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Generalized j-Factorial Functions, Polynomials, and Applications

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Abstract

The paper generalizes the traditional single factorial function to integer-valued multiple factorial (j-factorial) forms. The generalized factorial functions are defined recursively as triangles of coefficients corresponding to the polynomial expansions of a subset of degenerate falling factorial functions. The resulting coefficient triangles are similar to the classical sets of Stirling numbers and satisfy many analogous finite-difference and enumerative properties as the well-known combinatorial triangles. The generalized triangles are also considered in terms of their relation to elementary symmetric polynomials and the resulting symmetric polynomial index transformations. The definition of the Stirling convolution polynomial sequence is generalized in order to enumerate the parametrized sets of j-factorial polynomials and to derive extended properties of the j-factorial function expansions.

The generalized j-factorial polynomial sequences considered lead to applications expressing key forms of the j-factorial functions in terms of arbitrary partitions of the j-factorial function expansion triangle indices, including several identities related to the polynomial expansions of binomial coefficients. Additional applications include the formulation of closed-form identities and generating functions for the Stirling numbers of the first kind and r-order harmonic number sequences, as well as an extension of Stirling's approximation for the single factorial function to approximate the more general j-factorial function forms.

1 Notational Conventions

Donald E. Knuth's article *Two Notes on Notation* [33] establishes several of the forms of standardized notation employed in the article. In particular Knuth's notation for the *Stirling* *number* triangles and the *Stirling polynomial* sequences [24; 32] are used to denote these forms and reasonable extensions of these conventions are used to denote the generalizations of the forms established by this article. The usage of notation for standard mathematical functions is explained inline in the text where the context of the relevant forms apply [46]. The following is a list of the other main notational conventions employed throughout the article.

- **Indexing Sets:** The *natural numbers* are denoted by the set notation $\mathbb{N}$ and are equivalent to the set of non-negative integers, where the set of integers is denoted by the similar blackboard set notation for $\mathbb{Z}$. The standard set notation for the real numbers ($\mathbb{R}$) and complex numbers ($\mathbb{C}$) is used as well to denote scalar and approximate constant values.
- **Natural Logarithm Functions:** The *natural logarithm* function is denoted $\operatorname{Log}(z)$ in place of $\ln(z)$ in the series expansion properties involving the function. Similarly $\operatorname{Log}(z)^k$ denotes the natural logarithm function raised to the k^{th} power.
- **Iverson's Convention:** The notation $[\mathtt{condition}]_\delta$ for a boolean-valued input condition represents the value 1 (or 0) where the input condition evaluates to `True` (or `False`). Iverson's convention is used extensively in the *Concrete Mathematics* reference and is a comparable replacement for *Kronecker's delta* function for multiple pairs of arguments. For example, the notation $[n = k]_\delta$ is equivalent to $\delta_{n,k}$ and the notation $[n = k = 0]_\delta$ is equivalent to $\delta_{n,0}\delta_{k,0}$.
- **Sequence Enumeration and Coefficient Extraction:** The notation $\langle g_n \rangle \mapsto \{g_0,\ g_1,\ g_2,\ \ldots\}$ denotes a sequence indexed over the natural numbers. Given the generating function $F(z)$ representing the formal power series (also generating series expansion) that enumerates $\langle f_n \rangle$, the notation $[z^n]F(z) := f_n$ denotes the series coefficients indexed by $n \in \mathbb{N}$.
- **Fixed Parameter Variables:** For an indexing variable n, the notation N_c is employed to represent a fixed parameter in a formula or generating function that is treated as a constant and that is only assigned the explicit value of the respective non-constant indexing variable after all other variables and indices have been input and processed symbolically in a relevant form. In particular the fixed N_c variable should be treated as a constant parameter in series or generating function closed-forms, even when the non-constant form of n refers to a particular coefficient index in the series expansion. The footnote on p. 18 clarifies the context and particular utility of the fixed parameter usage in a specific example inline in the text.

2 Introduction

The parametrized *multifactorial* (*j-factorial*) functions studied in this article generalize the standard classical *single factorial* [A000142] and *double factorial* [A001147; A000165; and A006882] functions and are characterized by the analogous recursive property in (2.1).

$$n!_{(j)} := n\ (n-j)!_{(j)}\ [n \geq j]_\delta + [0 \leq n < j]_\delta \tag{2.1}$$

The classical *generalized falling factorial* function, $(z|\alpha)^{\underline{n}}$, studied extensively by Adelberg and several others [3; 15; 16; 28; 35; 64], can be defined analytically in terms of the *gamma function* in (2.2) and by the equivalent product expansion form in (2.3). The function can also be expressed by the equivalent finite-degree polynomial expansion in z with coefficients given in terms of the unsigned triangle of *Stirling numbers of the first kind* [A094638] in equation (2.4).

$$(z|\alpha)^{\underline{n}} := \frac{\alpha^n \, \Gamma(\frac{z}{\alpha})}{\Gamma(\frac{z}{\alpha} - n)} + [n = 0]_\delta \tag{2.2}$$

$$= \prod_{i=0}^{n-1} (z - i\alpha) + [n = 0]_\delta \tag{2.3}$$

$$= \sum_{k=1}^{n} \begin{bmatrix} n \\ k \end{bmatrix} (-\alpha)^{n-k} z^k + [n = 0]_\delta \tag{2.4}$$

The definitions of (2.2) and (2.3) extend a well-known simplified case of the *falling factorial* function for $\alpha = 1$, commonly denoted by the equivalent forms $x^{\underline{n}} = x!/(x-n)!$ and $(x)_n$ [24; 50]. Other related factorial function variants include the *generalized factorials of t of order n and increment h*, denoted $t^{(n,\, h)}$, considered by C. Charalambides [17, §1], the forms of the *Roman factorial* and *Knuth factorial* functions defined by Loeb [40], and the *q-shifted factorial* functions defined by McIntosh [42] and Charalambides [17, (2.2)].

This article explores forms of the polynomial expansions corresponding to a subset of the generalized, integer-valued falling factorial functions defined by (2.3). The finite-difference forms studied effectively generalize the Stirling number identity of (2.4) for the class of *degenerate falling factorial* expansion forms given by $(x-1|\alpha)^{\underline{n}}$ when α is a positive integer. The treatments offered in many standard works are satisfied with the analytic gamma function representation of the full falling factorial function expansion. In contrast, the consideration of the generalized factorial functions considered by this article is motivated by the need for the precise definition of arbitrary sequences of the coefficients that result from the variations of the finite-degree polynomial expansions in z originally defined by equation (2.4).

For example, in a motivating application of the research it is necessary to extract only the even powers of z in the formal polynomial expansion of the double factorial function over z. The approach to these expansions is similar in many respects to that of Charalambides' related article where the expansion coefficients of the *generalized q-factorial* functions are treated separately from the forms of the full factorial function products [17, §3 and §5]. The coefficient-based definition of the falling factorial function variants allows for a rigorous and more careful study of the individual finite-degree expansions that is not possible from the purely analytic view of the falling factorial function given in terms of the full product expansion and infinite series representations of the gamma function [cf. (2.2)].

The exploration of the j-factorial function expansions begins in §3.1 by motivating the recursive definition of the coefficient triangles for the polynomial expansions of the (degenerate) factorial functions defined in the form of equation (2.4). The article then expands the properties of the factorial function expansions in terms of finite-difference identities and enumerative properties in §3, relations to transformations of *elementary symmetric functions* in §4, and in the forms of the j*-factorial polynomials* that generalize the sequence of *Stirling polynomials* in §5. A number of interesting applications and examples are considered as well, with particular emphasis on the forms discussed in §4.3 and §6.

3 Finite Difference Representations for the j-Factorial Function Expansion Coefficients

3.1 Triangle Definitions

Consider the coefficient triangles indexed over $n, k \in \mathbb{N}$ and defined recursively by equations (3.1) and (3.2) [35, cf. (1.2)].

$$\begin{bmatrix} n \\ k \end{bmatrix}_\alpha = (\alpha n + 1 - 2\alpha) \begin{bmatrix} n-1 \\ k \end{bmatrix}_\alpha + \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_\alpha + [n = k = 0]_\delta \tag{3.1}$$

$$\begin{Bmatrix} n \\ k \end{Bmatrix}_\alpha = (\alpha k + 1 - \alpha) \begin{Bmatrix} n-1 \\ k \end{Bmatrix}_\alpha + \begin{Bmatrix} n-1 \\ k-1 \end{Bmatrix}_\alpha + [n = k = 0]_\delta \tag{3.2}$$

The "*triangular*" recurrences defining the α*-factorial triangles* are special cases of a more general form in equation (3.35) that includes well-known classical combinatorial sets [24, Ch. 6] such as the *Stirling cycle numbers* (first kind) [A008275; and A094638], also defined by (3.1) when $\alpha := 1$, the *Stirling subset numbers* (second kind) [A008277], the *Eulerian numbers* for permutation *"ascents"* [A066094], and the *"second-order" Eulerian numbers* [A008517] [cf. §6.2.4]. The unsigned triangles corresponding to a positive integer parameter α are unimodal over each row and are strictly increasing at each fixed column for sufficiently large row index n [62, §4.5]. The signed coefficient analog of the triangle in (3.1) is defined recursively as (3.3) and may be expressed in terms of the unsigned triangle by the conversion formulas in equation (3.4).

$$\left[\begin{bmatrix} n \\ k \end{bmatrix}\right]_\alpha = (2\alpha - \alpha n - 1) \left[\begin{bmatrix} n-1 \\ k \end{bmatrix}\right]_\alpha + \left[\begin{bmatrix} n-1 \\ k-1 \end{bmatrix}\right]_\alpha + [n = k = 0]_\delta \tag{3.3}$$

$$\left[\begin{bmatrix} n \\ k \end{bmatrix}\right]_\alpha = (-1)^{n-k} \begin{bmatrix} n \\ k \end{bmatrix}_\alpha \qquad \iff \qquad \begin{bmatrix} n \\ k \end{bmatrix}_\alpha = (-1)^{n-k} \left[\begin{bmatrix} n \\ k \end{bmatrix}\right]_\alpha \tag{3.4}$$

The implicit interpretation of the (3.1) triangle as expansion coefficient sets is demonstrated by considering the motivation for the following procedure. Let the polynomial $p_n(x)$ correspond to the n^{th} distinct polynomial expansion of the α*-factorial* function, $(x-1)!_{(\alpha)}$, and define the polynomial coefficients as $[x^{k-1}]p_n(x) := \begin{bmatrix} n \\ k \end{bmatrix}_\alpha$. Observe that provided a polynomial (row) index n, the coefficient forms corresponding to subsequent polynomial expansions are formed by the multiplication of a linear factor in x with the existing polynomial. The next equation defines the triangle of coefficients that result from the expansions of this form [17, cf. Thm. 3.2].

$$[x^{k-1}]p_n(x) = (\alpha n + 1 - 2\alpha)\ [x^{k-1}]p_{n-1}(x) + [x^{k-2}]p_{n-1}(x)$$

It follows that for rows indexed by $n \in [1, \infty) \subseteq \mathbb{N}$ and columns indexed by $k \in [1, \infty) \subseteq \mathbb{N}$, the polynomial expansions yield an identical recursive definition of the α-factorial coefficient triangles to that given by (3.1) [cf. (3.5)].

In order to evaluate the factorial function expansions numerically, consider that the range of natural numbers that correspond to any distinct α-factorial polynomial expansion in s is a function of α: there is exactly one $n \in \mathbb{N}$ for each single factorial function expansion, two $n \in \mathbb{N}$ for each double factorial function expansion, and so on for each positive integer value of α.

3.2 Finite-Difference Properties

Many analogs of the classical Stirling number identities and related combinatorial properties for the triangles in equations (3.1) and (3.2) are generalized by the analogous forms in the following discussions [cf. §3.4.2]. A number of additional identities and forms related to the Stirling number forms, including relations to the *Bell numbers* [A000110], *Lah numbers* [A008297], *multi-poly Bernoulli numbers*, and *Tanh numbers* [A111593], are discussed in the references by Agoh and several others [9; 8; 10; 47] [16, (3.5)] [24, Table 202] [26, §2 and §3] [28, §3.1] [55, Thm. 2.1 and (2.4)]. The discussions and properties of the α-factorial polynomials given in §5 also provide a number of identities involving *generalized Bernoulli polynomials* and other functions that may be applied to the forms of (3.1) and (3.2). The next section in §4 contains detailed discussions of the (3.1) triangle properties as well.

The initial characteristic finite-difference properties defining the Stirling number sets are mirrored by the generalized α-factorial triangles as given by the following equations in (3.5) and (3.6) [34, §1.2.6].

$$(x-1|\alpha)^{\underline{n}} = \sum_{k=0}^{n-1} \left[\begin{matrix} n \\ k+1 \end{matrix}\right]_{\alpha} (-1)^{n-1-k} x^k \tag{3.5}$$

$$x^n := \sum_{k=0}^{n} \left\{\begin{matrix} n \\ k \end{matrix}\right\}_{\alpha} (x-1|\alpha)^{\underline{k}} \tag{3.6}$$

Depending on the application it may be convenient to define the parameter α in equations (3.1) and (3.2) over the rational numbers. This slight generalization in form will still result in the correct form for the interpretation given by the product expansion of (3.5). For example, the book *Concrete Mathematics* considers the form of $r^{\underline{k}}(r-1/2)^{\underline{k}}$ related to the *central binomial coefficients* [24, §5.3]. One additional special case of equation (3.1) occurs when $\alpha := 0$ where the form provides the recursive definition for *Pascal's triangle*. The case degrades nicely in the context a 0-factorial function where the polynomial factors in the expansion remain constant in form over all n. The form of equation (3.5) for this case then effectively defines the *binomial theorem* in reverse: $\sum_k \left[\begin{smallmatrix} n \\ k \end{smallmatrix}\right]_0 s^{k-1} = (s+1)^{n-1}$.

