right| \;=\; (-1)^{n-k} \left[{n \atop k}\right]. \tag{2.43}$$

# 3 Applications: Eulerian families

In this section we will study three types of Eulerian numbers: a) the Eulerian family with parameters $(0, 1; 1, -1)$; b) the second-order Eulerian family with parameters $(0, 1; 2, -1)$; and c) the $\nu$-order Eulerian family, characterized by the parameters $(0, 1; \nu, -1)$ for integers $\nu \geq 2$. For each of them, we will consider different choices for the other parameters $(\gamma, \gamma')$. All these families are of Type I, so we will use Theorems 1.5 and 2.1. Notice that Eq. (1.7) implies that $(X, Y) = (x, y)$ for all of them. In terms of the parameters $r, r', s, s', \sigma$ [cf., (1.7)/(1.11)], these families are characterized by $r = 0$, $r' = -\nu \neq 0$, $s = \gamma$, $s' = \nu + \gamma' - 1$, and $\sigma = -1$, with $\nu \in \mathbb{N}$. Then, the polynomial generating function $P_n(x)$ (2.2)/(2.4)/(2.23) is given by:

$$P_n(x) = n! \, \frac{(1-x)^{s+s'+n\nu}}{x^s} \operatorname{Res} \left( \frac{z^{s-1}}{(1-z)^{1+s+s'-\nu}} \left[ \log \frac{z\,\widehat{Q}^0(z)}{x\,\widehat{Q}^0(x)} \right]^{-n-1} ; z = x \right) \tag{3.1a}$$

$$= \frac{(1-x)^{s+s'-n\nu}}{x^s} \lim_{z \to x} \frac{\partial^n}{\partial z^n} \left( \frac{z^{s-1}(z-x)^{n+1}}{(1-z)^{1+s+s'-\nu}} \left[ \log \frac{z\,\widehat{Q}^0(z)}{x\,\widehat{Q}^0(x)} \right]^{-n-1} \right), \tag{3.1b}$$

where in this case $\widehat{Q}^0(x) = \exp[Q^0_{0,-\nu,-1}(x)]$ [cf., (2.5)].

## 3.1 The Eulerian family

The Eulerian numbers $\left\langle {n \atop k} \right\rangle$ [23, entry A173018], [15, Section 6.2] are defined as follows:[3] for $n \in \mathbb{N}$, $\left\langle {n \atop k} \right\rangle$ is the number of permutations $\pi = \pi_1 \pi_2 \cdots \pi_n$ of $[n]$ with $k$ ascents (an ascent is an index $i \in [n-1]$ such that $\pi_{i+1} > \pi_i$). For $n = 0$, we define $\left\langle {0 \atop k} \right\rangle = \delta_{k,0}$. Then, the numbers $\left\langle {n \atop k} \right\rangle$ are non-vanishing only for $0 \leq k \leq n-1$. They satisfy the recursion (1.1) with $(\alpha, \beta; \alpha', \beta') = (0, 1; 1, -1)$ and $(\gamma, \gamma') = (1, 0)$.

Now let $r$ be a positive integer. Several authors [24, p. 215], [12, Chapter II, p. 17], [21], [3, Problems 17 and 18, p. 38] have considered the *r-Eulerian numbers*[4] $\left\langle {n \atop k} \right\rangle_r$, defined for $n \in \mathbb{N}$ to be the number of permutations of $[n]$ with $k$ ascents of size *at least* $r$ (i.e., indices $i \in [n-1]$ such that $\pi_{i+1} \geq \pi_i + r$). Indeed, for $r = 1$, they reduce to the ordinary Eulerian numbers. Again, we define $\left\langle {0 \atop k} \right\rangle_r = \delta_{k,0}$. For $r = 2$, these numbers appear in Ref. [23, entry A120434] with a different indexing. It is not difficult to see that $\left\langle {n \atop k} \right\rangle_r = n! \delta_{k,0}$ for $0 \leq n \leq r$, and that, for $n \geq r$ it is nonvanishing only for $0 \leq k \leq n - r$. The $r$-Eulerian numbers satisfy the recursion [12, Eq. (8)]:

$$\left\langle {n \atop k} \right\rangle_r = (k+r) \left\langle {n-1 \atop k} \right\rangle_r + (n-k-r+1) \left\langle {n-1 \atop k-1} \right\rangle_r \qquad \text{for } n \geq r. \tag{3.2}$$

[3]Please, note that we use the indexing and notation of GKP [15], and these differ both from the traditional indexing [7, 24], [23, entry A008292], and the traditional notation $A(n, k) = \left\langle {n \atop k-1} \right\rangle$.

[4]The notation for the $r$-Eulerian numbers is rather non-standard: e.g., $A_r(n, k)$ [21,24], ${}^r A_{n,k}$ [12], or $A(n, k, r)$ [3]. In addition, the indexing is not always the same, as for the ordinary Eulerian numbers.

Owing to this restriction on $n$, the above recursion does not meet all the conditions of Question 1.1. However, we can bring the $r$-Eulerian numbers into the framework of Question 1.1 by defining the *shifted $r$-Eulerian numbers*:

$$\left\langle {n \atop k} \right\rangle_{(r)} \;=\; \begin{cases} \dfrac{1}{(r-1)!} \left\langle {n+r-1 \atop k} \right\rangle_r & \text{for } n \geq 0, \\ 0 & \text{for } n < 0. \end{cases} \tag{3.3}$$

These numbers satisfy the recursion

$$\left\langle {n \atop k} \right\rangle_{(r)} \;=\; (k+r) \left\langle {n-1 \atop k} \right\rangle_{(r)} + (n-k) \left\langle {n-1 \atop k-1} \right\rangle_{(r)} + [n=k=0]\,, \tag{3.4}$$

for any $n \geq 0$. Indeed, for $r=1$ we recover the ordinary Eulerian numbers $\left\langle {n \atop k} \right\rangle_{(1)} = \left\langle {n \atop k} \right\rangle$.

At this point, we can remove the restriction that $r \in \mathbb{N}$, by taking the recursion (3.3) as the definition of the shifted $r$-Eulerian numbers $\left\langle {n \atop k} \right\rangle_{(r)}$ and regarding $r$ as an indeterminate parameter. Then, $\left\langle {n \atop k} \right\rangle_{(r)}$ is a polynomial in $r$ of degree $n-k$, with integer coefficients.

**Remark.** Carlitz [4, 5] and Dillon and Roselle [10] introduced a different generalization $A^{(r)}_{n,k}$ of the Eulerian numbers. These numbers satisfy the recurrence [10, Eq. (2.3)], [5, Eq. (2.10)]:

$$A^{(r)}_{n,k} \;=\; (k+1)\, A^{(r)}_{n-1,k} + (n-k+r-1)\, A^{(r)}_{n-1,k-1} + [n=k=0]\,, \tag{3.5}$$

where we have adapted their indexing to ours. For $r=1$ they correspond to the ordinary Eulerian numbers; for $r=2$ they are the ordinary Eulerian numbers with $A^{(2)}_{n,n} = 1$ (rather than 0); for $r = 3,4,5,6$ they correspond to the 2–restricted,. . . , 5–restricted Eulerian numbers in [23] (entries A144696, A144697, A144698, and A144699, respectively). The numbers $A^{(r)}_{n,k}$ belong to the Eulerian family $(0,1;1,-1)$ with $(\gamma,\gamma') = (1, r-1)$, and their EGF is [cf., Theorem (1.5)]:

$$F(x,y) \;=\; e^{(1-x)\,y} \left( \frac{1-x}{1-x\,e^{(1-x)\,y}} \right)^r . \tag{3.6}$$

Let us now introduce a class of "generalized Eulerian numbers" that contains the shifted $r$-Eulerian numbers, as well as the generalization by Carlitz, and Dillon and Roselle (3.5). Let us consider a pair of indeterminates $s$ and $t$, and define the $(s,t)$-Eulerian numbers by the recursion

$$\left\langle {n \atop k} \right\rangle_{(s,t)} \;=\; (k+s) \left\langle {n-1 \atop k} \right\rangle_{(s,t)} + (n-k+t) \left\langle {n-1 \atop k} \right\rangle_{(s,t)} + [n=k=0]\,. \tag{3.7}$$

Then, $\left\langle{n \atop k}\right\rangle_{(s,t)}$ is a polynomial of degree $n$ in $s, t$ with leading term $\binom{n}{k} s^{n-k} t^k$. When $(s,t) = (r,0)$, it reduces to the shifted $r$-Eulerian numbers $\left\langle{n \atop k}\right\rangle_{(r)}$, and when $(s,t) = (1, r-1)$ to the numbers $A_{n,k}^{(r)}$ (3.5).

