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JACOBI'S GENERATING FUNCTION FOR JACOBI POLYNOMIALS

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ABSTRACT. An idea of Hermite is used to give a simple proof of Jacobi's generating function for Jacobi polynomials.

One of the standard ways to prove the orthogonality of Legendre polynomials is to take their generating function

$$(1 - 2xr + r^2)^{-1/2} = \sum_{n=0}^{\infty} P_n(x) r^n \tag{1}$$

and show that the integral

$$\int_{-1}^{1} (1 - 2xr + r^2)^{-1/2} (1 - 2xs + s^2)^{-1/2} \, dx$$

is a function of the variable rs. See, for example, Courant-Hilbert [**1**, pp. 85–86]. This proof was given by Legendre [**6**, p. 250].

Jacobi gave a generating function for a more general set of orthogonal polynomials $P_n^{(\alpha,\beta)}(x)$. These polynomials can be defined by

$$(1 - x)^{\alpha}(1 + x)^{\beta} P_n^{(\alpha,\beta)}(x) = \frac{(-1)^n}{2^n n!} \frac{d^n}{dx^n}\left[(1 - x)^{n+\alpha}(1 + x)^{n+\beta}\right]. \tag{2}$$

It is easy to use (2) and integration by parts to prove

$$\int_{-1}^{1} P_n^{(\alpha,\beta)}(x) P_m^{(\alpha,\beta)}(x)(1 - x)^{\alpha}(1 + x)^{\beta} \, dx = 0, \qquad m \neq n, \alpha,\beta > -1. \tag{3}$$

Jacobi's generating function is

$$\sum_{n=0}^{\infty} P_n^{(\alpha,\beta)}(x) r^n = 2^{\alpha+\beta} R^{-1} (1 - r + R)^{-\alpha} (1 + r + R)^{-\beta} \tag{4}$$

where $R = (1 - 2xr + r^2)^{1/2}$. His original proof used Lagrange's extension of Taylor's theorem. A second proof of this generating function was given a few years later by Tchebychef [**9**]. His proof was modeled on Legendre's proof mentioned above. He seems to have found this generating function independently from Jacobi, since he does not mention Jacobi's paper. Tchebychef's proof is a very complicated one which involves a number of changes of variables to reduce the integral

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$$\int_{-1}^{1} R^{-1}(1 - r + R)^{-\alpha}(1 + r + R)^{-\beta}S^{-1}$$
$$\cdot (1 - s + S)^{-\alpha}(1 + s + S)^{-\beta}(1 - x)^{\alpha}(1 + x)^{\beta}\,dx \qquad (5)$$

to a function of $(r \cdot s)$. Here $S = (1 - 2xs + s^2)^{1/2}$. Once this has been done then the coefficient of r^n in (4), which is clearly a polynomial of degree n, can be identified with $P_n^{(\alpha,\beta)}(x)$ as defined in (2) by observing that both take the value $\Gamma(n + \alpha + 1)/[\Gamma(\alpha + 1)n!]$ at $x = 1$.

Hermite [4, p. 26] found a much easier way to prove the orthogonality of Legendre polynomials from the generating function (1). Stieltjes [4, p. 28] remarked that Hermite's proof was very simple and seemed preferable to that of Legendre. This simplicity carries over to Jacobi polynomials.

To see how this idea can even suggest Jacobi's generating function we first give Hermite's proof. He considered

$$I_k = \int_{-1}^{1} x^k(1 - 2xr + r^2)^{-1/2}\,dx \qquad (6)$$

and set $(1 - 2xr + r^2)^{1/2} = 1 - ry$. The integral is then

$$I_k = \int_{-1}^{1}\left[y + \frac{r(1 - y^2)}{2}\right]^k dy$$

and this is a polynomial of degree k in r. But it is also

$$I_k = \sum_{n=0}^{\infty} r^n \int_{-1}^{1} x^k P_n(x)\,dx,$$

and for this to reduce to a polynomial of degree k we must have

$$\int_{-1}^{1} x^k P_n(x)\,dx = 0, \qquad n = k + 1, k + 2, \ldots.$$

This is equivalent to

$$\int_{-1}^{1} P_k(x)P_n(x)\,dx = 0, \qquad k \neq n. \qquad (7)$$

Hermite's proof can be read backwards as a proof of the generating function (1) if the orthogonality relation is known and an appropriate normalization is given. In this case it is $P_n(1) = 1$, which is clearly satisfied by the polynomials defined by (1).