The Stirling number inversion identities [24, Table 264] are generalized by the forms of equations (3.7) and (3.8).

$$\sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_\alpha \begin{Bmatrix} k \\ m \end{Bmatrix}_\alpha (-1)^{n-k} = [m = n]_\delta \tag{3.7}$$

$$\sum_{k=0}^{n} \begin{Bmatrix} n \\ k \end{Bmatrix}_\alpha \begin{bmatrix} k \\ m \end{bmatrix}_\alpha (-1)^{n-k} = [m = n]_\delta \tag{3.8}$$

The orthogonality relation for the Stirling numbers is preserved in these properties for the generalized triangles and defines the analogous result of corollary (3.9) [47, cf. (1.1)] [24].

$$f(n) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_\alpha (-1)^k g(k) \quad \Longleftrightarrow \quad g(n) = \sum_{k=0}^{n} \begin{Bmatrix} n \\ k \end{Bmatrix}_\alpha (-1)^k f(k) \tag{3.9}$$

The identities of equations (3.10) and (3.11) generalize classical Stirling number properties as well [24, Table 265].

$$\sum_{k=0}^{n} \begin{bmatrix} n+1 \\ k+1 \end{bmatrix}_\alpha \begin{Bmatrix} k \\ m \end{Bmatrix}_\alpha (-1)^{m-k} = \frac{\alpha^{n-m}\ \Gamma\left(n + \frac{1}{\alpha}\right)}{\Gamma\left(m + \frac{1}{\alpha}\right)} [n \geq k]_\delta \tag{3.10}$$

$$\sum_{k=0}^{n+1} \begin{Bmatrix} n+1 \\ k+1 \end{Bmatrix}_\alpha \begin{bmatrix} k \\ m \end{bmatrix}_\alpha (-1)^{m-k} = \sum_{j=0}^{n-k} \binom{k-1+j}{k-1} \alpha^j + [k = 0]_\delta \tag{3.11}$$

The generalized first triangle in (3.1) can be expressed in terms of the Stirling numbers of the first kind through the following identities.

$$\begin{bmatrix} n \\ k \end{bmatrix}_\alpha = \sum_{j=0}^{n-k} \begin{bmatrix} n-1 \\ k-1+j \end{bmatrix} \binom{k-1+j}{k-1} \alpha^{n-k-j} + [n = k = 1]_\delta \tag{3.12}$$

$$\begin{bmatrix} x \\ x-n \end{bmatrix}_\alpha = \sum_{k=0}^{n} \begin{bmatrix} x \\ x-k \end{bmatrix} \frac{\alpha^k\ \Gamma(x-k)\ (1-\alpha)^{n-k}}{\Gamma(x-n)\ \Gamma(n-k+1)}$$

The generalized second triangle in (3.2) can be expressed in terms of the *Stirling numbers of the second kind* through the following equations [8, (3.4) and (3.5)].

$$\begin{aligned} \begin{Bmatrix} n \\ k \end{Bmatrix}_\alpha &= \sum_{j=0}^{n-k} \begin{Bmatrix} k-1+j \\ k-1 \end{Bmatrix} \binom{n-1}{k-1+j} \alpha^j \\ &= \sum_{i=0}^{k-1} \sum_{j=0}^{n-k} \binom{n-1}{k-1+j} \binom{k-1}{i} \frac{(-1)^i\ \alpha^j\ (k-1-i)^{k-1+j}}{(k-1)!} \\ &= \sum_{i=1}^{k-1} \sum_{j=0}^{n-k} \binom{n-1}{k-1+j} \binom{k-2}{i-1} \frac{(-1)^{k-1+i}\ \alpha^j\ i^{k-2+j}}{(k-2)!} \end{aligned} \tag{3.13}$$

Let the *linear differential operator*, $\{D^k\}[f(N_c)]$, be defined such that for integer $k \geq 1$ the operator denotes the k^{th} partial derivative of f with respect to N_c and for all other k,

$\{D^k\}[f(N_c)] := f(N_c)\,[k=0]_\delta$. Additional properties of the generalized first triangle are then given in pairs below for the finite $n \to N_c$ and corresponding finite-difference *"binomial derivative"* formulas.

$$\begin{aligned}\begin{bmatrix} n \\ k \end{bmatrix}_\alpha &= \left\{ \frac{D^{k-1}}{(k-1)!\ \alpha^{k-1}} + \frac{D^{k-2}}{(k-2)!\ \alpha^{k-2}} \right\} \left[\sum_{i=0}^{n-3} (\alpha N_c + 1 - 2\alpha)^{i+1} (-\alpha)^{n-3-i} \begin{bmatrix} n-2 \\ i+1 \end{bmatrix} \right] \\ &= \sum_{i=0}^{n-3} \sum_{r=0}^{i+1} \begin{bmatrix} n-2 \\ i+1 \end{bmatrix} \binom{i+1}{r} \binom{r}{k-1} (-1)^{n-3-i} \alpha^{n+r-2-i-k} N_c^{r+1-k} (1-2\alpha)^{i+1-r} \\ &\quad + \sum_{i=0}^{n-3} \sum_{r=0}^{i+1} \begin{bmatrix} n-2 \\ i+1 \end{bmatrix} \binom{i+1}{r} \binom{r}{k-2} (-1)^{n-3-i} \alpha^{n+r-1-i-k} N_c^{r+2-k} (1-2\alpha)^{i+1-r}\end{aligned} \tag{3.14}$$

$$\begin{aligned}\begin{bmatrix} n \\ k \end{bmatrix}_\alpha &= \left\{ \frac{D^{k-1}}{(k-1)!\ \alpha^{k-1}} + \frac{D^{k-2}}{(k-2)!\ \alpha^{k-2}} \right\} \left[\sum_{i=0}^{n-2} (\alpha N_c + 2 - 2\alpha)^{i} (-1)^{n-i} \begin{bmatrix} n-1 \\ i+1 \end{bmatrix}_\alpha \right] \\ &= \sum_{i=0}^{n-2} \sum_{r=0}^{i} \begin{bmatrix} n-1 \\ i+1 \end{bmatrix}_\alpha \binom{i}{r} \binom{r}{k-1} (-1)^{n-i} \alpha^{r+1-k} N_c^{r+1-k} 2^{i-r} (1-\alpha)^{i-r} \\ &\quad + \sum_{i=0}^{n-2} \sum_{r=0}^{i} \begin{bmatrix} n-1 \\ i+1 \end{bmatrix}_\alpha \binom{i}{r} \binom{r}{k-2} (-1)^{n-i} \alpha^{r+2-k} N_c^{r+2-k} 2^{i-r} (1-\alpha)^{i-r}\end{aligned} \tag{3.15}$$

The second of the listed "binomial derivative" formulas may be considered as an "*involution of sorts*" since the coefficient form of $\left[\begin{smallmatrix} n \\ k \end{smallmatrix}\right]_\alpha$ is given in terms of coefficients from the same triangle. The identity is also of particular interest since the involution-like phrasing results in the applications for the α-factorial polynomials discussed in §5.3.2 and §5.3.4 that are based on the alternate forms of the involution identities derived in §4.2.1.

Finally, the sums of the first coefficient triangle rows indexed over the integer $n \geq 1$ have the generalized forms given by the next pair of equations in (3.16) and (3.17) [24, cf. (6.9)]

$$\sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_\alpha = \alpha^{n-1} \frac{\Gamma\left(n-1+\frac{2}{\alpha}\right)}{\Gamma\left(\frac{2}{\alpha}\right)} \tag{3.16}$$

$$\sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_\alpha (-1)^{n-k} = \sum_{k=0}^{n} \left[\begin{bmatrix} n \\ k \end{bmatrix}\right]_\alpha = 0 \tag{3.17}$$

3.3 Enumerative Properties

The primary motivation for considering the initial enumerations given in this section is to generalize the important generating function identities for the classical Stirling number triangles summarized in Table 351 of *Concrete Mathematics* [24, §7.4]. In general, the resulting generalizations for the triangles in (3.1) and (3.2) are more complex than that of the original Stirling number identities, though the results are key in characterizing the behavior of the generalized α-factorial function expansions. Additional enumerative properties for the triangles are established in the discussions of §4, §5, and §6.1.

To begin with, consider the next generalization of the classical identity of (3.18) [24, (7.48)] in equation (3.19) defined over the upper index $m \in [1, \infty) \subseteq \mathbb{N}$.

$$\sum_{n\geq 0} \begin{bmatrix} m \\ n \end{bmatrix} z^n = z^{\overline{m}} = z(z+1)\cdots(z+m-1) \tag{3.18}$$

$$\sum_{n\geq 0} \begin{bmatrix} m \\ n \end{bmatrix}_\alpha z^{n-1} = \frac{\alpha^{m-1}\Gamma(m-1+\frac{z+1}{\alpha})}{\Gamma(\frac{z+1}{\alpha})} \tag{3.19}$$

Next, let the function f_m be defined by equation (3.20).

$$f_m := \sum_{k=0}^{m}\sum_{j=0}^{k}\sum_{i=0}^{j} \left\{ \begin{matrix} m \\ k \end{matrix} \right\}_\alpha \begin{bmatrix} k \\ j \end{bmatrix} \left\{ \begin{matrix} j+1 \\ i+1 \end{matrix} \right\} \frac{(-\alpha)^{k-j}\; i!\; z^i}{(1-z)^i} \tag{3.20}$$

The form of the identity in equation (3.21) [24, (7.46)] can be extended by the forms of (3.22) and (3.23), as well as by the form of equation (3.24) for positive integer m [3] [22, cf. $A_{k,i}$].

$$\sum_{n=0}^{\infty} n^m\; z^n = \sum_{k=0}^{m} \left\{ \begin{matrix} m \\ k \end{matrix} \right\} \frac{k!\; z^k}{(1-z)^{k+1}} \tag{3.21}$$

$$\sum_{n=0}^{\infty} (n-\alpha+1)^m\; z^n = \sum_{k=0}^{m} \frac{[(1-z)^{k+1}]f_{m+1}}{(k+1)\;(1-z)^{k+1}} \tag{3.22}$$

$$\sum_{n=0}^{\infty} (n-\alpha+1)^m\; z^n = \sum_{j=1}^{m} \left\{ \begin{matrix} m \\ m+1-j \end{matrix} \right\}_{1/\alpha} \frac{(-\alpha)^{j-1}(m-j)!}{(1-z)^{m+1-j}} \tag{3.23}$$

$$\sum_{n=0}^{\infty} (\alpha n+\beta)^m\; z^n = \sum_{n=0}^{\infty} \left(\sum_{k=0}^{m} \binom{m}{k} \alpha^k \beta^{m-k} n^k \right) z^n \tag{3.24}$$

The classical "*double*" generating functions enumerating the original Stirling number triangles are defined by the following forms as [24, (7.54) and (7.55)]

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \left\{ \begin{matrix} n \\ m \end{matrix} \right\} w^m \frac{z^n}{n!} = e^{w(e^z-1)} \quad \text{and} \quad \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \begin{bmatrix} n \\ m \end{bmatrix} w^m \frac{z^n}{n!} = \frac{1}{(1-z)^w}$$

and yield the generalizations to the α-factorial triangle cases given in respective order by the next equations in (3.25) and (3.26).

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \left\{ \begin{matrix} n \\ m \end{matrix} \right\}_\alpha \frac{w^m z^n}{(n-1)!} = wze^z \left(e^{w(e^{\alpha z}-1)/\alpha} - 1 \right) \tag{3.25}$$

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \begin{bmatrix} n \\ m \end{bmatrix}_\alpha \frac{w^m z^n}{n!} = \frac{(1-\alpha z)^{-(w+1)/\alpha}}{(\alpha-w-1)} \left((\alpha-1)(1-\alpha z)^{(w+1)/\alpha} + w(\alpha z-1) \right) \tag{3.26}$$

It follows from equation (3.26) that for $m, n \geq 1$ and for all $k \in \mathbb{N}$, the results of equations (3.27) and (3.28) hold for the generalized triangle coefficients [cf. §6.1].

$$\sum_{k=1}^{m} \begin{bmatrix} n \\ k \end{bmatrix}_{\alpha} \frac{s^{k-1}}{n!} = [w^m z^n] \left(\frac{(1-z\alpha)^{-\frac{1+sw}{\alpha}} \left((\alpha-1)(1-z\alpha)^{\frac{1+sw}{\alpha}} - sw(1-\alpha z) \right)}{s(w-1)(1-\alpha+sw)} \right) \tag{3.27}$$

$$\sum_{k=1}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_{\alpha} \frac{s^{k-1}}{n!} = [z^n w^k] \left(\frac{\left(1-\frac{\alpha z}{w}\right)^{-\frac{1+sw}{\alpha}} \left(w\left(1-\frac{\alpha z}{w}\right)^{\frac{1+sw}{\alpha}} + \alpha z - w \right)}{(w-1)(1-\alpha+sw)} \right) \tag{3.28}$$

The result of the identity in (3.28) corresponds to the coefficients on a prescribed diagonal index of the full generating function in equation (3.27) [54, cf. §6.3]. Both of the identities are related to the α-factorial function expansion polynomial properties discussed in §6.1.

A generalization of the classical identity in (3.29) [24, (7.49)] can be derived from the form of (3.13) and is given by equation (3.30).

$$\sum_{n=0}^{\infty} \begin{Bmatrix} n \\ m \end{Bmatrix} \frac{z^n}{n!} = \frac{(e^z-1)^m}{m!} \tag{3.29}$$

$$\sum_{n=0}^{\infty} \begin{Bmatrix} n \\ m \end{Bmatrix}_{\alpha} \frac{z^n}{(n-1)!} = \frac{\alpha^{1-m}}{(m-1)!} \, z e^z (e^{\alpha z}-1)^{m-1} \tag{3.30}$$

It can be shown from equation (3.15) that for positive $m \in \mathbb{N}$ the following identity holds where the α-factorial polynomial $\sigma_n^{\alpha}(x)$ is defined formally in §5.2.1.

$$\frac{(m-1)!}{(n-1)!} \begin{bmatrix} n+1 \\ m \end{bmatrix}_{\alpha} = \sum_{i=0}^{n-1} \sum_{r=0}^{i} \left(\frac{(-1)^{n-1-i} \sigma_{n-1-i}^{\alpha}(N_c)\, \alpha^{r+1-m} (N_c+1)^{r+1-m} (2-2\alpha)^{i-r}}{(r+1-m)!\, (i-r)!} \right) \times$$
$$\times \left(1 + \frac{\alpha(N_c+1)(m-1)}{(r+2-m)} \right)$$

The identity provides the alternate generating function form for the analog of equation (3.31) [24, (7.50)] given in equation (3.32).

$$\sum_{n=0}^{\infty} \begin{bmatrix} n \\ m \end{bmatrix} \frac{z^n}{n!} = \frac{1}{m!} \operatorname{Log}\left(\frac{1}{1-z} \right)^m \tag{3.31}$$

$$\sum_{n=0}^{\infty} \begin{bmatrix} n+1 \\ m \end{bmatrix}_{\alpha} \frac{z^n}{(n-1)!} = \frac{(m-1+z)}{(m-1)!} \, z^{m-1} e^z \left(\frac{\alpha z e^{\alpha z}}{e^{\alpha z}-1} \right)^{N_c} \tag{3.32}$$

An alternate extension of (3.31) [24, (7.50)] is given by equation (3.33) and is derived from the forms in (5.24) and (5.25) where $t := 1$ and $\mathcal{S}_1(\alpha z) = -\operatorname{Log}(1-\alpha z)/(\alpha z)$ [cf. §5.3.5].

$$\sum_{n=0}^{\infty} \begin{bmatrix} n \\ m \end{bmatrix}_{\alpha} \frac{z^n}{n!} = \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} \frac{(-1)^{m+k} \operatorname{Log}(1-\alpha z)^{m+k} (1-\alpha)^k}{\alpha^{m+k} (m+k)\, k!\, (m-1)!} \right) z^n \tag{3.33}$$

3.4 Relations to Generalized Stirling Numbers

The triangles defined in §3.1 may be compared to several of the treatments given in the referenced literature on Stirling numbers and the related properties of single factorial function expansions. Particular generalizations and variations on the classical Stirling number triangles are discussed in the references by Adelberg and others [11; 30; 44; 43] [3, §7] [16, §3 and §4] [35, §1.3 and §2.4; cf. §3.1] [58, §4, (5.23), and (5.24)] [64, §3]. Combinatorial interpretations and examples for the *generalized Stirling number* sets are discussed elsewhere by Lang [37]. Additionally, several Stirling number forms and properties are defined in terms of the differences of more generalized factorial functions in the work by Charalambides and Koutras [15, §4].

3.4.1 Finite-Difference Properties of the Non-Central Stirling Numbers

The form of the *non-central Stirling numbers of the first kind* is discussed in Koutras' work [35, §1]. This section explores analogous properties of the (3.1) triangle for the non-central Stirling numbers of the first kind. The properties given for the row sums of the non-central triangles [35, Remark 3] are analogous in form to (3.16) and (3.17). There are several additional properties for the non-central Stirling numbers of the first kind, $s_a(n;\ k)$, that are similar to the forms of §3.2 and result from expanding the non-central Stirling triangle recurrences in a manner similar to the derivation of the properties for the (3.1) triangle forms.