**Remark.** Carlitz and Scoville [6] introduced a different 2–parameter generalization of the ordinary Eulerian numbers. Starting from the ordinary Eulerian numbers, they defined the numbers

$$A(r,s) \;=\; \left\langle{r+s+1 \atop s}\right\rangle, \quad \text{for } r,s \geq 0. \tag{3.8}$$

Notice that the $s$-th row of the square array $A(r,s)$ corresponds to the $s$-th column of the triangular array of the ordinary Eulerian numbers, starting at the first non-zero element. The $A(r,s)$ satisfy the recurrence:

$$A(r,s) \;=\; (s+1)\, A(r-1,s) + (r+1)\, A(r,s-1) + [r=s=0]\,, \tag{3.9}$$

for $r,s \geq 0$. In general, $A(r,s) \neq 0$ when $s > r$, as for the ordinary Eulerian numbers; in this case, we use the symmetry relation $A(r,s) = A(s,r)$, which is a consequence of the well-known symmetry relation for the Eulerian numbers:

$$\left\langle{n \atop k}\right\rangle \;=\; \left\langle{n \atop n-k-1}\right\rangle, \qquad \text{for all } n \geq 1 \text{ and } 0 \leq k \leq n. \tag{3.10}$$

Clearly, the above recursion (3.9) does not fit into the general framework of Question 1.1. The generalized numbers $A(r,s|\alpha,\beta)$ introduced by Carlitz and Scoville satisfy the recurrence [6, Eq. (1.9)]:

$$A(r,s|\alpha,\beta) \;=\; (s+\alpha)\, A(r-1,s|\alpha,\beta) + (r+\beta)\, A(r,s-1|\alpha,\beta)\,, \tag{3.11}$$

where $\alpha, \beta$ are two indeterminates, and $A(r,s|\alpha,\beta)$ satisfy the relation $A(r,s|\alpha,\beta) = A(s,r|\beta,\alpha)$. Again, this generalization does not belong to the class of recurrences defined in Question 1.1.

The $(s,t)$-Eulerian numbers $\left\langle{n \atop k}\right\rangle_{(s,t)}$ (3.7) belong to the Eulerian family $(\alpha,\beta;\alpha',\beta') = (0,1;1,-1)$ with $(\gamma,\gamma') = (s,t)$. Or equivalently, $r = 0, r' = -1, \sigma = -1, s = s$, and $s' = t$. Then, their EGF is [cf., Theorem (1.5)]:

$$F(x,y) \;=\; e^{s\,(1-x)\,y} \left( \frac{1-x}{1-x\, e^{(1-x)\,y}} \right)^{s+t}. \tag{3.12}$$

Let us now study the $(s,t)$-*Eulerian polynomials* defined for $n \in \mathbb{Z}_0$ as:

$$A_n^{(s,t)}(x) \;=\; \sum_{k=0}^{n} \left\langle{n \atop k}\right\rangle_{(s,t)} x^k\,. \tag{3.13}$$

These polynomials are given by (3.1):

$$\begin{aligned} A_n^{(s,t)}(x) &= n!\,\frac{(1-x)^{s+t+n}}{x^s}\,\frac{1}{2\pi i}\int_{Z_X(\Gamma)} \frac{z^{s-1}}{(1-z)^{s+t}}\left[\log\frac{z}{x}\right]^{-n-1} dz & (3.14a)\\ &= n!\,\frac{(1-x)^{s+t+n}}{x^s}\,\mathrm{Res}\left(\frac{z^{s-1}}{(1-z)^{s+t}}\left[\log\frac{z}{x}\right]^{-n-1}, z=x\right) & (3.14b)\\ &= \frac{(1-x)^{s+t+n}}{x^s}\,\lim_{z\to x}\frac{\partial^n}{\partial z^n}\left(\frac{z^{s-1}(z-x)^{n+1}}{(1-z)^{s+t}}\left[\log\frac{z}{x}\right]^{-n-1}\right), & (3.14c)\end{aligned}$$

where we have taken into account that $Q^0_{0,-1,-1}(x) = 0$ [cf., (1.14c)], and (3.14a) comes from the proof of Theorem 2.1.

A Rodrigues-like formula for the $(s,t)$-Eulerian polynomials can be obtained from the integral (3.14a) by performing the change of variable $z = e^u$ and writing $x = e^v$. After some algebra, we obtain

$$A_n^{(s,t)}(e^v) = \frac{(1-e^v)^{s+t+n}}{e^{sv}}\,\frac{d^n}{dv^n}\frac{e^{sv}}{(1-e^v)^{s+t}}\,. \qquad (3.15)$$

**Remark.** It is straightforward to find, by using (3.15), a relation between the ordinary Eulerian polynomials $A_n = A_n^{(1,0)}$ and the traditional Eulerian polynomials $A_n^{(0,1)}$:

$$A_n(x) = x^n\,A_n^{(0,1)}(1/x)\,. \qquad (3.16)$$

An equivalent formula for these polynomials can be obtained by using (3.12)/(3.13) and performing the change of variable $y \mapsto u = (1-x)y$:

$$A_n^{(s,t)}(x) = (1-x)^{s+t+n}\,n!\,[u^n]\frac{e^{s\,u}}{(1-x\,e^u)^{s+t}}\,. \qquad (3.17)$$

This formula allows us to obtain a closed expression for $A_n^{(s,t)}$ by expanding the denominator in (3.17):

$$\begin{aligned} A_n^{(s,t)}(x) &= (1-x)^{s+t+n}\,n!\,[u^n]\sum_{j\ge 0}\frac{(s+t)^{\overline{j}}\,x^j}{j!}\,e^{u\,(s+j)}\\ &= (1-x)^{s+t+n}\sum_{j\ge 0}\frac{(s+t)^{\overline{j}}}{j!}\,(s+j)^n\,x^j\,. \qquad (3.18)\end{aligned}$$

Notice that we have used $(-1)^j(s+t)^{\overline{j}}/j!$ instead of $\binom{-s-t}{j}$ because $s,t$ are indeterminates, so that $s+t$ can take a non-integer value. From (3.18), we easily obtain

**Proposition 3.1** *The $(s,t)$-Eulerian polynomials (3.13) satisfy for any $n \ge 0$ and arbitrary parameters $s,t$ the relation*

$$\frac{x\,A_n^{(s,t)}(x)}{(1-x)^{n+s+t}} = \sum_{k\ge 1}\frac{(s+t)^{\overline{k-1}}}{(k-1)!}\,(k+s-1)^n\,x^k\,. \qquad (3.19)$$

This proposition generalizes the well-known formula for the ordinary Eulerian polynomials:

$$\frac{x\, A_n(x)}{(1-x)^{n+1}} \;=\; \sum_{k\geq 1} k^n\, x^k\,. \tag{3.20}$$

A closed expression for $\left\langle {n \atop k} \right\rangle_{(s,t)}$ can be obtained from (3.18) by expanding the binomial $(1-x)^{n+s+t}$, and changing variables in the sums:

$$\begin{aligned} A_n^{(s,t)}(x) &= \sum_{i,j\geq 0} (-1)^i\, \frac{(n+s+t)^{\underline{i}}}{i!}\, \frac{(s+t)^{\overline{j}}}{j!}\, (s+j)^n\, x^{i+j} \\ &= \sum_{k\geq 0} x^k \sum_{j=0}^{k} (-1)^{k-j}\, \frac{(n+s+t)^{\underline{k-j}}}{j!\,(k-j)!}\, (s+t)^{\overline{j}}\, (s+j)^n\,. \end{aligned} \tag{3.21}$$

Then, we conclude that

**Theorem 3.2** *The $(s,t)$-Eulerian numbers that satisfy the recursion (3.7) are equal to*

$$\left\langle {n \atop k} \right\rangle_{(s,t)} \;=\; \frac{1}{k!} \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j}\, (n+s+t)^{\underline{k-j}}\, (s+t)^{\overline{j}}\, (s+j)^n \tag{3.22}$$

*for $n \geq 0$ and $0 \leq k \leq n$.*

The following corollary recollects some particular cases of the above theorem. (The proofs are trivial.)

**Corollary 3.3** *Let $n \geq 0$, and $0 \leq k \leq n$. Then:*

- *The $(s,-s)$-Eulerian numbers take the form*

$$\left\langle {n \atop k} \right\rangle_{(s,-s)} \;=\; (-1)^k \binom{n}{k}\, s^n\,. \tag{3.23}$$

- *The shifted $r$-Eulerian numbers $(s,t) = (r,0)$ are given by*

$$\left\langle {n \atop k} \right\rangle_{(r)} \;=\; \frac{1}{k!} \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j}\, (n+r)^{\underline{k-j}}\, r^{\overline{j}}\, (r+j)^n\,. \tag{3.24}$$

- *The ordinary Eulerian numbers $(s,t) = (1,0)$ are given by [15, Eq. (6.38)]:*

$$\left\langle {n \atop k} \right\rangle \;=\; \sum_{j=0}^{k} (-1)^j \binom{n+1}{j}\, (k-j+1)^n\,. \tag{3.25}$$

- *The ordinary Eulerian numbers with the traditional ordering $A(n,k)$ [7] are given by $(s,t)=(0,1)$:*

$$A(n,k) \;=\; \sum_{j=0}^{k} (-1)^j \binom{n+1}{j} (k-j)^n \;=\; \left\langle {n \atop k-1} \right\rangle . \tag{3.26}$$

**Remarks.** 1. Harris and Park [17, Eq. (10)] generalized the Eulerian numbers by modifying (3.25) in the following way:

$$A_{n,k}(\delta) \;=\; \sum_{j=0}^{k} (-1)^j \binom{n+1}{j} (k-j+\delta)^n \,, \tag{3.27}$$

for $n \in \mathbb{Z}_0$ and $0 \le k \le n$. The sequence for $\delta = 0$ corresponds corresponds to the ordinary Eulerian numbers with the traditional indexing (3.26) [23, entry A008292]; for $\delta = 1$ corresponds to our definition of the Eulerian numbers [15], and for $\delta = 2$, corresponds to the entry A180246 in [23].