To carry over this proof to Jacobi polynomials, assume they are defined by (2), and the orthogonality relation (3) has been proven. Consider a generating function

$$f(x, r) = \sum_{n=0}^{\infty} P_n^{(\alpha,\beta)}(x)r^n$$

and look at the integral

$$I_k = \int_{-1}^{1} x^k f(x, r)(1 - x)^{\alpha}(1 + x)^{\beta}\,dx.$$

As above we want this to be a polynomial of degree k in the variable r. The change of variables used above gives

$$I_k = \int_{-1}^{1}\left[y + \frac{r(1-y^2)}{2}\right]^k f(x,r)\left[\frac{1-r+1-ry}{2}\right]^\alpha \cdot\left[\frac{1+r+1-ry}{2}\right]^\beta (1-ry)(1-y)^\alpha(1+y)^\beta\,dy.$$

This is clearly a polynomial of degree k if

$$f(x,r) = 2^{\alpha+\beta}(1-r+1-ry)^{-\alpha}(1+r+1-ry)^{-\beta}(1-ry)^{-1},$$

where

$$(1-2xr+r^2)^{1/2} = 1-ry.$$

So if

$$f(x,r) = 2^{\alpha+\beta}(1-r+R)^{-\alpha}(1+r+R)^{-\beta}R^{-1} \tag{8}$$

then

$$f(x,r) = \sum_{n=0}^{\infty} Q_n(x)r^n \tag{9}$$

where $Q_n(x)$ is a polynomial of degree n in x which satisfies

$$\int_{-1}^{1} x^k Q_n(x)(1-x)^\alpha(1+x)^\beta\,dx = 0, \qquad n = k+1, k+2, \ldots.$$

Thus

$$Q_n(x) = a_n P_n^{(\alpha,\beta)}(x)$$

for some constant a_n, since there is only one set of polynomials that are orthogonal with respect to a given measure after they have been normalized. When $x = 1$ (8) and (9) give

$$\sum_{n=0}^{\infty} Q_n(1)r^n = (1-r)^{-\alpha-1}$$

so

$$Q_n(1) = \frac{(\alpha+1)_n}{n!} = \frac{\Gamma(n+\alpha+1)}{\Gamma(\alpha+1)n!}.$$

It is also easy to see from (2) that

$$P_n^{(\alpha,\beta)}(1) = \frac{(\alpha+1)_n}{n!}$$

so

$$Q_n(x) = P_n^{(\alpha,\beta)}(x).$$

I agree completely with Stieltjes that this proof of Hermite is superior to that of Legendre. Whether it is used to preference to Jacobi's original proof is a matter of taste. Szegö [**8**, §4.4] gives four proofs of this generating function.

The first and second proofs use Cauchy's theorem in a standard way that is familiar from proofs of Lagrange's theorem, and so are essentially equivalent to Jacobi's original proof. The third proof is Tchebychef's proof and the fourth is Jacobi's. There are other proofs. Rainville [**7**, §140] gives one that relies on the reduction of an Appell function F_4, and Carlitz [**1**] gives a simple proof which uses the method of proving that a special F_4 can be reduced to a product without explicitly mentioning multiple hypergeometric series. This is analogous to what Szegö did in his first proof, he gives a direct proof of this generating function by a method that could be used to prove a general theorem that could then be used to obtain (4).

Hermite's proof is buried in one of the most beautiful set of letters that has been written [**3**], [**4**]. Stieltjes was twenty-five when he first wrote Hermite, who was almost sixty. One can trace the development of Stieltjes as a mathematician and see how Hermite aided his education by supplying information that Stieltjes had not learned yet, by mentioning problems, and giving encouragement to a very talented young man. Not only are these letters worth reading for their historical and pedagogical value, as this note shows there is interesting mathematics there that has not been rediscovered. They are worth further study.

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