To begin with, consider the recurrence form and corresponding conversion formula for the unsigned non-central Stirling numbers as follows [35, cf. (1.2) and (1.3)] [cf. §3.1]:

$$\begin{aligned}\bar{s}_a(n;\ k) &= (n+a-1)\ \bar{s}_a(n-1;\ k) + \bar{s}_a(n-1;\ k-1) + [n=k=0]_\delta \\ &= (-1)^{n-k}\ s_{-a}(n;\ k) = |\bar{s}_{-a}(n;\ k)|.\end{aligned}$$

By expanding the recurrence definitions it is possible to express both the signed and unsigned triangles of the non-central Stirling numbers of the first kind through the next identities [35, cf. §1.2].

$$\begin{aligned}s_a(n;\ k) &= \sum_{j=0}^{n-k} \begin{bmatrix} n \\ n-j \end{bmatrix} \binom{n-j-1}{k-1} (-1)^j a^{n-k-j} \\ s_a(n;\ k) &= \sum_{j=0}^{n-k} \binom{k-1+j}{k-1} (-1)^{n-k} n^j s_{-a}(n;\ j+k) \\ \bar{s}_a(n;\ k) &= \sum_{j=0}^{n-k} \binom{k-1+j}{k-1} n^j s_a(n;\ j+k)\end{aligned}$$

The form of the *non-central Stirling numbers of the second kind* is also discussed in Koutras' work [35, §2]. The following discussion briefly explores analogous properties of the (3.2) triangle for the non-central Stirling numbers of the second kind. Similar to several of the identities in terms of powers of the input variables in §3.2 and to the properties noted for the non-central numbers of the first kind, the next pair of identities extend the properties given by Koutras [35, cf. §2.2] for the non-central Stirling numbers of the second kind, $S_a(n;\ k)$.

$$\begin{aligned} S_a(n;\ k) &= \sum_{i=0}^{n-k} \left\{ {n-i \atop k} \right\} \binom{n-1}{i} (-a)^i \\ &= \sum_{i=0}^{n-k} \sum_{j=0}^{n-k-i} \left\{ {n-i-j \atop k} \right\} \binom{n-1}{i} \binom{n-i-1}{j} (-1)^{n-k-j}\ a^i\ (k+1)^j \end{aligned}$$

3.4.2 Comparison to the Unified Generalizations of the Classical Stirling Number Triangles

The articles authored by Hsu et. al. [28; 29] offer unified approaches to a number of separate generalizations of the classical Stirling cycle and subset triangles (first and second kinds, respectively). The work of Hsu and Shiue provides a more comprehensive discussion of unified properties that are analogous to the forms discussed in this article and so will be the focus of the Stirling number form comparisons addressed by this section. The α-factorial triangles of the first kind in (3.1) and second kind in (3.2) satisfy many similar properties to the *unified Stirling numbers*, though there are key distinctions in the forms from the treatment given in the references.

To begin with, the particular manner that the first triangle (3.1) may be considered as a generalization of the classical set of Stirling numbers of the first kind is precisely the context of the author's first memorable encounter with these numbers: *as factorial function expansions*. It appears that by considering the Stirling number generalizations as factorial function expansion coefficients some sense of the direct combinatorial meaning attached to the original triangle is obscured. In this case, the motivations of this article for generalizing the Stirling triangles gives an alternate, if separate, meaning to these triangles [11, cf. *r-Stirling numbers*].

The motivation for constructing the triangles discussed in §3.1 provides the non-dual triangle pair interpretations between the triangles of (3.1) and (3.2) that yield the properties analogous to the classical forms offered by the last sections and also to the unified generalizations discussed by Hsu's work. The specific Stirling-number-like relation between the first (3.1) and second (3.2) triangles defined by equations (3.5) and (3.6) is the key difference between the forms introduced by this article and the unified forms. Unlike both the classical Stirling triangles and the unified definitions, the (3.1) and (3.2) triangles do *not* conform to the typical *dual*, or "*conjugate*", relationship formed by the original triangles [24, cf. (6.33)] [55]. In contrast, the pair definition of $\{S^1,\ S^2\}$ given in the reference by Hsu [28] requires that the generalized Stirling triangles satisfy a symmetric relationship for the pair-based identities offered within that text. The distinction is particularly apparent when considering the relations of the separate α-factorial polynomial sequences of the first and second kinds in §5.3.1 [24, cf. Table 272; §6.2 and §6.5].

In place of the unified set pairs, the following pair of identities may be defined in terms of the *Gould polynomials*, $G_n(x;\ a,\ 1) := x\,(x-an-1)_{n-1}$, also denoted $x^{[a;\ n]}$, through equation (3.34) [15, cf. §4.1] [50, §1.4] [44, §1].

$$x^{[a;\ n]} = \sum_{k=0}^{n} t(n;\ k)\ x^k \quad \text{and} \quad x^n = \sum_{k=0}^{n} T(n;\ k)\ x^{[a;\ k]} \tag{3.34}$$

As formulated in §3.2 and §3.3, the pair of triangles defined by this article in the forms of (3.1) and (3.2), as well as the alternate generalization suggested by equation (3.34), still satisfy the orthogonality relations analogous to the unified form properties [15, cf. §5.1] [28, §1.3] and result in analogous enumerations compared to the generalized Stirling number sets that are defined in comparable forms by each of the unification articles [28, Thm. 2; Remarks 1 and 2] [29]. As noted in the text of *The Umbral Calculus* [50, §1.4], the case of the identities corresponding to the *central factorial polynomials*, denoted by the special case form of $x^{[n]} = x^{[-1/2;\ n]}$, is discussed in Riordan's book [48].

For comparison, note that the recurrence relation (3.35) provides a more general form of the (3.1) and (3.2) triangles, *and* the unified Stirling number triangles [28, Thm. 1], as well additional combinatorial triangles of interest such as the first and "second-order" Eulerian numbers noted in the discussion of §3.1.

$$\left|\begin{matrix} n \\ k \end{matrix}\right| = (\alpha n + \beta k + \gamma) \left|\begin{matrix} n-1 \\ k \end{matrix}\right| + (\alpha' n + \beta' k + \gamma') \left|\begin{matrix} n-1 \\ k-1 \end{matrix}\right| + [n = k = 0]_{\delta} \tag{3.35}$$

A more thorough consideration of the general and special case forms of the triangles defined by (3.35) is handled in the excellent reference on the topic [24, §5 and §6].

4 Symmetric Polynomial Transforms and Applications

Polynomial sequences and enumerative forms related to symmetric functions have a wide variety of combinatorial applications as discussed in several of the referenced works [11, §5] [23; 41] [39, cf. Prop. 2.1 (Proof 2) and Prop. 2.12] [54]. For example, the classically defined Stirling number triangles may be defined in terms of, and have several properties related to, symmetric polynomials [16, cf. §2] [43, cf. Prop. 2.1].

The key and defining properties of the (3.1) triangle are related to the standard symmetric functions phrased by the definitions for the *elementary symmetric polynomial index transformations* in §4.1. The results offered in the next several sections are of particular interest since many of the forms progress from the finite-difference-based properties for the α-factorial function expansions established by the previous section to fully analytic forms desired for the *distinct* triangle expansion coefficients.

4.1 Elementary Symmetric Polynomial Index Transforms

4.1.1 Index Transform Preliminaries

Let the *elementary symmetric polynomial* function [23, cf. §2] be defined by (4.1).

$$\begin{aligned} e_k(j) &:= [z^k]\left(\prod_{m=0}^{j}(1+z\ x_m)\right) \\ e_k(j) &= \sum_{0\leq<i_1<\cdots<i_k\leq j} x_{i_1}\cdots x_{i_k} + [k=0]_\delta\,[j\geq 1]_\delta \end{aligned} \tag{4.1}$$

The symmetric properties related to the definition of equation (4.1) are considered as symmetric index transforms of the index inputs from the transformation function x_m defined over the domain of natural numbers. Due to the distinction from the strictly symmetric polynomials given in terms of numbered distinct variables, the properties noted for the transformations do not necessarily result in *shift invariant* functions for the general case. The given symmetric interpretation can however be reconciled with the form of traditional symmetric polynomials by considering that the transformation function can be expanded in terms of the auxiliary notation for traditional elementary symmetric polynomials to obtain any desired shift invariant results as needed by applications external to the article [23, cf. §8].

The elementary symmetric function transforms considered by the applications in the context of this article are considered in a slightly more general form than the discussions of the next sections explicitly require. To begin with, define the special case index transform of the elementary symmetric function in (4.1) by the form of equation (4.2).

$$q_k(j) := [z^k]\left(\prod_{m=0}^{j}(1+z\ (g+cm))\right) \tag{4.2}$$

The symmetric index transform definition in (4.2) yields the exponential generating function in (4.3) that can be derived in terms of the constant parameters g and c that completely specify the transformation considered by the (4.2) form.

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{q_n(m)}{m!}\ w^m z^n = (1+gz)(1-cwz)^{\frac{-(1+(c+g)z)}{cz}} \tag{4.3}$$

The definitions of equations (4.2) and (4.3) yield the following identities where $\sigma_n(x)$ is a Stirling (convolution) polynomial [24, §6.2] and $B_n^{(a)}(z)$ is a generalized Bernoulli polynomial

[3; 13] [50, §2.2].

$$q_n(m) = \sum_{j=0}^{n} \binom{m+1-j}{n-j} \begin{bmatrix} m+1 \\ m+1-j \end{bmatrix} c^j \; g^{n-j}$$

$$\begin{aligned} q_n(m) &= \sum_{j=0}^{n} \left(c^j \; (m+1) \; \sigma_j \, (m+1) \left(1 - \frac{j}{m+1} \right) \right) \frac{(m+1)! \; g^{n-j}}{(m+1-n)! \; (n-j)!} \\ &= \quad \binom{m+1}{n} c^n \; B_n^{(m+2)} \left(m+1+\frac{g}{c} \right) \end{aligned}$$

4.1.2 Generalized Triangle-Related Symmetric Index Transforms

Consider the following identities phrased in similar recursive terms and form as the definition in equation (4.1) where $(a)_n = \Gamma(a+n)/\Gamma(a)$ denotes the *Pochhammer symbol*, or *rising factorial* function.

$$\begin{aligned} q_j &:= (1 + (\gamma + \alpha - \alpha \; j)z) \; q_{j-1} + [j=0]_\delta \qquad (4.4) \\ &= (-\alpha z)^{j-1}(1+\gamma z) \; \left(1 - \frac{1+\gamma z}{\alpha z} \right)_{j-1} \\ \sum_{j=0}^{\infty} q_j \; w^j \; &= \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} \left(\sum_{i=0}^{k} \binom{i+j-k}{j-k} \begin{bmatrix} j \\ i+j-k \end{bmatrix} \gamma^i \alpha^{k-i} \right) z^k w^j. \end{aligned}$$

The next generating function over z is given recursively by the form of equation (4.5) and is defined such that the requirement $[z^{i+1}]q_{n+i+1} = [z^n]M_i(z)$ holds in terms of the function of (4.4).

$$M_i(z) := \frac{(\gamma - \alpha i)M_{i-1}(z) - \alpha z M'_{i-1}(z)}{(1-z)} - \frac{(\alpha z - \gamma(1-z))}{(1-z)^3} [i=0]_\delta + \frac{[i=-1]_\delta}{(1-z)} \qquad (4.5)$$

The definition of (4.5) is similar to the polynomial forms considered by M. Ward in the generating functions for the triangle of Stirling numbers of the first kind [61, (2.41) and (2.5)]. Ward's article considers polynomials over z for each discrete index i that are defined by the following equation.

$$H_{i+1}(z) = (iz+i+1) \; H_i(z) + (1-z)zH'_i(z) + [i=0]_\delta$$

While the context of the recurrence forms differs between the separate works, the forms of the $H_i(z)$ polynomial recurrence may eventually have uses in suggesting additional properties for the $M_i(z)$ form of equation (4.5), if not in providing a full closed-form solution to the coefficient expressions related to the applications considered by this article.

The imposed equivalence condition implies that the recurrence definition in (4.5) can be defined in terms of the Stirling numbers of the first kind through equation (4.6)

$$M_i(z) = \sum_{n=0}^{\infty} \left(\sum_{k=0}^{i+1} \binom{n+k}{k} (-1)^{n+i+1} \alpha^{i+1-k} \gamma^k \begin{bmatrix} n+i+1 \\ n+k \end{bmatrix} \right) z^n, \qquad (4.6)$$

as well as in terms of the Stirling numbers on the series expansion index diagonals in the next equations of (4.7) and (4.8) [1].

$$[z^k]M_{i-k}(z) = \sum_{j=0}^{i-k+1} \left(\left[\begin{matrix} i+1 \\ j+k \end{matrix} \right] \binom{j+k}{k} (-\alpha)^{i-k+1-j}\ \gamma^j \right) - [k = i+1]_\delta \tag{4.7}$$

$$[z^k]M_{i-k}(z) = \sum_{j=0}^{i-k+1} \left(\frac{(j+k)\ (-\alpha)^{i-k+1-j}\ \gamma^j\ i!}{k!\ j!} \sigma_{i-k+1-j}\,(i+1) \right) - [k = i+1]_\delta \tag{4.8}$$

Let the shorthand $[z^n]M_i(z) := m_i(n)$ denote the coefficients that define the series expansion of the generating function (4.5). Then for the transform function $Q(j) := (\gamma - \alpha j)$ corresponding to $g := \gamma$ and $c := -\alpha$ in equation (4.2), the following symmetric index transform identities also hold for the coefficient forms over the indices $n \in \mathbb{N}$ and $i \in [-1,\ \infty] \subseteq \mathbb{Z}$.

$$[z^n]M_i(z) = \sum_{k=0}^{n} Q(k+i)\ m_{i-1}(k) + [i = -1]_\delta = q_{i+1}(n+i)$$

The specific symmetric-index-based property for the (3.1) triangle is given by equation (4.9) where $\gamma := (\alpha N_c + 1 - 2\alpha)$ [61, cf. §2].

$$\left[\begin{matrix} n \\ k \end{matrix} \right]_\alpha = [z^{k-2}]M_{n-k-1}(z) + [z^{k-1}]M_{n-k-2}(z) \tag{4.9}$$

Additionally, the identities of (4.10), (4.11), and (4.12) hold for the generalized triangle coefficients as given in terms of equation (4.9) when $n \in [3,\ \infty) \subseteq \mathbb{N}$ and where $\sigma_n\,(x)$ denotes the sequence of Stirling polynomials [24, §6.2].

$$\begin{aligned}
\left[\begin{matrix} n \\ k \end{matrix} \right]_\alpha &= \sum_{0 \leq i \leq 1} \left(\sum_{j=0}^{n-k-i} \left[\begin{matrix} n-2 \\ j+k+i-2 \end{matrix} \right] \binom{j+k+i-2}{k+i-2} \frac{(-\alpha)^{n-k-i-j}\ \gamma^j}{n!} \right) && (4.10) \\
&= \sum_{0 \leq i \leq 1} \left(\sum_{j=0}^{n-k-i} \frac{(j+k+i-2)\ (-\alpha)^{n-k-i-j}\ \gamma^j}{n(n-1)\ (k+i-2)!\ j!} \sigma_{n-k-i-j}\,(n-2) \right) && (4.11) \\
&= [z^n] \left(\frac{z^k\,((k-1)(\alpha(n-2)+2)z + (\alpha(n-2)+1)z^2 + (k-1)(k-2))}{n(n-1)(n-2)\ (\alpha z)^{2-n}\,(e^{\alpha z}-1)^{n-2}\, e^{-(\alpha(n-2)+1)z}\ \Gamma(k)} \right) && (4.12)
\end{aligned}$$

The last listing of the identities given for the (3.1) triangle coefficients are actually special case forms of the $M_i(z)$ series coefficients on the index diagonals defined by the forms of equations (4.7) and (4.8).