2. If $(s,t) = (0,-1)$ we obtain also a simple formula for the corresponding $(s,t)$-Eulerian numbers:

$$\left\langle {n \atop k} \right\rangle_{(0,-1)} \;=\; (-1)^k \binom{n-1}{k-1} + [n=k=0] \,. \tag{3.28}$$

To finish this section, we want to obtain useful formulas for the $(s,t)$-Eulerian polynomials when $s+t$ is a *non-negative integer* $s+t \in \mathbb{Z}_0$. As the case $s+t=0$ is already given by (3.23), we assume in the rest of this section that $s+t \in \mathbb{N}$. Let us now go back to Eq. (3.18), and rewrite it as:

$$A_n^{(s,t)}(x) \;=\; (1-x)^{s+t+n} \sum_{k\ge 0} (-1)^k \binom{-s-t}{k} (s+k)^n \, x^k \,. \tag{3.29}$$

It is possible to express (3.29) in terms of the Lerch transcendent function defined by [14, Eq. 9.550, p. 1039][5]

$$\Phi(z,v,w) \;=\; \sum_{k=0}^{\infty} \frac{z^k}{(k+w)^v} \,, \qquad |z|<1 \,, w \ne 0,-1,-2,\ldots \tag{3.30}$$

In our case, the index $v = -n \ge 0$, so we do not need to impose the restriction $w \ne 0,-1,-2,\ldots$. Therefore, we will use the Lerch function (3.30) evaluated at $w=-n$:

$$\Phi(x,-n,s) \;=\; \sum_{k=0}^{\infty} (k+s)^n \, x^k \;=\; \left( s + x \frac{\partial}{\partial x} \right)^n \frac{1}{1-x} \,, \quad |x| \;<\; 1 \,, \tag{3.31}$$

[5] Notice that our arguments for the Lerch function are $(z,v,w)$, while those in the definition of [14] are $(z,s,v)$. Therefore, the variable $v$ plays a different role in each case.

which is now valid for any value of $s$. Notice that $\Phi(x, -n, s)$ is a rational function of $x$ with a pole of degree $n + 1$ at $x = 1$.

**Remark.** MATHEMATICA provides two different Lerch functions [28]: `HurwitzLerchPhi` coincides with the standard definition (3.30); but `LerchPhi` gives a slightly different definition.

Let us now consider the function

$$f(x) \;=\; \sum_{k=0}^{\infty} (-1)^k \binom{-t-s}{k} (k+s)^n \, x^k \,, \quad |x| \;<\; 1 \,, \tag{3.32}$$

so that $A_n^{(s,t)}(x) \;=\; (1-x)^{s+t+n} f(x)$. The power series that defines $f(x)$ is the Hadamard product of the power series for the functions

$$f_1(x) \;=\; \frac{1}{(1-x)^{s+t}} \,, \tag{3.33a}$$

$$f_2(x) \;=\; \Phi(x, -n, s) \,, \tag{3.33b}$$

defined for $|x| < 1$. Hence, for $|x| < 1$ we can use the multiplication formula for $z$-transforms [16], [7, p. 85] to get

$$f(x) \;=\; \frac{1}{2\pi i} \int_\Gamma f_2(z)\, f_1\!\left(\frac{x}{z}\right) \frac{dz}{z} \;=\; \frac{1}{2\pi i} \int_\Gamma \frac{z^{t+s-1}\, \Phi(z, -n, s)}{(z-x)^{s+t}}\, dz \tag{3.34}$$

$$= \; \operatorname{Res}\left( \frac{z^{t+s-1}\, \Phi(z, -n, s)}{(z-x)^{s+t}} \,; z = x \right) , \tag{3.35}$$

where the contour $\Gamma$ is any simple closed curve [of index $+1$] contained in the region $|x| < |z| < 1$ of the complex plane (notice that we are taking $s + t - 1 \in \mathbb{Z}_0$). We then get

$$A_n^{(s,t)}(x) \;=\; (1-x)^{n+s+t} \operatorname{Res}\left( \frac{z^{t+s-1}\, \Phi(z, -n, s)}{(z-x)^{s+t}} \,; z = x \right) \tag{3.36a}$$

$$= \; \frac{(1-x)^{n+s+t}}{(s+t-1)!} \frac{\partial^{s+t-1}}{\partial x^{s+t-1}} \left( x^{s+t-1} \Phi(x, -n, s) \right) \tag{3.36b}$$

$$= \; \frac{(1-x)^{n+s+t}}{(s+t-1)!} \frac{\partial^{s+t-1}}{\partial x^{s+t-1}} \left( x^{s+t-1} \left( s + x \frac{\partial}{\partial x} \right)^n \frac{1}{1-x} \right) . \tag{3.36c}$$

The ordinary Eulerian polynomials [15] can be obtained by setting $(s, t) = (1, 0)$. In this case we recover from (3.36) the well-known result

$$A_n(x) \;=\; (1-x)^{n+1}\, \Phi(x, -n, 1) \;=\; \frac{(1-x)^{n+1}}{x} \operatorname{Li}_{-n}(x) \tag{3.37a}$$

$$= \; (1-x)^{n+1} \left( 1 + x \frac{\partial}{\partial x} \right)^n \frac{1}{1-x} \,, \tag{3.37b}$$

where, for any $s \in \mathbb{C}$, $\mathrm{Li}_s$ denotes the standard polylogarithm [20]:

$$\mathrm{Li}_s(z) \;=\; \sum_{k\ge 1} \frac{z^k}{k^s}\,, \qquad |z| < 1\,. \tag{3.38}$$

Notice that we are interested in those polylogarithms with non-positive integer values of $s = -n \le 0$ (in the same way as for the Lerch function):

$$\mathrm{Li}_{-n}(z) \;=\; \sum_{k\ge 1} z^k\, k^n\,, \qquad |z| < 1\,, \tag{3.39}$$

so that $\mathrm{Li}_{-n}(z)$ is a rational function in $z$ with a pole of order $n+1$ at $z=1$. In addition, for $(s,t) = (0,1)$ we obtain the traditional Eulerian polynomials:

$$A_n^{(0,1)}(x) \;=\; (1-x)^{n+1}\,\Phi(x,-n,0) \;=\; (1-x)^{n+1}\,\mathrm{Li}_{-n}(x) \tag{3.40a}$$

$$\;=\; (1-x)^{n+1} \left( x \frac{\partial}{\partial x}\right)^n \frac{1}{1-x}\,. \tag{3.40b}$$

## 3.2 The second-order Eulerian family

The second-order Eulerian numbers $\left\langle\!\left\langle {n \atop k} \right\rangle\!\right\rangle$ [23, entry A008517], [13], [15, Section 6.2] are defined as follows:[6] for $n \in \mathbb{N}$, $\left\langle\!\left\langle {n \atop k} \right\rangle\!\right\rangle$ is the number of permutations of the multiset $\{1,1,2,2,\ldots,n,n\}$ that have $k$ ascents and in which, for all $m \in [n]$, all numbers between two occurrences of $m$ are greater than $m$. (Indeed, the two occurrences of $n$ should go together.) For $n=0$, we define $\left\langle\!\left\langle {0 \atop k} \right\rangle\!\right\rangle = \delta_{k,0}$. Then, the numbers $\left\langle\!\left\langle {n \atop k} \right\rangle\!\right\rangle$ are non-vanishing only for $0 \le k \le n-1$. They satisfy the recursion (1.1) with $(\alpha,\beta;\alpha',\beta') = (0,1;2,-1)$ and $(\gamma,\gamma') = (1,-1)$.

We now introduce a class of "generalized second-order Eulerian numbers" that generalizes the ordinary second-order Eulerian numbers in the same way as the $(s,t)$-Eulerian numbers $\left\langle {n \atop k} \right\rangle_{(s,t)}$ (3.7) generalize the ordinary Eulerian numbers $\left\langle {n \atop k} \right\rangle = \left\langle {n \atop k} \right\rangle_{(1,0)}$. Let us consider a pair of indeterminates $s$ and $t$, and define the *second-order* $(s,t)$-*Eulerian numbers* by the recursion

$$\left\langle\!\!\left\langle {n \atop k} \right\rangle\!\!\right\rangle_{(s,t)} \;=\; (k+s) \left\langle\!\!\left\langle {n-1 \atop k} \right\rangle\!\!\right\rangle_{(s,t)} + (2n-k+t-1) \left\langle\!\!\left\langle {n-1 \atop k} \right\rangle\!\!\right\rangle_{(s,t)} + [n=k=0]\,. \tag{3.41}$$

Then, $\left\langle\!\left\langle {n \atop k} \right\rangle\!\right\rangle_{(s,t)}$ is a polynomial of degree $n$ in $s,t$ with leading term $\binom{n}{k} s^{n-k} t^k$. It reduces to $\left\langle\!\left\langle {n \atop k} \right\rangle\!\right\rangle$ when $(s,t) = (1,0)$. Notice that the choice $(s,t) = (0,1)$ leads to the second-order Eulerian numbers with the traditional indexing [23, entry A008517], [13].