[1] The *Mathematica* package `Stirling.m` provides additional recursive forms for similar sums involving the Stirling or Eulerian numbers and hypergeometric terms through the routine `FindRec` implemented by the software package cited in the references section [49].

4.2 Properties and Enumeration Results for the Generalized j-Factorial Function Expansion Coefficient Triangles

4.2.1 Initial Properties

Let the function $Q_\alpha(j) := (1-\alpha(j+2))$ and define the next symmetric index transform by the following equation corresponding to the case where $x_m := Q_\alpha(m)$ in the product expansion for the function $e_k(j)$ of equation (4.1).

$$q_k^\alpha(j) := [z^k]\left(\prod_{m=0}^{j}(1+z\ Q_\alpha(m))\right)$$

In general, for integer index $n \geq 3$, the triangle in (3.1) can be defined in terms of the given expansion forms and symmetric properties through the following equations.

$$p_n^\alpha(0) := \prod_{m=0}^{n-3}(\alpha n+1-(m+2)\alpha) = \sum_{k=0}^{n-2}\begin{bmatrix}n-1\\k+1\end{bmatrix}_\alpha (-1)^n\ (2\alpha-\alpha n-2)^k$$

$$p_n^\alpha(0) = \prod_{m=0}^{n-3}(\alpha n+Q_\alpha(m)) = \sum_{k=0}^{n-2} q_{n-2-k}^\alpha(n-3)\ (\alpha n)^k \tag{4.13}$$

$$p_n^\alpha(j) = \sum_{i=j}^{n-2}\binom{i}{j}\frac{[n^i]\ p_n^\alpha(0)}{\alpha^j}\ n^{i-j} = \sum_{i=j}^{n-2}\binom{i}{j}\ q_{n-2-i}^\alpha(n-3)\ (\alpha n)^{i-j} \tag{4.14}$$

The coefficients of $p_n^\alpha(0)$ in (4.14) are given by the next form of equation (4.15).

$$[n^i]\ p_n^\alpha(0) = \sum_{k=i}^{n-2}\begin{bmatrix}n-1\\k+1\end{bmatrix}_\alpha\binom{k}{i}\ (-1)^{n+i}\ \alpha^i\ (2\alpha-2)^{k-i} \tag{4.15}$$

An identity for $p_n^\alpha(j)$ resulting from the coefficient form of the last equation is given by (4.16).

$$p_n^\alpha(j) = \sum_{i=0}^{n-2-j}\sum_{k=0}^{n-2}\begin{bmatrix}n-1\\k+1\end{bmatrix}_\alpha\binom{i+j}{j}\binom{k}{i+j}\ (-1)^{n-2-j-i}\ \alpha^i\ n^i\ (2\alpha-2)^{k-j-i} \tag{4.16}$$

The symmetric-based form for the triangle coefficients of (3.1) then results in the next expression given in equation (4.17).

$$\begin{bmatrix}n\\k\end{bmatrix}_\alpha = p_n^\alpha(k-1)+p_n^\alpha(k-2) \tag{4.17}$$

The same construct involved in the formulation of the so termed "binomial derivative" identities in §3.2 applies to the identities given in this section. Observe that if the forms are treated as polynomials in n with discrete powers independent of the input variable, equation (4.14) may be considered as a multiple derivative with respect to n. However, note that differentiating the same polynomial with the degree of the lead term defined in terms if the input variable as the expression n^{n-k} gives a result that differs from the desired forms considered by the finite-difference approaches in this context. This distinction may be summarized by comparing the form of $\frac{\partial}{\partial x}[x^p]$ versus the form that results from the variable-dependent expression in evaluating $\frac{\partial}{\partial x}\left[x^{x-k}\right]$ for fixed $p, k \in \mathbb{N}$.

4.2.2 An Exponential Generating Function

Let the function $R_\alpha(j) := (\alpha N_c + 1 - (j+2)\alpha)$ and define the following symmetric index transform corresponding to the case where $x_m := R_\alpha(m)$ in the function $e_k(j)$ of equation (4.1).

$$r_k^\alpha(j) := [z^k]\left(\prod_{m=0}^{j}(1 + z\ R_\alpha(m))\right)$$

The form of of the triangle coefficients in (3.1) is related to the given index transform through the next equation.

$$\begin{bmatrix} n \\ k \end{bmatrix}_\alpha = r_{n-1-k}^\alpha(n-3) + r_{n-k}^\alpha(n-3)$$

Next, let the function $\mathrm{QPS} : \mathbb{N} \to \mathbb{Z}$ be defined and enumerated as in the form of equations (4.18) and (4.19).

$$\mathrm{QPS}(m) := z^{j+2}\ (z+1)\prod_{j=0}^{m}(1 + R_\alpha(j)z^{-1}) \tag{4.18}$$

$$\mathrm{QPS}(m) \ = \frac{(-1)^{m+1}\alpha^{m+1}z(z+1)\Gamma(m+3-N_c-\frac{z+1}{\alpha})}{\Gamma(2-N_c-\frac{z+1}{\alpha})} \tag{4.19}$$

Additionally, define the exponential generating function for the function $\mathrm{QPS}(m)$ by equation (4.20).

$$\widehat{\mathrm{QPS}}(w;\ z) := \sum_{j=0}^{\infty}\frac{\mathrm{QPS}(j)}{j!}\ w^j = z(z+1)(z+1+(N_c-2)\ \alpha)\ (1+\alpha w)^{N_c-3+\frac{z+1}{\alpha}} \tag{4.20}$$

The form of the (3.1) triangle coefficients resulting from the terms in the series expansion of the exponential generating function can then be expressed by equation (4.21).

$$\begin{bmatrix} n \\ k \end{bmatrix}_\alpha = [z^k]\,\mathrm{QPS}(n-3) = (n-3)!\ [z^k w^{n-3}]\widehat{\mathrm{QPS}}(w;\ z) \tag{4.21}$$

The coefficients in the series expansion of $\widehat{\mathrm{QPS}}(w;\ z)$ involved in evaluating the form of the equation in (4.21) are given by

$$[w^m z^n]\widehat{\mathrm{QPS}}(w;\ z) = \sum_{i+j+k=n} q_1(i;\ n)q_2(j;\ n)q_3(k;\ n) \tag{4.22}$$

where the functions $q_i(m;\ n)$ in equation (4.22) are defined as follows [25, cf. §8]:

$$\begin{aligned}
q_1(m;\ n) =& [w^m]\,\mathrm{Log}(1+\alpha w)^{n-3} \\
q_2(m;\ n) =& \frac{\alpha^{m+1-n}}{\Gamma(n)}\binom{N_c-3+\frac{1}{\alpha}}{m} \\
q_3(m;\ n) =& \frac{2\alpha^m(\alpha N_c-2\alpha+1)\Gamma(m-M_c-1)(-\psi^{(0)}\,(m-M_c-1)+\psi^{(0)}\,(1-M_c))+1)}{\Gamma(m+1)\ \Gamma(1-M_c)} \\
&+\frac{(-1)^{m+1}\alpha^{m+2}}{m}\ (n-1)(N_c-2).
\end{aligned}$$

Note that the powers of the natural logarithm function in $q_1(m;\ n)$ may be considered in terms of standard Stirling number properties through equation (3.31) for negative integer exponents, the duality identity given by $\left[{n \atop k}\right] = \left\{{-k \atop -n}\right\}$ [24, (6.33)], and the polynomial identities of Table 272 as listed in the *Concrete Mathematics* reference [24, §6.2].

The discussion on fixed variable notation outlined in §1 is also relevant to the forms defined by the section. In particular, one notable caveat point of the fixed variable notation appears in evaluating inputs to the gamma function, $\Gamma(z)$, and *polygamma function*, $\psi^{(k)}(z)$, in the function $q_3(m;\ n)$ defined through equation (4.22). The fixed parameter M_c in this case *cannot* be evaluated at exact integer values until after corresponding non-constant variable index m has completely defined the symbolic expansions in the identity. A computer-based application such as *Mathematica* can be used to verify the special utility of the expansions that result from this example case of the fixed variable notation for the index m in terms of the parameter M_c.

4.2.3 Other Forms of the Generalized Triangle Coefficients

Consider the product function definition and expansion identity given by the form of the following equations.

$$\widetilde{p}(n) := \frac{(-1)^n \alpha^{n-2}\gamma}{n!} \prod_{m=1}^{n-2}\left(m - \frac{\gamma}{\alpha}\right) = \frac{(-1)^{n+1}\alpha^{n-1}\Gamma(n-1-\frac{\gamma}{\alpha})}{\Gamma\left(-\frac{\gamma}{\alpha}\right)\Gamma(n+1)}$$

$$\widetilde{p}(n) = \sum_{j=1}^{n} \left[{n-1 \atop j}\right] \frac{(-1)^{n-1-j}}{n!} \alpha^{n-1-j}\gamma^j$$

The k^{th} derivatives of the function $\widetilde{p}(n)$ can be expressed by the following forms where $\sigma_n(x)$ denotes the sequence of Stirling polynomials [cf. §5] and $E_n(x) = \int_1^\infty e^{-xt}t^{-n}dt$ denotes the *exponential integral* function.

$$\begin{aligned}
\frac{\partial^k}{\partial\gamma^k}\left[\frac{\widetilde{p}(n)}{k!}\right] &= \frac{\alpha^{n-1}}{n!} \sum_{j=1}^{n} \binom{j}{k} \left[{n-1 \atop j}\right] \frac{(-1)^{n-1-j}\ \gamma^{j-k}}{\alpha^j} \\
&= \sum_{j=0}^{n-2} \frac{(j+1)\ \gamma^{j+1-k}}{k!\ (j+1-k)!}\ \frac{(-\alpha)^{n-2-j}}{n}\ \sigma_{n-2-j}(n-1) \\
&= [z^n]\left(\frac{\mathrm{Log}(1+\alpha z)E_{-k}\left((1+\frac{\gamma}{\alpha})\,\mathrm{Log}\left(\frac{1}{1+\alpha z}\right)\right) + \alpha z(1+\alpha z)^{\frac{\gamma}{\alpha}}}{(n-1)\,\mathrm{Log}(1+\alpha z)^{-k}\ \alpha^{k+1}\Gamma(k+1)}\right) + \delta_{n,1}\delta_{k,0}
\end{aligned}$$

Next, consider the generating function for $\widetilde{p}(n)$ defined by the form of the following equation.

$$P(z) := \sum_{n=0}^{\infty} \widetilde{p}(n) z^n = \frac{(1+\alpha z)^{1+\frac{\gamma}{\alpha}}}{(\alpha+\gamma)}$$

The form of the k^{th} derivatives of $\widetilde{p}(n)$ can then be enumerated precisely by considering the properties of the derivatives of the ordinary generating function $P(z)$ as given in terms of the equations below [50, cf. §4.4]:

$$\begin{aligned}
\frac{\partial^{(k)}}{\partial\gamma^{(k)}}\left[\frac{\widetilde{p}(n)}{k!}\right] &:= [z^n]\ \frac{\partial^{(k)}}{\partial\gamma^{(k)}}\left[\frac{P(z)}{\Gamma(k+1)}\right] \\
&= [z^n]\left(\sum_{j=0}^{k}\frac{(-1)^{k+j}\operatorname{Log}(1+\alpha z)^j}{j!\ \alpha^j(\alpha+\gamma)^{k+1-j}}\ (1+\alpha z)^{1+\frac{\gamma}{\alpha}}\right) \\
&= [z^n]\left(\frac{(-1)^k\ \Gamma(k+1;\ -(1+\frac{\gamma}{\alpha})\operatorname{Log}(1+\alpha z))}{(\alpha+\gamma)^{k+1}\Gamma(k+1)}\right).
\end{aligned}$$

For positive $k \in \mathbb{N}$, the k^{th} derivative of $P(z)$, denoted d_k, satisfies the recurrence relation given in equation (4.23) and can be enumerated by the corresponding ordinary generating function in equation (4.24).

$$d_k = \frac{-k}{(\alpha+\gamma)}d_{k-1} + \frac{(1+\alpha z)^{1+\frac{\gamma}{\alpha}}\operatorname{Log}(1+\alpha z)^k}{\alpha^k(\alpha+\gamma)} - \frac{(1+\alpha z)^{1+\frac{\gamma}{\alpha}}}{(\alpha+\gamma)^2}[k=1]_\delta \tag{4.23}$$

$$\begin{aligned}
\sum_{k=1}^{\infty} d_k\ t^k = \frac{e^{-(t-1)(\alpha+\gamma)/t}}{\alpha t(\alpha+\gamma)^2}\Big((1+\alpha z)^{\frac{\alpha+\gamma}{\alpha}}((\alpha+\gamma)\operatorname{Log}(1+\alpha z)-\alpha)+\alpha e^{\alpha+\gamma}(\alpha+\gamma)^2\ \times \\
\times\int_1^t \frac{e^{-\frac{\alpha+\gamma}{t}}(1+\alpha z)^{\frac{\alpha+\gamma}{\alpha}}\operatorname{Log}(1+\alpha z)(\operatorname{Log}(1+\alpha z)t-\operatorname{Log}(1+\alpha z)t+\alpha)}{\alpha(\alpha-\operatorname{Log}(1+\alpha z)t)}\,dt\Big)
\end{aligned} \tag{4.24}$$

Identities for special case columns of the generalized triangle coefficients in (3.1) may then be derived by computing the iterated derivatives of $\widetilde{p}(n)$ from equation (4.25) and in the coefficient form of equation (4.26) where the second form is derived from the recursive definition provided by (4.23) [1, cf. Prop. 2; Prop. 3; and (17)].

$$\frac{1}{n!}\begin{bmatrix}n+1\\k\end{bmatrix}_\alpha = \frac{\partial^{k-1}}{\partial\gamma^{k-1}}\left[\frac{\widetilde{p}(n)}{(k-1)!}\right]+\frac{\partial^{k-2}}{\partial\gamma^{k-2}}\left[\frac{\widetilde{p}(n)}{(k-2)!}\right] \tag{4.25}$$

$$= [z^n]\left(\frac{n\operatorname{Log}(1+\alpha z)E_{1-k}\left(-(n+\frac{1}{\alpha})\operatorname{Log}(1+\alpha z)\right)+(1+\alpha z)^{\frac{1}{\alpha}+n}}{\alpha^{k-1}\operatorname{Log}(1+\alpha z)^{1-k}\ \Gamma(k)}\right) \tag{4.26}$$

The parameter γ in (4.25) must again be evaluated after the symbolic computation of the partial derivative terms and in terms of the substitution defined by $\gamma := (\alpha N_c + 1 - \alpha)$.

The definition of equation (4.25) may also be used to construct additional "*horizontal*" and "*vertical*" recurrence relations for the (3.1) triangle coefficients that are analogous to the forms defined for the classical Stirling number triangles by the work of Charalambides and Singh [16, p. 2543]. The triangle identity in equation (4.25) suggests a systematic procedure for recursively computing more specific closed-form expressions for fixed columns of the (3.1) triangle in terms of the gamma, incomplete gamma, polygamma, exponential integral, and related functions. In particular, these closed-forms include the special cases for the classically-defined triangle of Stirling numbers of the first kind as specifically detailed in several applications related to sequences of *harmonic numbers* that are discussed next and in §4.3.2.