The second-order $(s,t)$-Eulerian numbers $\left\langle\!\left\langle {n \atop k} \right\rangle\!\right\rangle_{(s,t)}$ (3.41) belong to the second-order Eulerian family $(0,1;2,-1)$ with $(\gamma,\gamma') = (s,t-1)$. Or equivalently, $r=0, r' =$

[6] We remind the reader again that we use the indexing and notation of GKP [15]. Both of them differ from the traditional indexing [23, entry A008517], [13], and the notation $B_{n,k} = \left\langle\!\left\langle {n \atop k-1} \right\rangle\!\right\rangle$ used in [13].

$-2, \sigma = -1, s = s$, and $s' = t$. Then, $Q_{0,-2,-1}(x) = \log x - x$ [cf., (1.14c)], and their EGF is [cf., Corollary 4.5 of Paper I and Theorem (1.5)]:

$$F(x,y) \;=\; \left( \frac{T\left(e^{y(1-x)^2}\, T^{-1}(x)\right)}{x} \right)^{s} \left( \frac{1-x}{1 - T\left(e^{y(1-x)^2}\, T^{-1}(x)\right)} \right)^{s+t}, \qquad (3.42)$$

where $T$ is the tree function, defined as follows [8, 9]:

$$T(z)\, e^{-T(z)} \;=\; z\,. \qquad (3.43)$$

Equivalently, this function satisfies $T^{-1}(z) = ze^{-z}$, and it is closely related to the Lambert $W$ function [8, 9]: $T(z) = -W(-z)$. If $x$ is real, then for $0 < x \le 1/e$ there are two possible real values of $T(x)$. We will denote the branch satisfying $T(x) \le 1$ the principal branch of the tree function, and, as we will only use this branch throughout this paper, we will denote it simply as $T(x)$. (In the literature, this principal branch is denoted $T_0(x)$, while the other branch satisfying $T(x) \ge 1$ is denoted as $T_{-1}(x)$. When $z$ is a complex variable, then $T(z)$ has infinitely many branches [8].) The power series defining the (principal branch of) the tree function is given by [8, 9]:

$$T(z) \;=\; \sum_{n=1}^{\infty} \frac{n^{n-1}}{n!}\, z^n\,. \qquad (3.44)$$

This power series converges for $|z| < 1/e$, as it has a branch cut on the line $[1/e, \infty)$ [8, 9].

Let us now study the *second-order* $(s,t)$*-Eulerian polynomials* $B_n^{(s,t)}$ defined for $n \in \mathbb{Z}_0$ as:

$$B_n^{(s,t)}(x) \;=\; \sum_{k=0}^{n} \left\langle\!\!\left\langle {n \atop k} \right\rangle\!\!\right\rangle_{(s,t)} x^k\,. \qquad (3.45)$$

These polynomials are given by (3.1):

$$B_n^{(s,t)}(x) \;=\; n!\, \frac{(1-x)^{s+t+2n}}{x^s} \frac{1}{2\pi i} \int_{Z_X(\Gamma)} \frac{z^{s-1}}{(1-z)^{s+t-1}} \left[ \log \frac{ze^{-z}}{xe^{-x}} \right]^{-n-1} dz \qquad (3.46a)$$

$$\;=\; n!\, \frac{(1-x)^{s+t+2n}}{x^s} \operatorname{Res}\left( \frac{z^{s-1}}{(1-z)^{s+t-1}} \left[ \log \frac{ze^{-z}}{xe^{-x}} \right]^{-n-1}, z = x \right) \qquad (3.46b)$$

$$\;=\; \frac{(1-x)^{s+t+2n}}{x^s} \lim_{z\to x} \frac{\partial^n}{\partial z^n} \left( \frac{z^{s-1}(z-x)^{n+1}}{(1-z)^{s+t-1}} \left[ \log \frac{z\, e^{-z}}{x\, e^{-x}} \right]^{-n-1} \right). \qquad (3.46c)$$

A Rodrigues-like formula for the second-order $(s,t)$-Eulerian polynomials can also be obtained from the integral (3.46a) by performing the change of variables $ze^{-z} = e^u$ and $xe^{-x} = e^v$. Therefore, $z = T(e^u)$ and $x = T(e^v)$, where $T$ is the tree function (3.44). As the derivative of the tree function is [8, 9]:

$$T'(x) \;=\; \frac{T(x)}{x\,(1 - T(x))}\,, \qquad (3.47)$$

we obtain, after some algebra, the final result:

$$B_n^{(s,t)}(T(e^v)) \;=\; \frac{(1-T(e^v))^{s+t+2n}}{T(e^v)^s} \frac{d^n}{dv^n} \frac{T(e^v)^s}{(1-T(e^v))^{s+t}} \,. \qquad (3.48)$$

An equivalent expression for the polynomials $B_n^{(s,t)}$ can be obtained from (3.42)/(3.45) by performing the change of variable $y \mapsto u = (1-x)^2 y$:

$$B_n^{(s,t)}(x) \;=\; n!\, \frac{(1-x)^{s+t+2n}}{x^s} [u^n] \frac{\left(T\left(T^{-1}(x)\, e^u\right)\right)^s}{\left(1-T\left(T^{-1}(x)\, e^u\right)\right)^{s+t}} \,. \qquad (3.49)$$

Using this formula is not very difficult to obtain explicit closed form for both the second-order $(s,t)$-Eulerian polynomials and the second-order $(s,t)$-Eulerian numbers. We start by expanding $(1-T(\xi))^{-(s+t)}$ in powers of $T(\xi)$, where $\xi = T^{-1}(x)\, e^u = x\, e^{-x+u}$:

$$B_n^{(s,t)}(x) \;=\; n!\, \frac{(1-x)^{s+t+2n}}{x^s} \sum_{j=0}^{\infty} \frac{(s+t)^{\overline{j}}}{j!} [u^n] T(\xi)^{s+j} \,. \qquad (3.50)$$

One important property of the tree function (3.44) is that its powers can be computed in closed form [9, Eq. (10)]:

$$T(z)^s \;=\; \sum_{k=0}^{\infty} \frac{s\,(k+s)^{k-1}}{k!}\, z^{s+k} \,. \qquad (3.51)$$

Inserting this expression in (3.50), we obtain after an additional change of variables:

$$B_n^{(s,t)}(x) \;=\; n!\, \frac{(1-x)^{s+t+2n}}{x^s} \sum_{p=0}^{\infty} \sum_{j=0}^{p} \frac{(s+t)^{\overline{j}}\,(s+j)\,(p+s)^{p-j-1}}{j!\,(p-j)!} [u^n] \xi^{s+p} \,, \qquad (3.52)$$

where $\xi = xe^{-x+u}$. After some more straightforward algebra we find that

$$\begin{aligned} B_n^{(s,t)}(x) \;&=\; (1-x)^{s+t+2n} \sum_{p=0}^{\infty} \frac{x^p}{p!} e^{-x(p+s)} \\ &\quad \times \sum_{j=0}^{p} \binom{p}{j} (s+t)^{\overline{j}}\,(s+j)\,(p+s)^{n+p-j-1} \,. \end{aligned} \qquad (3.53)$$

From this equation we easily obtain the following proposition (which resembles Proposition 3.1 for the $(s,t)$-Eulerian polynomials $A_n^{(s,t)}$):

**Proposition 3.4** *The second-order $(s,t)$-Eulerian polynomials (3.45) satisfy for any $n \ge 0$ and arbitrary parameters $s, t$ the relation*

$$\begin{aligned} \frac{x\, B_n^{(s,t)}(x)}{(1-x)^{2n+s+t}} \;&=\; \sum_{k\ge 1} \frac{x^k}{(k-1)!}\, e^{-x(k+s-1)} \\ &\quad \times \sum_{j=0}^{k-1} \binom{k-1}{j} (s+t)^{\overline{j}}\,(s+j)\,(k+s-1)^{n+k-j-2} \,. \end{aligned} \qquad (3.54)$$

When $(s,t)=(1,0)$, we obtain the following relation for the ordinary second-order Eulerian polynomials $B_n(x) = B_n^{(1,0)}(x)$:

$$\frac{x\, B_n(x)}{(1-x)^{2n+1}} \;=\; \sum_{k\geq 1} \frac{k^{n+k-1}}{(k-1)!} \left(x e^{-x}\right)^k . \tag{3.55}$$

This equation resembles Eq. (3.20) for the ordinary Eulerian polynomials $A_n(x)$ (3.13). The proof of this results makes use of the following

**Lemma 3.5** *Let $n$ be an arbitrary positive integer. Then:*

$$1 \;=\; \sum_{j=0}^{n} \binom{n}{j} j!\, j\, \frac{1}{n^{j+1}} , \tag{3.56a}$$

$$1 \;=\; \sum_{j=0}^{n} \binom{n}{j} (j+1)!\, \frac{1}{(n+1)^{j+1}} . \tag{3.56b}$$

**Remark.** The proof of this lemma is postponed to Appendix A. Notice also that Eq. (3.56b) is obviously true also for $n=0$.

If we now expand the exponential $e^{-x(p+s)}$ in powers of $x$ in (3.53), and perform an additional change of variables in the sums, we arrive at:

$$\begin{aligned} B_n^{(s,t)}(x) \;&=\; (1-x)^{s+t+2n} \sum_{r=0}^{\infty} \frac{x^r}{r!} \sum_{p=0}^{r} \binom{r}{p} (-1)^{r-p} \\ &\qquad \times \sum_{j=0}^{p} \binom{p}{j} (s+t)^{\overline{j}}\, (s+j)\, (p+s)^{n+r-j-1} . \end{aligned} \tag{3.57}$$

Finally, we have to expand the term $(1-x)^{s+t+sn}$ in powers of $x$. After some algebra and a final change of variables we arrive at

$$\begin{aligned} B_n^{(s,t)}(x) \;&=\; \sum_{k\geq 0} \frac{x^k}{k!} \sum_{r=0}^{k} \binom{k}{r} (s+t+2n)^{\underline{k-r}} \sum_{p=0}^{r} \binom{r}{p} (-1)^{k-p} \\ &\qquad \times \sum_{j=0}^{p} \binom{p}{j} (s+t)^{\overline{j}}\, (s+j)\, (p+s)^{n+r-j-1} . \end{aligned} \tag{3.58}$$

We then conclude

**Theorem 3.6** *The second-order $(s,t)$-Eulerian numbers that satisfy the recursion (3.41) are equal to*

$$\begin{aligned} \left\langle\!\!\left\langle {n \atop k} \right\rangle\!\!\right\rangle_{(s,t)} \;&=\; \frac{1}{k!} \sum_{r=0}^{k} \binom{k}{r} (s+t+2n)^{\underline{k-r}} \sum_{p=0}^{r} \binom{r}{p} (-1)^{k-p} \\ &\qquad \times \sum_{j=0}^{p} \binom{p}{j} (s+t)^{\overline{j}}\, (s+j)\, (p+s)^{n+r-j-1} \end{aligned} \tag{3.59}$$

*for $n \geq 0$ and $0 \leq k \leq n$.*

The following corollary recollects some particular cases of the above theorem; its proof can be found in Appendix A.

**Corollary 3.7** *Let $n \geq 0$, and $0 \leq k \leq n$. Then:*

*(a) The second-order $(s,-s)$-Eulerian numbers take the form*

$$\left\langle\!\!\left\langle {n \atop k} \right\rangle\!\!\right\rangle_{(s,-s)} = s \sum_{r=0}^{k} \frac{1}{r!} \binom{2n}{k-r} \sum_{p=0}^{r} \binom{r}{p} (-1)^{k-p} (p+s)^{n+r-1} . \tag{3.60}$$

*(b) The ordinary second-order Eulerian numbers $(s,t) = (1,0)$ are given by*

$$\left\langle\!\!\left\langle {n \atop k} \right\rangle\!\!\right\rangle = \sum_{r=0}^{k} (-1)^{k-r} \binom{1+2n}{k-r} \left\{ {n+r+1 \atop r+1} \right\} , \tag{3.61}$$

*where the numbers $\left\{ {n \atop k} \right\}$ are the standard Stirling subset numbers [15].*

*(c) The ordinary second-order Eulerian numbers with the traditional ordering $B_{n,k}$ are given by $(s,t) = (0,1)$:*

$$B_{n,k} = \left\langle\!\!\left\langle {n \atop k} \right\rangle\!\!\right\rangle_{(0,1)} = \sum_{r=0}^{k} (-1)^{k-r} \binom{1+2n}{k-r} \left\{ {n+r \atop r} \right\} . \tag{3.62}$$

**Remarks.** 1. In [23, entry A008517] a closed formula is given for the $B_{n,k}$ [due to Johannes W. Meijer (2009) as cited in that reference]:

$$B_{n,k} = \left\langle\!\!\left\langle {n \atop k} \right\rangle\!\!\right\rangle_{(0,1)} = \sum_{j=0}^{n-k} (-1)^{j} \binom{1+2n}{j} \left[ {2n-k-j+1 \atop n-k-j+1} \right] , \tag{3.63}$$

where the numbers $\left[ {n \atop k} \right]$ are the Stirling cycle numbers [15]. (See also Paper I and references therein.)

2. In the same reference [23, entry A008517], another closed formula is given for $B_{n,k}$ in terms of the Ward numbers $t_1(n,k) = \left\{\!\!\left\{ {n+k \atop k} \right\}\!\!\right\}$, where the $\left\{\!\!\left\{ {n \atop k} \right\}\!\!\right\}$ are the associated Stirling subset numbers [11], [23, entry A134991] (see also Paper I and references therein):

$$B_{n,k} = \left\langle\!\!\left\langle {n \atop k} \right\rangle\!\!\right\rangle_{(0,1)} = \sum_{j=0}^{n} (-1)^{k-j} \binom{n-j}{k-j} t_1(n,j) \tag{3.64a}$$

$$= \sum_{j=0}^{n} (-1)^{k-j} \binom{n-j}{k-j} \left\{\!\!\left\{ {n+j \atop j} \right\}\!\!\right\} . \tag{3.64b}$$

This formula is due to Tom Copeland (2011) as cited in [23, entry A008517].

In the same vein as for the $(s,t)$-Eulerian polynomials $A_n^{(s,t)}$ [cf., (3.36)], we may try to use the Hadamard product in (3.53) to obtain useful expressions for the second-order $(s,t)$-Eulerian polynomials $B_n^{(s,t)}$ when $s+t \in \mathbb{Z}_0$. However, this procedure is in general rather involved for $s+t \in \mathbb{N}$ owing to the appearance of complicated double series. Therefore, we have preferred to focus on a couple of "easy" (but still interesting) cases: $(s,-s)$ and $(1,0)$.

Let us start with the first case $t=-s$. Then, the second sum in (3.53) only has one contribution from the value $j=0$ (due to the term $0^{\overline{j}}$). Therefore, we can express the corresponding polynomial as

$$B_n^{(s,-s)}(x) = \frac{(1-x)^{2n}}{x^s} \left(T^{-1}(x)\right)^s \sum_{k\geq 0} \frac{s\,(s+k)^{n+k-1}}{n!} \left(T^{-1}(x)\right)^k \quad (3.65a)$$

$$= \frac{(1-x)^{2n}}{x^s} \left(T^{-1}(x)\right)^s f\left(T^{-1}(x)\right), \quad (3.65b)$$

where $T^{-1}(x) = xe^{-x}$, and the function $f$ is defined as:

$$f(x) = \sum_{k\geq 0} \frac{s\,(s+k)^{n+k-1}}{n!} x^k. \quad (3.66)$$

We now define the auxiliary functions

$$f_1(x) = \sum_{k\geq 0} \frac{s\,(k+s)^{k-1}}{n!} x^k = \left(\frac{T(x)}{x}\right)^s, \quad (3.67a)$$

$$f_2(x) = \sum_{k\geq 0} (k+s)^n x^k = \Phi(x,-n,s), \quad (3.67b)$$

where we have used the definition (3.31) for the Lerch function with negative second entry $\Phi(x,-n,s)$, and the expression (3.51) for the $s$-th power of the tree function $T(x)^s$. Then, we have that $f$ is given by the Hadamard product

$$f(x) = \frac{1}{2\pi i} \int_\Gamma \Phi\left(\frac{x}{z}, -n, s\right) \frac{T(z)^s}{z^{s+1}}\, dz \quad (3.68)$$

and therefore,

$$B_n^{(s,-s)}(x) = \frac{(1-x)^{2n}}{x^s} \left(T^{-1}(x)\right)^s \frac{1}{2\pi i} \int_\Gamma \Phi\left(\frac{T^{-1}(x)}{z}, -n, s\right) \frac{T(z)^s}{z^{s+1}}\, dz \quad (3.69a)$$

$$= \frac{(1-x)^{2n}}{x^s} \left(T^{-1}(x)\right)^s$$

$$\times \operatorname{Res}\left(\frac{T(z)^s}{z^{s+1}} \Phi\left(\frac{T^{-1}(x)}{z}, -n, s\right), z = T^{-1}(x)\right), \quad (3.69b)$$

where the contour $\Gamma$ is contained in the region $|T^{-1}(x)| < |z| < 1/e$ of the complex $z$-plane.