4.3 Harmonic Number Applications

4.3.1 Generating Functions and Other Identities for the Sequence of First-Order Harmonic Numbers

The sequence of *first-order harmonic numbers* [A001008; and A002805] is defined classically through the identity [24]

$$H_n := H_{n-1} + \frac{1}{n} + [n=1]_\delta = \sum_{k=1}^{n} \frac{1}{k}.$$

The first-order harmonic numbers also satisfy the following properties in terms of the symmetric-index-transform-related function forms in §4.1.2 where $\sigma_n(x)$ denotes the sequence of Stirling polynomials [32] [24, §6.2] [cf. §5].

$$\begin{aligned}
H_n &= \frac{1}{n!}\begin{bmatrix} n+1 \\ 2 \end{bmatrix} = \frac{1}{n!}\left([z^0]M_{n-2}(z) + [z^1]M_{n-3}(z)\right) \\
&= \frac{1}{n!}\sum_{k=1}^{n-1}\begin{bmatrix} n-1 \\ k \end{bmatrix}(-1)^{n-1-k}\alpha^{n-1-k}\gamma^{k-1}(\gamma+k) \\
&= \sum_{k=0}^{n-2}\frac{(-\alpha)^{n-2-k}}{n}\sigma_{n-2-k}(n-1)(1+\gamma+k)\frac{\gamma^k}{k!}
\end{aligned}$$

The harmonic number sequence corresponds to the case of the (3.1) triangle properties for Stirling numbers of the first kind when $\alpha := 1$ and where $\gamma := \alpha(N_c+1)+1-2\alpha = N_c$.

The form of the last property for H_n provides an enumeration for the harmonic number sequence that is derived from the ordinary generating function for the Stirling (convolution) polynomial sequence from (5.1) [24, (6.50)] and that results in a series expansion for that enumeration expressed in terms of the *Nörlund polynomials* $B_n^{(x)}$ as given by the following equations [cf. §5.3.3; (4.32) and (3.32)].

$$\begin{aligned}
H_S(z) &:= \sum_{n=1}^{\infty} H_n z^n = z + \frac{(-1)^{N_c}\, z^{N_c+1}(e^z-1)\,(N_c\, z + N_c + 1)(e^{-z}-1)^{-N_c}}{(N_c-1)\, N_c} \\
[z^n]H_S(z) &:= H_n = [n=1]_\delta + \frac{(-1)^{n-2}\,(n+1)B_{n-2}^{(n)}}{n!} + \sum_{j=3}^{n}\frac{(-1)^{n-j}\,(1+nj)}{n(n-1)(j-1)!(n-j)!}B_{n-j}^{(n)}
\end{aligned}$$

It follows from the symmetric identities that the harmonic number sequence is enumerated by the ordinary generating function in (4.27), the exponential generating functions in equations (4.28) and (4.29), and by the *doubly*, or *twice*, exponential generating function in (4.30), where $L_n(x)$ denotes the sequence of *Laguerre polynomials* [50, §3.1], $\psi^{(0)}(z)$ denotes the polygamma function (*digamma function*), γ_{E} denotes the *Euler-Mascheroni constant*, and $I_k(z)$ denotes the *modified Bessel function of the first kind* [24, §5.5] [50, cf. §1.7].

$$H(z) := \sum_{n=1}^{\infty} H_n z^n = \frac{(1+z)^{N_c+1}((N_c+1)\operatorname{Log}(1+z)+N_c)}{(N_c+1)^2} \tag{4.27}$$

$$\widehat{H}(z) := \sum_{n=1}^{\infty} \frac{H_n}{n!} z^n = \frac{-\left(4N_c+4(N_c+1)^2 z+(N_c+1)^2 z^2\right)}{4(N_c+1)^2} \tag{4.28}$$

$$+ \frac{N_c}{(N_c+1)^2} L_{N_c+1}(-z) + \frac{1}{(N_c+1)} \frac{\partial}{\partial N_c} L_{N_c+1}(-z)$$

$$= \frac{e^z(z+1)(\gamma_{\mathrm{E}} N_c + N_c \psi^{(0)}(N_c+2) + \psi^{(0)}(N_c+2) + \gamma_{\mathrm{E}} - 1)}{(N_c+1)^2} \tag{4.29}$$

$$\widetilde{H}(z) := \sum_{n=1}^{\infty} \frac{H_n}{(n!)^2} z^n = \frac{(\psi^{(0)}(N_c+2)+\gamma_{\mathrm{E}}-1+(\psi^{(0)}(N_c+2)+\gamma_{\mathrm{E}})N_c)}{(N_c+1)^2\left(I_0(2\sqrt{z})+\sqrt{z}I_1(2\sqrt{z})\right)^{-1}} \tag{4.30}$$

The forms of equations (4.29) and (4.30) are derived from (4.28) by noting the particular series expansion forms for the Laguerre polynomials as follows:

$$\begin{aligned}
[z^n] L_{N_c+1}(-z) &= \frac{(-1)^n \; N_c \Gamma(n-N_c-1)}{(N_c+1)^2 \; \Gamma(n+1) \; \Gamma(-N_c-1)} \\
&= \frac{(N_c+1)}{(n!)^2} \left(\prod_{j=0}^{n-2} (N_c-j) \right) [n \geq 1]_\delta + [n=0]_\delta \\
&= \frac{\Gamma(N_c+2)}{(n!)^2} [n \geq 1]_\delta + [n=0]_\delta \\
[z^n] \frac{\partial}{\partial N_c} \left[L_{N_c+1}(-z) \right] &= \frac{(-1)^{n+1} \; \Gamma(n-N_c-1)(\psi^{(0)}(-N_c-1)\, \psi^{(0)}(n-N_c-1))}{\Gamma(n+1) \; \Gamma(-N_c)} \\
&= \frac{1-(n+1)\psi^{(0)}(n+1)}{\Gamma(n+1)}.
\end{aligned}$$

Given the form of equation (4.27), it is possible to formulate additional identities for the first-order harmonic numbers from the series coefficients in that enumeration of the sequence. Specifically, for $k \in [1, \infty) \subseteq \mathbb{N}$ and for the index $N_c \mapsto K_c$, the following properties are derived from the ordinary generating function for $H(z)$.

$$\begin{aligned}
H_k := [z^k] H(z) &= \sum_{j=1}^{k} \binom{K_c+1}{k-j} \frac{(-1)^{j+1}}{j\;(K_c+1)} + \frac{K_c}{(K_c+1)^2} \binom{K_c+1}{k} \\
&= \sum_{j=1}^{k} \binom{k+1}{k-j} \frac{(-1)^{j+1}}{j\;(k+1)} + \frac{k}{k+1} \\
&= \gamma_E - \frac{1}{k+1} + \psi^{(0)}(k+2)
\end{aligned}$$

The article by Adamchik also notes the related identity for the first-order harmonic numbers in terms of the Stirling triangle coefficients given by the next equation [1, §4] [24, cf. (6.58)].

$$\begin{bmatrix} n \\ 2 \end{bmatrix} = (n-1)!\ H_{n-1} = (n-1)!\ (\gamma + \psi(n))$$

The enumeration of (4.28) provides another identity for the sequence given by the following equation over the positive integer index for n [cf. (4.31); §4.3.2].

$$\frac{H_n}{n!} = \frac{(-1)^n\ \Gamma(n-N_c-1)}{(N_c+1)\ \Gamma(-N_c-1)\ \Gamma(n+1)^2} \left(\frac{N_c}{N_c+1} + \psi^{(0)}\left(-N_c-1\right) - \psi^{(0)}\left(n-N_c-1\right)\right)$$

4.3.2 Properties of the Stirling Numbers of the First Kind

The first result for the triangle of Stirling numbers of the first kind corresponds to the fixed column index $k := 2$ [A000254] and is a well-known identity related to the sequence of first-order harmonic numbers [24, §6.3]. The identity is rephrased by equation (4.31) in terms of the Stirling triangle result of (4.25) when $\alpha := 1$ and for the same fixed $k := 2$.

$$H_n = \frac{\partial}{\partial\gamma}\left[\widetilde{p}(n)\right] + \widetilde{p}(n) = \frac{(-1)^n\ \Gamma(n-N_c-1)(\psi^{(0)}\left(n-N_c-1\right) - \psi^{(0)}\left(-N_c\right) - 1)}{\Gamma(n+1)\ \Gamma(-N_c)} \tag{4.31}$$

The original harmonic number identity is equivalent to the property of the generalized Bernoulli polynomials discussed in the book *The Umbral Calculus* [50, pp. 99–100]. The Bernoulli polynomial form of the identity is stated in equation (4.32) as it appears in the reference.

$$H_n = 1 + \cdots + \frac{1}{n} = \frac{(-1)^{n-1}}{(n-1)!}\ B_{n-1}^{(n+1)}(0) \tag{4.32}$$

The second result is another well-known identity that relates the Stirling triangle for the fixed column index $k := 3$ [A000399] [24; 34] and the first-order and second-order harmonic number sequences [A001008; A002805; A007406; and A007407]. This classical result is restated by equation (4.33) and is then rephrased by the closed-form identity in terms of the polygamma function in (4.34).

$$\frac{1}{2}\left(H_n^2 - H_n^{(2)}\right) = \frac{1}{n!}\begin{bmatrix} n+1 \\ 3 \end{bmatrix} = \frac{\partial^2}{\partial\gamma^2}\left[\frac{\widetilde{p}(n)}{2}\right] + \frac{\partial}{\partial\gamma}\left[\widetilde{p}(n)\right] \tag{4.33}$$

$$\begin{aligned} \frac{1}{2}\left(H_n^2 - H_n^{(2)}\right) = \frac{(-1)^{n+1}\Gamma(n-N_c-1)}{2\ \Gamma(n+1)\ \Gamma(-N_c)} \Bigg(& \psi^{(1)}\left(n-N_c-1\right) - \psi^{(1)}\left(-N_c\right) \\ & + (\psi^{(0)}\left(-N_c\right) - \psi^{(0)}\left(n-N_c-1\right))(2 - \psi^{(0)}\left(n-N_c-1\right) + \psi^{(0)}\left(-N_c\right))\Bigg) \end{aligned} \tag{4.34}$$

Additional identities related to harmonic number sequences that are closely related to the form of equations (4.31) and (4.34) can be constructed in terms of the polygamma functions. These additional constructions can be derived from the column-based properties of the Stirling triangle as discussed in §4.2.3 in order to obtain related identities for the other fixed triangle columns as closed-form functions over the row index n. For example, when the column index is the fixed $k := 4$ [A000454], the next identity relates the Stirling number triangle and the integer-order *generalized harmonic number* sequences [A001008; A002805; A007406; A007407; A007408; and A007409] [1, §2].

$$\frac{1}{n!}\begin{bmatrix}n+1\\4\end{bmatrix} = \frac{1}{6}\left(H_n^3 - 3H_nH_n^{(2)} + 2H_n^{(3)}\right)$$

More generally, consider the function defined recursively by equation (4.35) where the generalized *r-order harmonic numbers* are defined as the finite partial sum $H_n^{(r)} := \sum_{k=1}^{n} k^{-r}$.

$$w(n;\ m) := \sum_{k=0}^{m-1} (1-m)_k\ H_{n-1}^{(k+1)}\ w(n;\ m-1-k) + [m=0]_\delta \tag{4.35}$$

From the definition provided in (4.35) it follows that the triangle of Stirling numbers of the first kind can be expressed in terms of the integer-order harmonic number sequences as in equation (4.36) where $k \in [1,\ n+1] \subseteq \mathbb{N}$ [1, cf. (5)] [44, cf. *hyperharmonic numbers*].

$$\frac{1}{n!}\begin{bmatrix}n+1\\k\end{bmatrix} = \frac{w(n+1;\ k-1)}{(k-1)!} \tag{4.36}$$

The forms obtained from (4.36) can then once again be rephrased in terms of the results in equations (4.25) and (4.26) through the procedure discussed in §4.2.3.

4.3.3 Properties of the r-Order Harmonic Number Sequences

The article by Adamchik provides a concise definition of the r-order harmonic numbers in terms of higher-order polygamma functions through the next equation for positive $r \in \mathbb{N}$ [1, §1; (14)].

$$H_n^{(r)} = \frac{(-1)^{r-1}}{(r-1)!}\ \left(\psi^{(r-1)}(n+1) - \psi^{(r-1)}(1)\right) = \int_0^1 \frac{\operatorname{Log}(t^{-1})^{r-1}}{(r-1)!}\frac{1-t^n}{1-t}\ dt \tag{4.37}$$

As remarked for the generalized (3.1) triangle forms defined by equation (4.26) in the previous section, the finite expansions in terms of polygamma functions that can be obtained for each separate fixed column index of the classical Stirling and generalized α-factorial triangles also serve as variations of the closed-form expansions for cases of the r-order harmonic number sequences. The following equations summarize the forms of the first several non-trivial cases of the positive integer-order harmonic number sequences given in terms the Stirling triangle coefficients for $\alpha := 1$.

$$\begin{aligned}
H_n^{(2)} &= \frac{1}{(n!)^2}\begin{bmatrix}n+1\\2\end{bmatrix}^2 - \frac{2}{n!}\begin{bmatrix}n+1\\3\end{bmatrix} \\
H_n^{(3)} &= \frac{1}{(n!)^3}\begin{bmatrix}n+1\\2\end{bmatrix}^3 - \frac{3}{(n!)^2}\begin{bmatrix}n+1\\2\end{bmatrix}\begin{bmatrix}n+1\\3\end{bmatrix} + \frac{3}{n!}\begin{bmatrix}n+1\\4\end{bmatrix} \\
H_n^{(4)} &= \frac{1}{(n!)^4}\begin{bmatrix}n+1\\2\end{bmatrix}^4 - \frac{4}{(n!)^3}\begin{bmatrix}n+1\\2\end{bmatrix}^2\begin{bmatrix}n+1\\3\end{bmatrix} + \frac{4}{(n!)^2}\begin{bmatrix}n+1\\2\end{bmatrix}\begin{bmatrix}n+1\\4\end{bmatrix} + \frac{2}{(n!)^2}\begin{bmatrix}n+1\\3\end{bmatrix}^2 \\
&\quad - \frac{4}{n!}\begin{bmatrix}n+1\\5\end{bmatrix}
\end{aligned}$$

In the general case, the closed-forms in terms of the polygamma functions that result from computations of these first several sequences of the r-order harmonic numbers from the identities of §4.2.3 are significantly more involved than that of equation (4.37). That being said, the closed-forms are interesting consequences of the identities in the previous several sections and the treatment of the constant parameter N_c yields distinct forms that simplify the behavior described by the row indexed results somewhat.

The forms of the r-order harmonic sequences enumerated from the expansion of these properties can be precisely expressed from the recurrence definition of equation (4.35). Let the exponential generating function over the single index m for the series terms from (4.35) be defined as in equation (4.38).

$$\widehat{W}_n(z) := \sum_{m=1}^{\infty} \frac{(-1)^m \ w(n+1;\ m)}{m!} \ z^m \tag{4.38}$$

It then follows that the generating function for the k-order harmonic sequences be defined by the forms of the next set of equations.

$$\begin{aligned} H_n(z) &:= \sum_{k=1}^{\infty} H_n^{(k)} \ z^k \\ &= \frac{z}{2} \ \frac{\partial}{\partial z}\left[\frac{\widehat{W}_n(z)^2}{2\ (2-\widehat{W}_n(z)^2)}\right] - z \ \frac{\partial}{\partial z}\left[\int_0^1 \frac{\widehat{W}_n(z)}{(1-x^2\ \widehat{W}_n(z)^2)} dx\right] \\ &= \frac{(4-4\widehat{W}_n(z)-4\widehat{W}_n(z)^2+4\widehat{W}_n(z)^3+\widehat{W}_n(z)^4)\ \widehat{W}_n'(z)\ z}{(\widehat{W}_n(z)-1)(\widehat{W}_n(z)+1)(\widehat{W}_n(z)^2-2)^2} \end{aligned} \tag{4.39}$$

The Stirling triangle defines the coefficients of (4.38) through the identity [1, cf. (5)]

$$w(n+1;\ m) = \begin{bmatrix} n+1 \\ m+1 \end{bmatrix} \frac{m!}{n!}$$

when the index inputs correspond to the standard bounds on positive non-zero entries of the triangle. However, it should be noted that in general the equivalence does not hold in both directions. That is, the Stirling triangle coefficients are defined by the enumeration of (4.35), but the form of the recurrence in equation (4.35) is not completely specified by the index cases corresponding to the Stirling triangle and the well-known classical series for that form. Finding a closed-form expression for the exponential generating function in (4.38), and then the resulting identities for the generating function in (4.39), remains an interesting open problem.