The second case corresponds to $(s,t) = (1,0)$, i.e., the ordinary second-order Eulerian polynomials $B_n$. Then, the second sum in (3.53) reduces to the form

$$\sum_{j=0}^{p} \binom{p}{j} (j+1)!\,(p+1)^{n+p-j-1}\,, \tag{3.70}$$

which by using Lemma 3.5 [cf., Eq. (3.56b)] reduces to $(p+1)^{n+p}$. Therefore, we obtain

$$\begin{aligned} B_n(x) &= \frac{(1-x)^{1+2n}}{x} \sum_{k\geq 0} \frac{(k+1)^{n+k}}{k!} \left(T^{-1}(x)\right)^{k+1} && \text{(3.71a)}\\ &= \frac{(1-x)^{1+2n}}{x} \sum_{j\geq 1} \frac{j^{n+j}}{j!} \left(T^{-1}(x)\right)^{j} && \text{(3.71b)}\\ &= \frac{(1-x)^{1+2n}}{x} f\left(T^{-1}(x)\right)\,, && \text{(3.71c)} \end{aligned}$$

where we have made the change of variables $k \mapsto j = k+1$. The function $f$ is now defined as

$$f(x) = \sum_{k\geq 1} \frac{k^{n+k}}{k!}\, x^k\,. \tag{3.72}$$

We introduce the auxiliary functions

$$\begin{aligned} f_1(x) &= \sum_{k\geq 1} k^{n+1}\, x^k = \mathrm{Li}_{-(n+1)}(x)\,, && \text{(3.73a)}\\ f_2(x) &= \sum_{k\geq 1} \frac{k^{k-1}}{k!}\, x^k = T(x)\,, && \text{(3.73b)} \end{aligned}$$

where $\mathrm{Li}_n$ is the standard polylogarithm (3.39). Then, $f$ is given by the Hadamard product

$$f(x) = \frac{1}{2\pi i} \int_\Gamma \mathrm{Li}_{-(n+1)}\left(\frac{T^{-1}(x)}{z}\right) \frac{T(z)}{z}\, dz \tag{3.74}$$

and therefore,

$$\begin{aligned} B_n(x) &= \frac{(1-x)^{1+2n}}{x} \frac{1}{2\pi i} \int_\Gamma \mathrm{Li}_{-(n+1)}\left(\frac{T^{-1}(x)}{z}\right) \frac{T(z)}{z}\, dz && \text{(3.75a)}\\ &= \frac{(1-x)^{1+2n}}{x} \operatorname{Res}\left[\mathrm{Li}_{-(n+1)}\left(\frac{T^{-1}(x)}{z}\right) \frac{T(z)}{z}\,, z = T^{-1}(x)\right]\,, && \text{(3.75b)} \end{aligned}$$

where the contour $\Gamma$ is contained in the region $|T^{-1}(x)| < |z| < 1/e$ of the complex $z$-plane.

## 3.3 The $\nu$-order Eulerian family

In Paper I we introduced another family of Eulerian numbers: the $\nu$-order Eulerian numbers corresponding to the family $(0,1;\nu,-1)$ with $(\gamma,\gamma')=(1,1-\nu)$ with $\nu\in\mathbb{N}$. We shall denote them as $\left\langle {n \atop k} \right\rangle^{(\nu)}$, and they satisfy the recurrence:

$$\left\langle {n \atop k} \right\rangle^{(\nu)} = (k+1)\left\langle {n-1 \atop k} \right\rangle^{(\nu)} + (\nu n - k + 1 - \nu)\left\langle {n-1 \atop k-1} \right\rangle^{(\nu)} + [n=k=0]\,. \quad (3.76)$$

This notation is motivated by the standard notation for derivatives, and the fact that the ordinary Eulerian numbers $\left\langle {n \atop k} \right\rangle$ corresponds to $\nu=1$ and the ordinary second-order Eulerian numbers $\left\langle\!\left\langle {n \atop k} \right\rangle\!\right\rangle$ to $\nu=2$. The third-order Eulerian numbers are defined in [23, entry A219512], and correspond to $\nu=3$. The EGF for the $\nu$-order Eulerian numbers is given by (see Section 5.1.6 of Paper I):

$$F(x,y) = \frac{1-x}{x}\,\frac{T_\nu\left(e^{y(1-x)^\nu}\,T_\nu^{-1}(x)\right)}{1-T_\nu\left(e^{y(1-x)^\nu}T_\nu^{-1}(x)\right)}\,, \quad (3.77)$$

where $T_\nu$ is a function given by

$$T_\nu^{-1}(z) = z\,e^{Q_\nu(z)}\,, \quad (3.78a)$$

$$Q_\nu(z) = \sum_{k=1}^{\nu-1} \binom{\nu-1}{k}\frac{(-z)^k}{k}\,. \quad (3.78b)$$

Indeed, when $\nu=1$, $T_\nu$ is the identity function; and when $\nu=2$, $T_\nu$ is the tree function (3.44).

As in the preceding sections, we can generalize these $\nu$-order Eulerian numbers, and define the *$\nu$-order $(s,t)$-Eulerian numbers* $\left\langle {n \atop k} \right\rangle^{(\nu)}_{(s,t)}$ as the solutions to the recurrence:

$$\left\langle {n \atop k} \right\rangle^{(\nu)}_{(s,t)} = (k+s)\left\langle {n-1 \atop k} \right\rangle^{(\nu)}_{(s,t)} + (\nu n - k + t + 1 - \nu)\left\langle {n-1 \atop k-1} \right\rangle^{(\nu)}_{(s,t)} + [n=k=0]\,, \quad (3.79)$$

where $(s,t)$ are indeterminates. These numbers belong to the Type–I family $(0,1;\nu,-1)$ with $(\gamma,\gamma')=(s,t+1-\nu)$; or equivalently to $r=0, r'=-\nu, \sigma=-1, s=s$, and $s'=t$. Indeed, the ordinary $\nu$-order Eulerian numbers (3.76) correspond to the choice $(s,t)=(1,0)$. The EGF function for the $\nu$-order $(s,t)$-Eulerian numbers is given by Theorem 1.5:

$$F(x,y) = \left(\frac{T_\nu\left(e^{y(1-x)^\nu}\,T_\nu^{-1}(x)\right)}{x}\right)^{s}\left(\frac{1-x}{1-T_\nu\left(e^{y(1-x)^\nu}T_\nu^{-1}(x)\right)}\right)^{s+t}\,. \quad (3.80)$$

Let us now study the *$\nu$-order $(s,t)$-Eulerian polynomials* defined for $n\in\mathbb{Z}_0$ as:

$$P_n(x) = \sum_{k=0}^{n}\left\langle {n \atop k} \right\rangle^{(\nu)}_{(s,t)} x^k\,. \quad (3.81)$$

We simply use $P_n$ to lighten the notation for this section; indeed, these polynomials also depend on the indices $s, t, \nu$. The polynomials $P_n$ are given by (3.1):

$$P_n(x) = n!\,\frac{(1-x)^{s+t+\nu n}}{x^s}\,\frac{1}{2\pi i}\int_{Z_X(\Gamma)} \frac{z^{s-1}}{(1-z)^{s+t+1-\nu}}\left[\log\frac{z e^{Q_\nu(z)}}{x e^{Q_\nu(x)}}\right]^{-(n+1)} dz \quad (3.82a)$$

$$= n!\,\frac{(1-x)^{s+t+\nu n}}{x^s}\,\mathrm{Res}\left(\frac{z^{s-1}}{(1-z)^{s+t+1-\nu}}\left[\log\frac{z e^{Q_\nu(z)}}{x e^{Q_\nu(x)}}\right]^{-(n+1)}, z = x\right) \quad (3.82b)$$

$$= \frac{(1-x)^{s+t+\nu n}}{x^s}\,\lim_{z\to x}\frac{\partial^n}{\partial z^n}\left(\frac{z^{s-1}(z-x)^{n+1}}{(1-z)^{s+t+1-\nu}}\left[\log\frac{z e^{Q_\nu(z)}}{x e^{Q_\nu(x)}}\right]^{-(n+1)}\right). \quad (3.82c)$$

A Rodrigues-like formula for the $\nu$-order $(s,t)$-Eulerian polynomials can also be obtained from the integral (3.82a) by performing the change of variables $z e^{Q_\nu(z)} = e^u$ and $x e^{Q_\nu(x)} = e^v$. Therefore, $z = T_\nu(e^u)$ and $x = T_\nu(e^v)$. As the derivative of $T_\nu(x)$ is given in closed form by the expression

$$T'_\nu(x) = \frac{T_\nu(x)}{x\,(1-T_\nu(x))^{\nu-1}}, \quad (3.83)$$

then we obtain:

$$P_n(T_\nu(e^v)) = \frac{(1-T_\nu(e^v))^{s+t+\nu n}}{T_\nu(e^v)^s}\,\frac{d^n}{dv^n}\frac{T_\nu(e^v)^s}{(1-T_\nu(e^v))^{s+t}}. \quad (3.84)$$

This is the obvious generalization for the corresponding formulas for the $(s,t)$-Eulerian and second-order $(s,t)$-Eulerian polynomials.

As in the previous sections, an equivalent representation of $P_n$ can be obtained directly from the EGF (3.80) after performing the change of variables $y \mapsto u = (1-x)^\nu y$:

$$P_n(x) = n!\,\frac{(1-x)^{s+t+\nu n}}{x^s}\,[u^n]\frac{\left(T_\nu\left(e^u\,T_\nu^{-1}(x)\right)\right)^s}{\left(1-T_\nu\left(e^u\,T_\nu^{-1}(x)\right)\right)^{s+t}}. \quad (3.85)$$

Going beyond this point is rather involved, as we need closed formulas for the powers of the functions $T_\nu(x)$; e.g., like Eq. (3.51) for the tree function $T(x) = T_2(x)$. This might be doable, as one can express the derivative of $T_\nu(x)$ in the form (3.83). Then, by following the same steps as in Refs. [8, 9], one might obtain a closed formula for $(T_\nu(x))^s$, and then, a closed formula for the $P_n(x)$ and the $\left\langle {n \atop k} \right\rangle^{(\nu)}_{(s,t)}$.