4.3.4 Example: Approximations of the Euler-Mascheroni Constant

The Euler-Mascheroni constant, γ_{E}, [A001620] is defined by the following limit of the difference between the n^{th} first-order harmonic number and natural logarithm evaluated at n.

$$\lim_{n\to\infty} (H_n - \mathrm{Log}(n)) = \lim_{n\to\infty} \gamma_n := \gamma_{\mathrm{E}} \approx 0.5772156649$$

The approximate values of the incremental constants, denoted γ_n, may be computed from the closed-form given by equation (4.31). A summary of computations of the approximations of the constant over increasing inputs n is given in Table 4.1.

Table 4.1: Euler Gamma Constant Approximations

n	$\log_{10}(n)$	H_n	$\mathrm{Log}(n)$	γ_n	$\lvert\gamma_{\mathrm{E}} - \gamma_n\rvert$
1	0	1.000000000	0	1.000000000	0.4227843351
10	1	2.928968254	2.302585093	0.6263831610	0.04916749607
100	2	5.187377518	4.605170186	0.5822073317	0.004991666750
1000	3	7.485470861	6.907755279	0.5777155816	0.0004999166667
10000	4	9.787606036	9.210340372	0.5772656641	0.00004999916667
100000	5	12.09014613	11.51292546	0.5772206649	$4.999991667 \times 10^{-6}$
1000000	6	14.39272672	13.81551056	0.5772161649	$4.999999167 \times 10^{-7}$

5 The j-Factorial Polynomials

5.1 Motivation and Background

The definition of the *Stirling convolution polynomial* sequence, denoted by $\sigma_n(x)$, and corresponding enumerations suggest a generalization that results in an extended set of parametrized α-factorial polynomial sequences, denoted by $\sigma_n^{\alpha}(x)$ [24, §6; cf. (7.52)] [32] [3, cf. §4] [50, cf. §4.8]. Additional polynomials related to the classical Stirling number triangles considered in the referenced literature include the *C-numbers* [16, (3.14)] and other combinatorial-based polynomial forms defined by the contexts of individual applications [11, §12] [22; 61].

The approach to defining the generalized polynomials is based on treating the triangle entries in equations (3.1) and (3.2) as polynomials in the upper coefficient index, as is used to define the polynomials corresponding to the existing expansions of the classical Stirling triangles. The polynomials corresponding to the Stirling numbers of the first kind are formed by the variant *Newton series* expansion identity in terms of the "second-order" Eulerian number triangle given by the next equation [A008517] [24, (6.44) and (6.45); cf. (6.43)] [32, cf. "Asymptotics"].

$$\begin{bmatrix} x \\ x-n \end{bmatrix} = \sum_{k=0}^{n} \left\langle\!\!\left\langle \begin{matrix} n \\ k \end{matrix} \right\rangle\!\!\right\rangle \binom{x+k}{2n}$$

It happens that considering the coefficient expansions with the known factors in the sum removed from the expansion of each binomial term results in forms that are particularly revealing with respect to the underlying structure of each triangle expressed by the generating function enumerating the polynomials. The Stirling (convolution) polynomial sequence is defined by the enumeration of equation (5.1) [24, §6.2].

$$\sum_{n=0}^{\infty} x\sigma_n(x)\, z^n := \sum_{n=0}^{\infty} \begin{bmatrix} x \\ x-n \end{bmatrix} \frac{(x-n-1)!}{(x-1)!}\, z^n = \left(\frac{ze^z}{e^z-1}\right)^x \tag{5.1}$$

The given generating function for the Stirling polynomial sequence satisfies the requirements of the more general form of *convolution polynomial* sequences [32] and so may be expressed in terms of the characteristic identities of these forms, including the table of convolution properties provided in the primary reference on the sequence [24, Table 272 and §6.5]. Additional properties of the sequence of Stirling polynomials in relation to the generalized α-factorial polynomial sequences are considered in §5.3.5.

5.2 j-Factorial Polynomial Definitions and Generating Functions

5.2.1 Generalized Polynomials of the First Kind

The α-factorial polynomial sequences are defined by a generalization of the Stirling convolution polynomial definition [24, §6.2] that results in the equivalent definitions given by equation (5.2).

$$\sigma_n^{\alpha}(x) := \begin{bmatrix} x \\ x-n \end{bmatrix}_{\alpha} \frac{(x-n-1)!}{x!} \qquad \Longleftrightarrow \qquad \begin{bmatrix} x \\ m \end{bmatrix}_{\alpha} = \frac{x!}{(m-1)!}\,\sigma_{x-m}^{\alpha}(x) \tag{5.2}$$

From equations (5.1) and (5.2) it is possible to derive the characteristic generating function forms for these parametrized α-factorial polynomial sequences from the results of §3.2 and §4 giving the generalized coefficients in terms of the Stirling triangle (omitted for length). The resulting enumerations are denoted by the generating function $S_\alpha(x;\ z)$ and are given by the equivalent forms in equations (5.3) and (5.4).

$$S_\alpha(x;\ z) := \sum_{n=0}^{\infty} \sigma_n^{\alpha}(x)\, z^n = e^{(1-\alpha)z} \left(\frac{\alpha z e^{\alpha z}}{e^{\alpha z}-1}\right)^x \tag{5.3}$$

$$= (\cosh((1-\alpha)z) + \sinh((1-\alpha)z)) \left(\frac{\alpha z\ (\cosh(\alpha z)+\sinh(\alpha z))}{\cosh(\alpha z)+\sinh(\alpha z)-1}\right)^x \tag{5.4}$$

As with the related generating function for the generalized Bernoulli polynomials, it should be possible to define the generating functions in terms of a complex-valued *Cauchy contour integral* about the origin as is given in Temme's work [57], though the form of (5.3) is sufficient to define the properties considered in the context of this article.

The generating function $S_\alpha(x;\ z)$ in (5.3) satisfies each of the following ordinary differential equation variations in (5.5) and (5.6).

$$S_1''(x-2;\ -\alpha z) + r_1(z)\ S_1'(x-2;\ -\alpha z) + r_0(z)\ S_1(x-2;\ -\alpha z) - \frac{S_\alpha(x;\ z)}{r_2(z)} = 0 \tag{5.5}$$

$$S_\alpha''(x-2;\ z) + \frac{p_1(z)}{p_2(z)} S_\alpha'(x-2;\ z + \frac{p_0(z)}{p_2(z)} S_\alpha(x-2;\ z) - \frac{S_\alpha(x;\ z)}{p_2(z)} = 0 \tag{5.6}$$

The coefficient functions $r_i(z)$ in equation (5.5) are defined as

$$\begin{aligned} r_0(z) &= \frac{(x-2)(-\alpha(x-2)z+x-1)}{z^2} \\ r_1(z) &= \frac{(x-2)(\alpha z-2)}{z} \\ r_2(z) &= \frac{z^2 e^{z(\alpha(x-2)+1)}}{(x-2)(x-1)} \end{aligned}$$

and the coefficient functions $p_i(z)$ in equation (5.6) are defined as

$$\begin{aligned} p_0(z) &= \frac{e^{\alpha z}(-\alpha(x-2)z+x-1)}{x-1} \\ p_1(z) &= \frac{(x-2)ze^{\alpha z}(\alpha z-2)\left(e^{\alpha z}(\alpha z-1)+1\right)}{(x-1)\left(\alpha xz+e^{\alpha z}((\alpha-1)z-x+2)-3\alpha z+x+z-2\right)} \\ p_2(z) &= (z^2 e^{\alpha z}(e^{\alpha z}\left(2x(\alpha z-1)+e^{\alpha z}\left(x(\alpha z-1)^2-2\alpha z(\alpha z-2)-3\right)+\alpha z(\alpha z-4)+6\right)+x-3)) \ / \ d_2(z) \\ d_2(z) &= ((x-1)(e^{2\alpha z}\left(-2(\alpha-1)(x-2)z+(\alpha-1)^2z^2+x^2-5x+6\right)+z^2(\alpha(x-3)+1)^2+ \\ &\quad e^{\alpha z}(z^2(\alpha(\alpha(3x-8)-2x+8)-2)-2(x-2)+z(\alpha(x-4)+2)-2(x-3)(x-2))+2(x-2)z(\alpha(x-3)+1)+x^2-5x+6)). \end{aligned}$$

A listing of the first several generalized polynomials of the first kind is provided in Table 5.1.

Table 5.1: The α-Factorial Polynomials of the First Kind

n	$\sigma_n^{\alpha}(x)$
0	1
1	$\frac{x\alpha}{2}-\alpha+1$
2	$\frac{x^2\alpha^2}{8}+\frac{1}{24}x\left(12\alpha-13\alpha^2\right)+\frac{1}{2}\left(\alpha^2-2\alpha+1\right)$
3	$\frac{x^3\alpha^3}{48}+\frac{1}{48}x^2\left(6\alpha^2-7\alpha^3\right)+\frac{1}{6}\left(-\alpha^3+3\alpha^2-3\alpha+1\right)+\frac{1}{24}x\left(7\alpha^3-13\alpha^2+6\alpha\right)$

5.2.2 Generalized Polynomials of the Second Kind

In the same manner as the previous section, consider the related polynomial definition for the (3.2) triangle given by the next equation in (5.7).

$$\widehat{\sigma}_n^{\alpha}(x) := \left\{ \begin{matrix} x \\ x-n \end{matrix} \right\}_{\alpha} \frac{(x-n-1)!}{x!} \qquad \Longleftrightarrow \qquad \left\{ \begin{matrix} x \\ m \end{matrix} \right\}_{\alpha} = \frac{x!}{(m-1)!} \widehat{\sigma}_{x-m}^{\alpha}(x) \tag{5.7}$$

A variant form of the polynomials defined by (5.7) that is inspired by the Stirling polynomial definitions offered by Adelberg [3, (4.6)] [22] is defined and enumerated by the interesting form in equation (5.8) [24, cf. (6.48) and (6.53)].

$$f_n^{\alpha}(x) = (n+x)\, \widehat{\sigma}_n^{\alpha}(n+x) \implies \sum_{n=0}^{\infty} f_n^{\alpha}(x)\, z^n = e^z \left(\frac{\alpha z}{e^{\alpha z}-1}\right)^{1-x} \tag{5.8}$$

A listing of the first several examples of the *variant polynomials of the second kind* is provided in Table 5.2.

Table 5.2: The Variant α-Factorial Polynomials of the Second Kind

n	$f_n^{\alpha}(x)$
0	1
1	$\frac{2-\alpha}{2}+\frac{x\alpha}{2}$
2	$\frac{x^2\alpha^2}{8}+\frac{1}{24}x\left(12\alpha-5\alpha^2\right)+\frac{1}{12}\left(\alpha^2-6\alpha+6\right)$
3	$\frac{x^3\alpha^3}{48}+\frac{1}{12}\left(\alpha^2-3\alpha+2\right)+\frac{1}{24}x^2\left(3\alpha^2-\alpha^3\right)+\frac{1}{48}x\left(\alpha^3-10\alpha^2+12\alpha\right)$

5.3 j-Factorial Polynomial Identities

5.3.1 Initial Polynomial Properties

As a first observation, the generating function defining the form of the polynomials can be shown through the methods of Graham and Knuth [24; 32] to produce polynomials of finite-degree such that $\deg\{\sigma_n^{\alpha}(x)\} = n$. This property is characteristic of polynomial sequences with generating functions in the general form of equations (5.3) and (5.8) involving a generating series raised to the power of the polynomial variable [50].

A simple pair of identities relating the polynomial parameter α multiplied by a scalar is formulated for each of the polynomial variations by the following equations. A notable special case of the first triangle polynomial identity results when $c = -1$.

$$\sigma_n^{c\alpha}(x) = \sum_{k=0}^{n} \frac{c^k(1-c)^{n-k}}{(n-k)!}\ \sigma_k^{\alpha}(x)$$

$$\widehat{\sigma}_n^{c\alpha}(n+x) = \sum_{k=0}^{n} \frac{(c-1)^{n-k}}{c^n\ (n-k)!}\frac{(k+x)}{(n+x)}\ \widehat{\sigma}_k^{c^2\alpha}(k+x)$$

The recursive identities in the next equations are immediate consequences of the respective definitions (5.2) and (5.7). These identities provide the most basic recursive properties for the sequences that involves a finite shift in each polynomial input [50, cf. §3.1].

$$(x+1)\ \sigma_n^{\alpha}(x+1) = (x-n)\sigma_n^{\alpha}(x) + (\alpha x + 1 - \alpha)\sigma_{n-1}^{\alpha}(x) \tag{5.9}$$

$$x\ \widehat{\sigma}_n^{\alpha}(x+1) = (x-n)\ \widehat{\sigma}_n^{\alpha}(x) + (\alpha x - \alpha n + 1)\ \widehat{\sigma}_{n-1}^{\alpha}(x)$$

The case of equation (5.9) when $\alpha := 1$ corresponds to the recursive definition of the original Stirling polynomial sequence [24, p. 311; §6].

Finally, the polynomials of the first and second kinds are related through the next pair of conversion formulas for a fixed parameter α. The given conversion identities follow from the discrete convolution of terms in the series expansions for the generating functions of equations (5.3) and (5.8) [24, cf. (6.33) and (6.48)] [cf. §3.4.2; §5.3.5].

$$\widehat{\sigma}_n^{\alpha}(n+x) = \frac{(1-x)}{(n+x)} \sum_{k=0}^{n} \frac{(\alpha x)^k}{k!}\ \sigma_{n-k}^{\alpha}(1-x)$$

$$\sigma_n^{\alpha}(x) = \sum_{k=0}^{n}\sum_{j=0}^{n-k} \frac{(\alpha(x-1)-1)^{n-k-j}}{x\ (n-k-j)!}(j+2-2x)(k+x)\ \widehat{\sigma}_j^{\alpha}(j+2-2x)\,\widehat{\sigma}_k^{\alpha}(k+x)$$

Given the coefficient definitions for each polynomial variation from equations (5.2) and (5.7), these conversion identities also provide new identities relating the triangles in (3.1) and (3.2) and extend the coefficient properties discussed in §3.2. The forms that result from the special case when $\alpha := 1$ offer an unusual pair of identities relating the classical Stirling coefficients as well as the classical triangles to the "second-order" Eulerian number triangle through closed-form double and quadruple sums [24, cf. Table 265; (6.43) and (6.44)].

5.3.2 Polynomial Results Based on Properties of the First Triangle

In the following discussion several forms of the $\sigma_n^{\alpha}(x)$ polynomials are derived from identities and resulting generating functions based on the properties of §4.2.1 [cf. §5.3.4].

First, by applying (4.16) to the identity in equation (4.17), the resulting form of the (3.1) triangle coefficients is given by the following equation in (5.10).

$$\begin{aligned}\left[\begin{matrix} x \\ x-n \end{matrix}\right]_{\alpha} \frac{(x-n-1)!}{x!} &= \sum_{k=0}^{x-2}\sum_{i=0}^{n-1} \frac{(\alpha X_c)^i\ (-1)^{n-1-i}\ \sigma_{x-2-k}^{\alpha}(X_c-1)\ (2\alpha-2)^{k-x}}{X_c\ (2\alpha-2)^{i-n-1}\ i!\ (n-1-i+k+2-x)!} \\ &+ \sum_{k=0}^{x-2}\sum_{i=0}^{n} \frac{(\alpha X_c)^i\ (x-n-1)\ (-1)^{n-i}\ \sigma_{x-2-k}^{\alpha}(X_c-1)\ (2\alpha-2)^{k-x}}{X_c(X_c-1)\ (2\alpha-2)^{i-n-2}\ i!\ (n-i+k+2-x)!}\end{aligned} \tag{5.10}$$

Let the functions $S_i(x;\ z)$ be defined as follows:

$$\begin{aligned}S_1(x;\ z) &:= \sum_{k=0}^{x-2}\sum_{n=0}^{\infty} \frac{-(2\alpha-2)^{k+2-x}\ \alpha^n\ X_c^{n-1}\ \sigma_{x-2-k}^{\alpha}(X_c-1)}{(X_c-1)\ \Gamma(n+1)\ \Gamma(k+2-x)}\ z^n \\ S_2(x;\ z) &:= \sum_{k=0}^{x-2}\sum_{n=0}^{\infty} \frac{(-1)^{x-k}\ (2+(X_c-2)\ \alpha)^n\ \sigma_{x-2-k}^{\alpha}(X_c-1)}{X_c\ (X_c-2)\ (1+2k-n+(k-n)(X_c-2)\ \alpha)^{-1}\ \Gamma(n+2)}\ z^{n+k+3-x} \\ S_3(x;\ z) &:= \sum_{k=0}^{x-2}\sum_{n=0}^{\infty} \frac{(-1)^{x+1-k}(2+(X_c-2)\alpha)^{n-1}\sigma_{x-2-k}^{\alpha}(X_c-1)}{X_c(X_c-1)(2+2k-n+(k-n+1)(X_c-2)\alpha)^{-1}\Gamma(n+1)}\ z^{n+k+2-x}.\end{aligned}$$

From each side of equation (5.10), and by the application of (5.2), it can be shown that

$$\begin{aligned}\sum_{n=0}^{\infty} \sigma_n^{\alpha}(x)\, z^n &= \sum_{k=0}^{x-2} \frac{-e^{\alpha X_c\ z}(2\alpha-2)^{k-x+2}\ \sigma_{x-2-k}^{\alpha}(X_c-1)}{X_c\ (X_c-1)\ \Gamma(k-x+2)} \\ &+ \sum_{k=0}^{x-2} \frac{e^{(2+(X_c-2)\alpha)z}\ (\alpha-1)^{k-x}\ ((1+(X_c-2)\alpha)\ z-(k+1))\ \sigma_{x-2-k}^{\alpha}(X_c-1)}{X_c\ (X_c-1)\ (z-\alpha z)^{k-x}\ z^2} \\ &+ \sum_{k=0}^{x-2} \frac{-e^{(2+(X_c-2)\alpha)z}\ (2\alpha-2)^{k-x+2}\ E_{x-1-k}((2-2\alpha)z)\ \sigma_{x-2-k}^{\alpha}(X_c-1)}{X_c\ (X_c-1)\ ((1+(X_c-2)\alpha)\ z-(k+1))^{-1}\ \Gamma(k+2-x)}\end{aligned}$$

where $E_k(x)$ denotes the exponential integral function. It follows from the given expressions that the next polynomial identity holds in terms of the functions $S_i(x;\ z)$.

$$\sigma_n^{\alpha}(x) = \frac{(-1)^{n+1}\ x\ (x+1)}{(x-n)}\ [z^{x+1-n}]\left(S_1(x+1;\ z)z^x + S_2(x+1;\ z)z^x + S_3(x+1;\ z)z^x\right)$$

Next, for $x \in [2,\ \infty) \subseteq \mathbb{N}$ and $n \in [0,\ x-2] \subseteq \mathbb{N}$, let the coefficients $c(x;\ n) := [t^x z^n] C(t;\ z)$ be defined by the following equation.

$$c(x;\ n) = \frac{(x-1)\,[n=0]_\delta}{X_c(X_c-1)} + \frac{-(2-2x+n+(1+n-x)(X_c-2)\alpha)}{X_c(X_c-1)(2+(X_c-2)\alpha)^{1-n}\ \Gamma(n+1)}\,[n \geq 1]_\delta$$

Starting from the form of equation (5.10), let the right hand side bivariate "double" generating function R be defined from the coefficient results of the following equations.

$$\begin{aligned} R(t;\ z) &:= \sum_{x=2}^{\infty}\sum_{n=0}^{\infty} \left[\begin{matrix} x \\ x-n \end{matrix}\right]_\alpha \frac{(x-n-1)!}{x!} = C(t;\ z)\ e^{(\alpha-1)tz} \left(\frac{-\alpha t z e^{-\alpha t z}}{e^{-\alpha t z}-1}\right)^{X_c-1} \\ [z^n]R(t;\ z) &= \sum_{k=0}^{n} \sigma_{n-k}^{\alpha}\,(X_c-1)\,X_c\,(-1)^{n-k} \cdot \left(\sum_{i=0}^{\infty} c(i;\ k)\ t^{i+n-k}\right) \\ [t^x z^n]R(t;\ z) &= \sum_{k=0}^{n} \sigma_{n-k}^{\alpha}\,(X_c-1)\,X_c\,(-1)^{n-k}\ c(x-n+k;\ k) \end{aligned}$$

The last of the coefficient identities for the function $R(t;\ z)$ provides the polynomial identity given by equation (5.11).

$$\begin{aligned} \sigma_n^{\alpha}(x) = {} & \frac{(-1)^n\ (x-1-n)}{x-1}\ \sigma_n^{\alpha}(x-1) \\ & + \sum_{k=1}^{n} \frac{(-1)^{n-k}(k+2x-2n-2-(n-x+1)(x-2)\alpha)}{(x-1)(\alpha x-2\alpha+2)^{1-k}\ \Gamma(k+1)}\ \sigma_{n-k}^{\alpha}(x-1) \end{aligned} \tag{5.11}$$

From the recurrence in equation (5.9) and the previous identities for the coefficients of the last generating function R, the next equations for the component functions $R(z)$ and $L(z)$ give the following ordinary differential equations in the respective polynomial generating functions of equation (5.3).

$$\begin{aligned} R(z) &:= \sum_{n=0}^{\infty} \left(\sigma_n^{\alpha}(X_c-1)\,(-1)^n + \sum_{k=1}^{n} \sigma_{n-k}^{\alpha}(X_c-1)\,X_c(-1)^{n-k}\ c(X_c-n+k;\ k)\right) z^n \\ &= \frac{e^{(2+(X-2)\alpha)z} z}{X_c-1} S'_\alpha(x-1;\ -z) - \frac{((\alpha z-1)\ X_c - 2\alpha z + z + 1)}{(X_c-1)\ e^{-(2+(X_c-2)\alpha)z}}\ S_\alpha(x-1;\ -z) \\ L(z) &:= \sum_{n=0}^{\infty} \left(\frac{(x-1-n)}{x}\ \sigma_n^{\alpha}(x-1) + \frac{(\alpha x+1-2\alpha)}{x}\ \sigma_{n-1}^{\alpha}(x-1)\right) z^n \\ &= -\frac{z}{x}\ S'_\alpha(x-1;\ z) + \frac{((\alpha x-2\alpha+1)z+x-1)}{x}\ S_\alpha(x-1;\ z) \end{aligned}$$

The full form of the ordinary differential equation satisfied by the polynomial sequence generating function, $S_\alpha(x-1;\ \pm z)$, then results from the component functions as the difference $L(z) - R(z) = 0$.

5.3.3 Generalized Bernoulli Polynomial Properties

The α-factorial polynomials polynomials of the first kind are closely related to the generalized Bernoulli polynomials, denoted by $B_n^{(\alpha)}(x)$, including the subset of Nörlund polynomials, denoted $B_n^{(\alpha)} = B_n^{(\alpha)}(0)$, through the generating function enumerating Bernoulli polynomials in equation (5.12). The generalized Bernoulli polynomials are studied extensively in the works of Adelberg and many others [3; 6; 13; 14; 45] [50, §2.2].

$$\sum_{n=0}^{\infty} \frac{B_n^{(\alpha)}(x)}{n!} z^n = e^{xz} \left(\frac{z}{e^z - 1}\right)^{\alpha} \tag{5.12}$$

One possible application of the identities relating the (3.1) and (3.2) triangles to the higher-order Bernoulli number sequences is to generalize the polynomial forms, and then the corresponding triangle coefficients through equations (5.2) and (5.7), to non-integral inputs for α and x [cf. §3.2; t in (5.24)]. Related generalizations of the classical Stirling numbers of the first kind are given in the work of Adamchik and others [1; 13; 39; 38] [11, (67) and (68)] [53, §3] [57, §3].

Note that while the generalized Bernoulli polynomial forms are interesting and suggest a number of external implications for the α-factorial polynomial sequences based on the vast literature on these forms, this particular treatment is not a significant influence on the exposition of the article. Rather, as with the treatment of the individual expansion coefficients, the interesting and important properties of the polynomials are discussed with emphasis on the forms provided for the generalized factorial function expansions defined in terms of these polynomials and more importantly on the motivating applications based on the forms as discussed in §4 and later in §6.

Identities for the generalized α-factorial polynomials of the first kind in equation (5.2) that are given in terms of the Nörlund polynomials and the higher-order (generalized) Bernoulli numbers include the following forms:

$$\begin{aligned}
\sigma_n^{\alpha}(x) &= \frac{1}{n!} \sum_{k=0}^{n} \binom{n}{k} (1 + (x-1)\alpha)^k \alpha^{n-k} B_{n-k}^{(x)} \qquad (5.13) \\
&= \frac{1}{n!} \sum_{k=0}^{n} \binom{n}{k} (1 + (x-2)\alpha)^k \alpha^{n-k} B_{n-k}^{(x)}(1) \\
\sigma_n^{\alpha}(x) &= \alpha^n \sum_{k=0}^{n} \frac{B_k^{(x)}(\frac{1}{\alpha} - 1)\, x^{n-k}}{k!\,(n-k)!} \\
\sigma_n^{\alpha}(x) &= \sum_{k=0}^{n} \frac{\alpha^k (1-\alpha)^{n-k}}{k!\,(n-k)!} B_k^{(x)}(x) \\
&= \sum_{k=0}^{n} \frac{\alpha^k (1-\alpha)^{n-k} (1+\alpha z)^{n-k}}{k!\,(n-k)!} B_k^{(x)}(\alpha x).
\end{aligned}$$

The generalized α-factorial polynomials of the second kind, $\widehat{\sigma}_n^{\alpha}(x)$, are expressed similarly in closed-form through the Nörlund polynomials by equation (5.14) and in terms of the generalized Bernoulli polynomials by equation (5.15).

$$\widehat{\sigma}_n^{\alpha}(x) = \sum_{k=0}^{n} \frac{B_k^{(1-x+n)} \alpha^k}{k!\,(n-k)!} \tag{5.14}$$

$$\widehat{\sigma}_n^{\alpha}(x) = \sum_{k=0}^{n} \frac{\alpha^k\,(1-\alpha)^{n-k}}{k!\,(n-k)!}\; B_k^{(1-x+n)}(1) \tag{5.15}$$

Each of these initial identities follows directly from the generating functions for the respective polynomial sequences compared to variations of the parameter forms in (5.12).

The sequences of Bernoulli polynomials, denoted by $B_n(x) := B_n^{(1)}(x)$, *Euler polynomials*, $E_n(x) := E_n^{(1)}(x)$, and *Hermite polynomials*, $H_n^{(v)}(x)$, each satisfying the treatment of an *Appell sequence* given in *The Umbral Calculus* reference, have an interesting property termed as the *multiplication theorem* [50, Thm. 2.5.10; §2.2]. A similar property for the α-factorial polynomial form is formulated by equation (5.16).

$$\sigma_n^{\alpha}(mx) = \sum_{i+j+k=n} \left(\sum_{r=0}^{m-2} \frac{\alpha^{j+k}\,(m-1)^{k-1}}{j!\,k!}\; \sigma_i^{\alpha}(x)\, B_j^{(mx-x-1)}\; B_k\left(x + \frac{r}{m-1}\right)\right) \tag{5.16}$$

The multiplication theorem also provides the α-factorial function identity given by the next equation where the input s is considered as a formal parameter in the distinct polynomial expansions that result from the respective triangle and polynomial forms over arbitrary index input [24, cf. (5.83)].

$$\frac{(s-1)!_{(\alpha)}}{(mx)!} = \sum_{k=1}^{mx} \begin{bmatrix} mx \\ k \end{bmatrix}_{\alpha} \frac{(-1)^{mx+k}\; s^{k-1}}{(mx)!} = \sum_{n=1}^{mx} \frac{(-1)^{mx+n}\; s^{n-1}}{(n-1)!}\; \sigma_{mx-n}^{\alpha}(mx)$$

Applications involving the α-factorial polynomials of the first kind often involve identities for the index diagonals defined as a fixed offset over both polynomial inputs. Given this observation it then becomes necessary to establish properties for the cases of these diagonal index forms. For well-defined indices $n \in \mathbb{N}$ and $k \in [0,\ x] \subseteq \mathbb{N}$, define the form of $\sigma_{x-k}^{\alpha}(x)$ by the following equations where $\Gamma(k;\ z)$ denotes the incomplete gamma function [cf. §5.3.5].

$$\begin{aligned} \sigma_{x-k}^{\alpha}(x) &= \sum_{j=0}^{x} \binom{x}{j} \frac{(1-\alpha+\alpha x)^j B_{x-j}^{(x)}}{\alpha^{j-x}\; x!}\, [k=0]_{\delta} + \sum_{j=0}^{x-k} \begin{bmatrix} x-1 \\ j+k-1 \end{bmatrix} \frac{\alpha^{x-k}\;(j+k-1)!}{\alpha^j\;(x-1)!\; j!}\, [k \geq 1]_{\delta} \\ &= [z^x] \left(\frac{(-1)^{k-1}\;\alpha^{x-k}\; z\;\operatorname{Log}(1-z)^{k-1}}{(1-z)^{\frac{1}{\alpha}}\;\Gamma(x-k+1)}\; \Gamma(x-k+1;\; -\operatorname{Log}(1-z)\;\alpha^{-1}) \right) \end{aligned}$$

The second diagonal identity results by applying the classical result for the exponential generating function enumerating the Stirling numbers of the first kind in equation (3.31) [24, Table 351] in the specific form of the equation

$$\sum_{n=1}^{\infty} \begin{bmatrix} n-1 \\ m \end{bmatrix} \frac{m!}{(n-1)!}\; z^n = (-1)^m\; z\,\operatorname{Log}(1-z)^m$$

to the first diagonal index property.