# 4 Other applications

We further show the power of the methods discussed in the paper by presenting a small number of examples. In all the cases we will use the integral representations derived in Section 2 and the residue formulas obtained in terms of them. The actual computation of these residues gives rise to three slightly different situations:

- Formulas of the generic type $P_n(x) = f_n(x) \dfrac{d^n}{dx^n} g_n(x)$.
- Formulas of the generic type $P_n(x) = f_n(x) \left. \dfrac{\partial^n}{\partial z^n} \right|_{z=x} g_n(x,z)$.
- Formulas of the generic type $P_n(x) = f_n(x) \lim_{z\to x} \dfrac{\partial^n}{\partial z^n} g_n(x,z)$.

In the previous expressions, $f_n$ and $g_n$ denote functions depending on the non-negative integer parameter $n$. In all cases, we obtain Rodrigues-like formulas for the corresponding polynomials.

## 4.1 The Lah families

Lah families (see Section 5.2.2 of Paper I) correspond to $(\alpha, \beta; \alpha', \beta') = (\beta, \beta; 0, 0)$ with $\beta \neq 0$. They belong to Type II, so Theorem 2.2 implies:

$$
\begin{aligned}
P_n(x) &= n!\, \beta^n\, e^{-\gamma' x/\beta}\, x^{-\gamma/\beta}\, \mathrm{Res}\left( \frac{z^{n+\gamma/\beta}\, e^{\gamma' z/\beta}}{(z-x)^{n+1}}\, ; z = x \right) && \text{(4.1a)} \\
&= \beta^n\, e^{-\gamma' x/\beta}\, x^{-\gamma/\beta}\, \frac{d^n}{dx^n} \left( x^{n+\gamma/\beta}\, e^{\gamma' x/\beta} \right) && \text{(4.1b)} \\
&= \sum_{k=0}^{n} \beta^{n-k}\, (\gamma')^k\, \frac{n!}{k!} \binom{n+\gamma/\beta}{n-k} x^k\,, && \text{(4.1c)} \\
&= \sum_{k=0}^{n} \beta^{n-k}\, (\gamma')^k \binom{n}{k} \left( n + \frac{\gamma}{\beta} \right)^{\underline{n-k}} x^k\,. && \text{(4.1d)}
\end{aligned}
$$

The last two formulas give closed expression for both the row polynomials and the Lah-family numbers.

The (signed) Lah family [24, pp. 43–44], [7, p. 156], [23, entry A008294] corresponds to the choice $(-1, -1; 0, 0)$ with $(\gamma, \gamma') = (1, -1)$. In this case we have

$$
\begin{aligned}
P_n(x) &= n!\, (-1)^n\, x\, e^{-x}\, \mathrm{Res}\left( \frac{z^{n-1}\, e^{z}}{(z-x)^{n+1}}\, ; z = x \right) && \text{(4.2a)} \\
&= (-1)^n\, x\, e^{-x}\, \frac{d^n}{dx^n} \left( x^{n-1}\, e^{x} \right) && \text{(4.2b)} \\
&= \sum_{k=0}^{n} (-1)^n\, \frac{n!}{k!} \binom{n-1}{k-1} x^k\,. && \text{(4.2c)}
\end{aligned}
$$

It is worth mentioning that the Laguerre polynomials $L_n(x) = P_n(x)/n!$ and its well known Rodrigues formula can be obtained from (4.1) by choosing $\beta = 1$ and $(\gamma, \gamma') = (0, -1)$.

## 4.2 The $t_4(n,k)$ family

This family (see Section 5.2.2 of Paper I) corresponds to $(\alpha, \beta; \alpha', \beta') = (2, 1; 0, 0)$. They also belong to Type II, so Theorem 2.2 implies:

$$P_n(x) = n!\, 2^{n+1}\, e^{-\gamma' x}\, x^{-\gamma}\, \mathrm{Res}\left(\frac{z^{2n+\gamma+1}\, e^{\gamma' z}}{\left(z^2 - x^2\right)^{n+1}}\,; z = x\right) \tag{4.3a}$$

$$= 2^{n+1}\, e^{-\gamma' x}\, x^{-\gamma}\, \frac{\partial^n}{\partial z^n}\bigg|_{z=x} \frac{z^{2n+\gamma+1}\, e^{\gamma' z}}{(z+x)^{n+1}} \tag{4.3b}$$

$$= \sum_{k=0}^{n} \left( \frac{(\gamma')^k}{k!} \sum_{\ell=0}^{n-k} \binom{2n-k-\ell}{n} \frac{(2n+\gamma+1)^{\underline{\ell}}}{\ell!} \frac{n!}{(-2)^{n-\ell-k}} \right) x^k\,. \tag{4.3c}$$

A Rodrigues-like formula for this family can be obtained directly from the integral representation from which (4.3a) comes:

$$P_n(x) = n!\, 2^{n+1}\, e^{-\gamma' x}\, x^{-\gamma} \int_{Z_X(\Gamma)} \frac{x^{2n+\gamma+1}\, e^{\gamma' z}}{(z^2 - x^2)^{n+1}}\, dz\,. \tag{4.4}$$

If we perform the change of variables $u = z^2$ and write $v = x^2$, then it is straightforward to get the following simple formula:

$$P_n(\sqrt{v}) = 2^n\, e^{-\gamma' \sqrt{v}}\, v^{-\gamma/2}\, \frac{d^n}{dv^n} \left(v^{n+\gamma/2}\, e^{\gamma' \sqrt{v}}\right)\,. \tag{4.5}$$

Equation (4.5) implies

$$P_n(x) = \sum_{k=0}^{n} \left( \frac{2^n (-\gamma')^k}{k!} \sum_{j=0}^{k} (-1)^j \binom{k}{j} \left(n + \frac{\gamma+j}{2}\right)^{\underline{n}} \right) x^k\,. \tag{4.6}$$

The choice $(\gamma, \gamma') = (-2, 1)$ corresponds to a generalization of the Stirling subset numbers and the Lah numbers; see point 12 of Section 2 in Paper I, where they are denoted $t_4(n,k)$.

## 4.3 Surj$(n,k)$ numbers

The Surj$(n,k)$ numbers [23, entry A019538], [2, Corollary 2.41] correspond to $(\alpha, \beta; \alpha', \beta') = (0, 1; 0, 1)$ and $(\gamma, \gamma') = (0, 0)$. Therefore, they are of type I and Theorem 2.1 implies:

$$P_n(x) = \frac{n!}{1+x} \frac{1}{2\pi i} \int_{Z_X(\Gamma)} \frac{1}{z} \left(\log \frac{z\,(1+x)}{x\,(1+z)}\right)^{-(n+1)} dz \tag{4.7a}$$

$$= \frac{n!}{1+x}\, \mathrm{Res}\left(\frac{1}{z} \left(\log \frac{z\,(1+x)}{x\,(1+z)}\right)^{-(n+1)}\,;\, z = x\right) \tag{4.7b}$$

$$= \frac{1}{1+x} \lim_{z \to x} \frac{\partial^n}{\partial z^n} \left( \frac{1}{z} \left( \frac{z - x}{\log \frac{z\,(1+x)}{x\,(1+z)}} \right)^{n+1} \right)\,. \tag{4.7c}$$

The flexibility provided by the availability of the integral representation (4.7a) for the polynomials $P_n(x)$ can be exploited to get interesting results. For instance, by performing the change of variables $z/(1+z) = e^u$ and writing $x/(1+x) = e^v$, we get the following Rodrigues-like formula for these polynomials:

$$P_n\left(\frac{e^v}{1-e^v}\right) \;=\; (1-e^v)\,\frac{d^n}{dv^n}\,\frac{1}{1-e^v}\,. \tag{4.8}$$

Notice that, by comparing (4.8) and (3.15), it is straightforward to prove that the $(1,0)$ and (0,1)-Eulerian polynomials satisfy [7, Theorem E, pp. 244]

$$A_n^{(0,1)}(x) \;=\; (1-x)^n\,P_n\left(\frac{x}{1-x}\right) \;=\; \sum_{k=0}^{n} k!\left\{ n \atop k \right\} x^k\,(1-x)^{n-k}\,, \tag{4.9}$$

$$A_n(x) \;=\; x^n\,A_n^{(0,1)}(1/x) \;=\; \sum_{k=0}^{n} k!\left\{ n \atop k \right\} (x-1)^{n-k}\,, \tag{4.10}$$

where we have used Eq. (3.16).

### 4.4 Stirling subset family

This family corresponds to $(\alpha,\beta;\alpha',\beta') = (0,1;0,0)$. It belongs to Type II, so Theorem 2.1 implies that

$$P_n(x) \;=\; n!\,x^{-\gamma}\,\frac{1}{2\pi i}\int_{Z_x(\Gamma)}\frac{z^{\gamma-1}\,e^{\gamma'(z-x)}}{(\log(z/x))^{n+1}}\,dz \tag{4.11a}$$

$$\;=\; n!\,e^{-\gamma' x}\,x^{-\gamma}\,\mathrm{Res}\left(\frac{z^{\gamma-1}\,e^{\gamma' z}}{(\log(z/x))^{n+1}}\,,;z=x\right) \tag{4.11b}$$

$$\;=\; e^{-\gamma' x}\,x^{-\gamma}\,\lim_{z\to x}\frac{\partial^n}{\partial z^n}\left(\frac{z^{\gamma-1}\,e^{\gamma' z}}{(\log(z/x))^{n+1}}\right)\,. \tag{4.11c}$$

Formula (4.11a) suggests the change of variables $z = e^u$ and to write $x = e^v$. We then obtain the Rodrigues-like formula for the row polynomials of the Stirling subset family:

$$P_n\left(e^v\right) \;=\; e^{-\gamma v-\gamma' e^v}\,\frac{d^n}{dv^n}e^{\gamma v+\gamma' e^v}\,. \tag{4.12}$$

Equation (4.12) implies

$$P_n(x) \;=\; \sum_{k=0}^{n}\left(\sum_{j=k}^{n}\binom{n}{j}\left\{ j \atop k \right\}\gamma^{n-j}\,(\gamma')^k\right)x^k\,. \tag{4.13}$$

Stirling subset numbers [15], [23, entry A008277] corresponds to $(\gamma,\gamma') = (0,1)$. In this case $P_n(x)$ are Touchard polynomials [26], and equation (4.12) reduces to the known identity:

$$P_n\left(e^v\right) \;=\; e^{-e^v}\,\frac{d^n}{dv^n}e^{e^v}\,. \tag{4.14}$$

In entry A008277 of Ref. [23], this identity is cited to be due to Peter Bala (2012).

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# A Some proofs

In this appendix we will prove some of the results stated in the text.

PROOF OF LEMMA 3.5.

Let us start with Eq. (3.56b), and define the following function for $z > 0$

$$S_n(z) \;=\; \sum_{j=0}^{n} \binom{n}{j} \frac{(j+1)!}{z^{j+1}} \;=\; \sum_{j=0}^{n} \frac{n!}{(n-j)!} \frac{j+1}{z^{j+1}} \,. \tag{A.1}$$

Our aim is to prove that $S_n(n+1) = 1$ [cf., (3.56b)]. We now introduce a new function $F_n$

$$F_n(z) \;=\; \int^z \frac{S_n(y)}{y}\, dy \;=\; -\sum_{j=0}^{n} \frac{n!}{(n-j)!} \frac{1}{z^{j+1}} \;=\; -\frac{n!}{z^{1+n}} \sum_{j=0}^{n} \frac{z^j}{j!} \,, \tag{A.2}$$

up to an additive unimportant constant. The last sum can be related to the incomplete gamma function $\Gamma(\alpha, x)$ [14, Eq. 8.352–2]:

$$\Gamma(n+1, z) \;=\; n!\, e^{-z} \sum_{j=0}^{n} \frac{z^j}{j!} \,, \qquad n \;\in\; \mathbb{N} \,. \tag{A.3}$$

Therefore,

$$F_n(z) \;=\; -e^z\, \Gamma(n+1, z)\, z^{-(n+1)} \,, \tag{A.4}$$

again up to an additive constant. We now differentiate this expression w.r.t. $z$:

$$\frac{dF_n(z)}{dz} \;=\; \frac{S_n(z)}{z} \;=\; \frac{1}{z} - e^z\, z^{-(n+1)}\, \Gamma(n+1, z) \left(1 - \frac{n+1}{z}\right) \,, \tag{A.5}$$

where we have used the following formula for the derivative of $\Gamma(n+1, z)$ w.r.t. $z$ [14, Eq. 8.356–4]:

$$\frac{d\Gamma(n+1, z)}{dz} \;=\; -z^n\, e^{-z} \,. \tag{A.6}$$

Then, we conclude that

$$S_n(z) \;=\; 1 - e^z\, z^{-n}\, \Gamma(n+1, z) \left(1 - \frac{n+1}{z}\right) \,, \tag{A.7}$$

which, at $z = n+1$ is equal to $S_n(n+1) = 1$, as we claimed in Eq. (3.56b).

The proof of Eq. (3.56a) is similar. We define a new function for $z > 0$

$$\widehat{S}_n(z) \;=\; \sum_{j=0}^{n} \binom{n}{j} j!\, \frac{j}{z^{j+1}} \;=\; n! \sum_{j=0}^{n} \frac{j}{(n-j)!}\, z^{-(1+j)}\,, \tag{A.8}$$

and we want to prove this time that $\widehat{S}_n(n) = 1$. We now introduce a new function $\widehat{F}_n(z)$ (modulo an additive constant):

$$\widehat{F}_n(z) \;=\; \int^z S_n(y)\, dy \;=\; -\frac{n!}{z^n} \sum_{j=0}^{n} \frac{z^{n-j}}{(n-j)!} \;=\; -\frac{n!}{z^n} \sum_{j=0}^{n} \frac{z^j}{j!}\,. \tag{A.9}$$

Using (A.3), we conclude that

$$\widehat{F}_n(z) \;=\; -\Gamma(n+1, z)\, e^z\, z^{-n}\,. \tag{A.10}$$

Differentiating this equation w.r.t. $z$, we obtain

$$\frac{d\widehat{F}_n(z)}{dz} \;=\; \widehat{S}_n(z) \;=\; 1 - e^z\, z^{-(n+1)}\, \Gamma(n+1, z)(z-n)\,, \tag{A.11}$$

which implies that $\widehat{S}_n(n) = 1$, as claimed in Eq. (3.56a). This completes the proof of the lemma. ∎

Proof of Corollary 3.7.

(a) When $t = -s$, the sum over $j$ is trivial due to the term $0^{\overline{j}} = \delta_{j,0}$. Then,

$$\left\langle\!\!\left\langle {n \atop k} \right\rangle\!\!\right\rangle_{(s,-s)} \;=\; \frac{s}{k!} \sum_{r=0}^{k} \binom{k}{r} (2n)^{\underline{k-r}} \sum_{p=0}^{r} \binom{r}{p} (-1)^k (p+s)^{n+r-1}\,. \tag{A.12}$$

Now because $2n$ is an integer, we can rearrange the binomial coefficients and $(2n)^{\underline{k-r}}$ to obtain (3.60).

(c) When $(s,t) = (0,1)$, we have that $1^{\overline{j}} = j!$, and then

$$\left\langle\!\!\left\langle {n \atop k} \right\rangle\!\!\right\rangle_{(0,1)} \;=\; \frac{1}{k!} \sum_{r=0}^{k} \binom{k}{r} (1+2n)^{\underline{k-r}} \sum_{p=0}^{r} \binom{r}{p} (-1)^{k-p} \times \sum_{j=0}^{p} \binom{p}{j} j!\, j\, p^{n+r-j-1}\,. \tag{A.13}$$

By using Eq. 3.56a, we obtain that the last sum is $p^{n+r}$ for any $r \geq 0$. Then, by re-arranging the binomials and $(1+2n)^{\underline{k-r}}$, we get

$$\left\langle\!\!\left\langle {n \atop k} \right\rangle\!\!\right\rangle_{(0,1)} \;=\; \sum_{r=0}^{k} \frac{1}{r!} \binom{1+2n}{k-r} \sum_{p=0}^{r} \binom{r}{p} (-1)^{k-p}\, p^{n+r}\,. \tag{A.14}$$

The last sum is related to the Stirling subset numbers [15, Eq. (6.19)]:

$$\left\{ {n \atop k} \right\} = \frac{1}{k!} \sum_{p=0}^{k} (-1)^{k-p} \binom{k}{p} p^n \,, \qquad n \ge 0\,, 0 \le k \le n\,. \tag{A.15}$$

After substituting (A.15) into (A.14), we recover (3.62).

(b) When $(s,t) = (1,0)$, we use the relation with the numbers in (c) above: $\left\langle\!\left\langle {n \atop k} \right\rangle\!\right\rangle = \left\langle\!\left\langle {n \atop k+1} \right\rangle\!\right\rangle_{(0,1)}$ for $n \ge 1$. We then obtain

$$\left\langle\!\!\left\langle {n \atop k} \right\rangle\!\!\right\rangle = \sum_{r=0}^{k+1} (-1)^{k-r+1} \binom{1+2n}{k+1-r} \left\{ {n+r \atop r} \right\}. \tag{A.16}$$

Notice that for $n \ge 1$, the term with $r = 0$ contributes with $(-1)^{k+1}\binom{1+2n}{k+1}\left\{ {n \atop 0} \right\} \propto \delta_{n,0}$, due to the Stirling number. Therefore, for $n \ge 1$, we can start the sum above at $r = 1$. But if we do that, then the above formula is also valid for $n = k = 0$: $\left\langle\!\left\langle {0 \atop 0} \right\rangle\!\right\rangle = \left\{ {1 \atop 1} \right\} = 1$. Then after changing variables $r \mapsto r-1$, we recover (3.61).

This completes the proof of the Corollary. ∎

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