The recursive form for the derivative of the polynomial $\sigma_n^{\alpha}(x)$ with respect to the variable x is derived below where $\langle B_n \rangle$ denotes the classical sequence of *Bernoulli numbers* [A000367; and A002445].

$$\begin{aligned}\frac{d}{dx}\left[\sigma_n^{\alpha}(x) - \sigma_{n-1}^{\alpha}(x)\right] &= \frac{d}{dx}\left[\alpha^n \sigma_n(x) + \sum_{k=0}^{n-1} \frac{(\alpha+n-1-k)\ \alpha^k\ (1-\alpha)^{n-1-k}}{(n-k)!}\ \sigma_k(x)\right] \\ &= [z^n]\left(e^{(1-\alpha)z}\left(\frac{\alpha z e^{\alpha z}}{e^{\alpha z}-1}\right)^x (1-z)\operatorname{Log}\left(\frac{\alpha z e^{\alpha z}}{e^{\alpha z}-1}\right)\right) \\ &= \frac{\alpha}{2}\ \sigma_{n-1}^{\alpha}(x) + \sum_{k=2}^{n} \frac{\alpha^{k-1}\ \sigma_{n-k}^{\alpha}(x)}{k(k-1)\ \Gamma(k+1)}\ \left(k^2 B_{k-1} - \alpha(k-1)B_k\right)\end{aligned}$$

These identities provide an analog to standard properties often cited for other well-known polynomial sequences. In particular, the identities parallel the form of a characteristic property of Appell sequences satisfying $s_n'(x) := n s_{n-1}(x)$ over all $n \in \mathbb{N}$.

Exponential generating functions and related coefficient identities for the polynomials of the first kind are given by equations (5.17), (5.18), and (5.19), where $\Gamma(k;\ z)$ denotes the incomplete gamma function [25, cf. §9] and $I_k(z)$ is the modified Bessel function of the first kind [24, §5.5] [50, cf. §1.7].

$$\frac{\sigma_n^{\alpha}(x)}{n!} = [t^n]\left(e^{(1+\alpha(x-1))t}\left(\frac{\alpha t}{e^{\alpha t}-1}\right)^x \frac{\Gamma(n+1;\ (1+\alpha(x-1)\ t))}{\Gamma(n+1)^2}\right) \tag{5.17}$$

$$\sum_{n=0}^{\infty} \frac{\sigma_n^{\alpha}(x)}{n!} z^n = \sum_{k=0}^{\infty} \frac{\alpha^k\ z^{k/2}}{\Gamma(k+1)}\ I_{-k}(2\sqrt{z})\ B_k^{(x)}(x-1) \tag{5.18}$$

$$\sum_{n=0}^{\infty} \frac{\sigma_n^{\alpha}(x)}{n!} z^n = \sum_{k=0}^{\infty}\sum_{j=0}^{k} \frac{(-\alpha\sqrt{z})^k\ (-1)^j}{(k-j)!}\ I_{-k}(2\sqrt{z})\ \sigma_j(x) \tag{5.19}$$

The form of the generalized Bernoulli polynomial in (5.18) is similar to that considered by equation (6.32) in §6.2.2 and leads to the special case form of equation (5.19).

5.3.4 Bernoulli Polynomial Enumerations Based on Properties of the First Triangle

It is possible to derive an alternate form for the α-factorial polynomial two-variable "*super*" generating series coefficients from the identities considered in §4.2.1. The derivation begins in a similar manner to that of §5.3.2 and yields a resulting generating function involving coefficient terms in the Nörlund polynomials and the exponential integral function, $E_n(z)$.

To begin with, observe that for integer $x \in [2,\ \infty) \subseteq \mathbb{N}$ and $n \in [0,\ x-2] \subseteq \mathbb{N}$ the following triangle identity holds.

$$\begin{bmatrix} x \\ x-n \end{bmatrix}_{\alpha} \frac{(x-n-1)!}{x!} = \sum_{i=0}^{x-2} \left(\frac{(\alpha X_c)^{n+1+i-x}}{X_c \ (n+1+i-x)!} + \frac{(\alpha X_c)^{n+2+i-x} \ (x-n-1)}{X_c \ (n+2+i-x)!} \right) \times \\ \times \left(\sum_{k=0}^{x-2-i} (-1)^{x-2-i-x} \sigma_{x-2-i-k}^{\alpha} (X_c - 1) \frac{(2-2\alpha)^k}{k!} \right)$$

It follows from the last identity that the polynomial result of the next equation also holds.

$$\sum_{m=2}^{\infty} \sum_{n=0}^{\infty} \sigma_n^{\alpha}(x)\, w^x z^n = \left(\frac{w^2 e^{(w-\alpha w+\alpha X_c)z}(1+(\alpha X_c-1)(w-1)z)}{(X_c-1)\ (w-1)^2} \right) \left(\frac{\alpha w z}{e^{\alpha w z}-1} \right)^{X_c-1} \tag{5.20}$$

The generating function from equation (5.20) is expressed by the product of a pair of series where the coefficients of the first series expansion may be expressed as follows:

$$[w^m] \left(\frac{w^2 \ e^{(w-\alpha w+\alpha X_c)z}(1+(\alpha X_c-1)(w-1)z)}{(X_c-1)\ (w-1)^2} \right) \\ = \begin{cases} \frac{e^{\alpha X_c z}(1+z-\alpha X_c z)}{X_c-1}, & \text{if } m=2; \\ \frac{(z-\alpha z)^m e^{(1+(X_c-1)\ \alpha)z}\left((m-2)E_{3-m}(z-\alpha z)(\alpha z-\alpha X_c z+m-1)+e^{(\alpha-1)z}(z-\alpha X_c z+m-1)\right)}{(X_c-1)(\alpha-1)^2 z^2 \Gamma(m-1)}, & \text{if } m \geq 3; \\ 0, & \text{otherwise.} \end{cases}$$

The generating function coefficients on the diagonal of the second series expansion are given by the Nörlund polynomial forms in the next equation.

$$\left(\frac{\alpha w z}{e^{\alpha w z}-1} \right)^{X_c-1} = \sum_{k=0}^{\infty} \frac{\alpha^k \ B_k^{(X_c-1)}}{k!} \ w^k z^k.$$

The full generating series expansion for the product in equation (5.20) is then defined by the coefficients of the formal variable z formed by a discrete convolution of the series terms provided from each of the separate generating function expansions.

5.3.5 Properties Related to the Stirling Polynomial Sequence

Several forms of the first triangle polynomials derived from the sequence of Stirling convolution polynomials are given in the following equations [cf. §5.3.3].

$$\sigma_n^{\alpha}(x) = \sum_{k=0}^{n} \frac{\alpha^k (1-\alpha)^{n-k}}{(n-k)!} \ \sigma_k(x) \tag{5.21}$$

$$\sigma_{x-k}^{\alpha}(x+1) = \frac{k!}{x!} \sum_{j=0}^{x-1} \begin{bmatrix} x \\ j+k \end{bmatrix} \binom{j+k}{k} \alpha^{x-k-j} + \frac{1}{x!} \ [k=0]_{\delta} [x \geq 0]_{\delta} \tag{5.22}$$

$$= \frac{1}{x} \sum_{j=0}^{x-1} \frac{(j+k)\ \alpha^{x-k-j}}{j!} \ \sigma_{x-k-j}(x) + \frac{1}{x!} \ [k=0]_{\delta} [x \geq 0]_{\delta} \tag{5.23}$$

The generating series for the cases of diagonal indices in the original Stirling polynomial sequence is defined implicitly by the form of equation (5.24) for a general real-valued parameter of t [24, §6.2] [32].

$$\mathcal{S}_t(z) = \left(\frac{-\operatorname{Log}(1 - z\mathcal{S}_t(z)^{t-1})}{z}\right)^{1/t} \implies [z^n]\mathcal{S}_t(z)^x = x\sigma_n(x + tn) \tag{5.24}$$

Properties of the α-factorial polynomial sequences in terms of the Stirling polynomials and the related generating series diagonals from equation (5.24) are given by the next pair of identities [cf. (3.33)].

$$\sigma_n^{\alpha}(x + tn) = [z^n]\left(\mathcal{S}_t(\alpha z)^x \sum_{k=0}^{n} \frac{z^k\mathcal{S}_t(\alpha z)^k(1-\alpha)^k}{(x+k)\ k!}\right) \tag{5.25}$$

$$\widehat{\sigma}_n^{\alpha}(x + tn) = \frac{(1 - x + (1-t)n)}{(x + tn)} \sum_{k=0}^{n} \frac{\alpha^k(x + (t-1)n)^k}{k!}\ \sigma_{n-k}^{\alpha}(1 - x + (1-t)n) \tag{5.26}$$

The first identity in (5.25) follows from the application of (5.24) to the sum in equation (5.21). The second identity in (5.26) follows from the variant polynomial form of $f_n^{\alpha}(x + (t-1)n)$ defined by equation (5.8). The identity in (5.26) can also be expressed in terms of the implicit generating function of equation (5.24) for the parameter value $t := -1$ through expansion of the sum in (5.25).

As remarked in §5.1, the Stirling polynomials are a special case of the finite-degree polynomial sequence forms defined and termed by Knuth as "*convolution polynomials*". The general convolution polynomial sequence form is defined by the series expansion of the generating function $F(z)^x$ corresponding to some (formal) power series $F(z)$ such that $F(0) = 1$ and where x denotes the domain of polynomial variable [32]. The $\sigma_n^{\alpha}(x)$ polynomials differ from the required convolution polynomial generating function series form by an exponential multiplier in (5.3), though the α-factorial polynomials satisfy similar convolution properties as given in the references [24, cf. Table 272]. In particular, the α-factorial polynomial sequences can be expressed in terms of the identities given by following equations when the integer $|\alpha| > 1$.

$$\sigma_n^{\alpha}(x + y) = \sum_{i+j+k=n} \frac{(\alpha-1)^i}{i!} \sigma_j^{\alpha}(x)\,\sigma_k^{\alpha}(y)$$

$$\sigma_n^{\alpha}(x_1 + \cdots + x_k) = \sum_{i_0+i_1+\cdots+i_k=n} \frac{(k-1)^{i_0}(\alpha-1)^{i_0}}{i_0!} \sigma_{i_1}^{\alpha}(x_1)\cdots\sigma_{i_k}^{\alpha}(x_k)$$

$$xn\ \sigma_n^{\alpha}(x + y) = \sum_{i+j+k=n} \left(\frac{x\ y\ (\alpha-1)^i\ j}{i!}\ \sigma_j^{\alpha}(x)\,\sigma_k^{\alpha}(y)\right) + (\alpha-1)y\ \sigma_{n-1}^{\alpha}(x + y)$$

These related identities can all be formally verified easily by performing simple operations on formal power series to the polynomial sequence generating generating functions. The special case when $\alpha := 1$ corresponding to the Stirling convolution polynomial sequence satisfies the original set of properties provided explicitly in the references [24, Table 272] [32].

5.3.6 Additional Properties of the Polynomials of the First Kind

The property given by equation (4.26) for the first coefficient triangle in (3.1) results in the next pair of polynomial identities.

$$\begin{aligned}\sigma_n^{\alpha}(x) &= [z^x]\left(\frac{\alpha^{n+1-x}z(1+\alpha z)^{\frac{1}{\alpha}+x-1}}{\operatorname{Log}(1+\alpha z)^{n+1-x}} + \frac{(x-1)\alpha z\ E_{n+1-x}\left((1-x-\alpha^{-1})\operatorname{Log}(1+\alpha z)\right)}{\alpha^{x-n}\ \operatorname{Log}(1+\alpha z)^{n-x}}\right) \\ &= [z^x]\left(\frac{z\alpha^{n+1-x}(1+\alpha z)^{\frac{1}{\alpha}+x-1}}{\operatorname{Log}(1+\alpha z)^{n+1-x}} + \frac{(x-1)\alpha z\ \Gamma\left(x-n;\ (1-x-\alpha^{-1})\operatorname{Log}(1+\alpha z)\right)}{(\alpha-\alpha x-1)^{x-n}}\right)\end{aligned}$$

Next, the form of the enumerative identity in (3.30) is used to derive the polynomial form of the next equation in (5.27).

$$\sigma_n^{\alpha}(1-x) = \sum_{k=0}^{n+2x-2} \frac{(x-1)!\ (-\alpha x)^{n+2x-2-k}}{(k+1-x)!\ (n+2x-2-k)!} \left\{ \begin{matrix} k+2-x \\ x \end{matrix} \right\}_{\alpha} \tag{5.27}$$

Finally, the definition in equation (5.7) combined with the form in equation (3.13) is used to derive the form of equation (5.28). Several polynomial-related identities for the second coefficient triangle (3.2) are also defined in terms of the generating function coefficients of equation (5.28) as given by the following equations.

$$\begin{aligned}\left\{ \begin{matrix} n \\ m \end{matrix} \right\}_{\alpha} &= \frac{(n-1)!}{(m-1)!}\ [z^n]\left(z^m e^z \left(\frac{-\alpha z e^{-\alpha z}}{e^{-\alpha z}-1}\right)^{1-m}\right) \\ &= \frac{(n-1)!}{(m-1)!} \sum_{j=0}^{n-m} \frac{(-\alpha)^j\ \sigma_j(1-m)}{(n-m-j)!} \\ &= \frac{(n-1)!}{(m-1)!} \sum_{i+j+k=n-m} \frac{(1-\alpha)^i\ (-\alpha)^{j+k}}{i!\ j!}\ B_j\ \sigma_k(-m) \\ &= \frac{(n-1)!}{(m-1)!} \sum_{j=0}^{n-m} \frac{(-1)^j\ (2-\alpha)^{n-m-j}}{(n-m-j)!}\ \sigma_j^{\alpha}(1-m) \\ &= \frac{(n-1)!}{(m-1)!} \sum_{i+j+k=n-m} \frac{2^i\ (-\alpha)^{i+j}}{i!\ j!}\ B_j\ \sigma_k^{(-\alpha)}(-m)\end{aligned} \tag{5.28}$$

These properties are also related to the polynomials of the second kind from equation (5.7) through the coefficient identities for the second triangle in (3.2).

6 Applications of the j-Factorial Polynomials

6.1 j-Factorial Function Expansions

The primary motivation in considering the generalized α-factorial polynomial sequences is to formulate a rigorous method for expanding and enumerating forms of the α-factorial functions, $(s-1)!_{(\alpha)}$, as polynomials of finite-degree in the formal input s. The α-factorial function expansions discussed in this section are directly related to the author's continued work on the divisibility of both multifactorial and binomial coefficient expansion polynomials that are partitioned by indices of an integer-valued arithmetic progression of the form $\langle qn+r \rangle$, or equivalently of the class of $n \in \mathbb{N}$ such that $n \equiv r \pmod{q}$ for some non-trivial $q, r \in \mathbb{N}$.

As an example, the initial motivation in considering the expansion coefficients of equation (3.1) involved finding generating function forms for the polynomial expansions of the form $\sum_k [s^{2k}](s-1)!_{(2)}$ evaluated as the functions in s corresponding to the distinct polynomial expansions of the double factorial function. More generally, the properties of the expansions of the forms $\sum_k [s^{qk+r}](s-1)!_{(\alpha)}$ and $\sum_{k \in \mathbb{S}} [s^k](s-1)!_{(\alpha)}$ for an arbitrary indexing set $\mathbb{S}$ motivate the next explorations of the generalized α-factorial polynomial and coefficient triangle properties.

Let the formal expansion polynomial variable for the α-factorial functions be fixed as the notation s and let the α-factorial function expansion polynomials corresponding to the arbitrary indexing set $\mathbb{S} \subseteq \mathbb{N}$ be defined as follows:

$$p_n^{\alpha}(s) := \sum_{k \in \mathbb{S}} \begin{bmatrix} n \\ k+1 \end{bmatrix}_{\alpha} (-1)^{n+k} s^k. \tag{6.1}$$

The special case where $\mathbb{S} := \mathbb{N}$ corresponds the full α-factorial function expansion of $(s-1)!_{(\alpha)}$. The next several sections generalize the forms of the expansions in (6.1) and establish other interesting properties related to these cases of the generalized α-factorial function expansions.

6.1.1 The Primary Result for Binomial Coefficients

Given the expansion variable offset $s_0 \in [0, \ \alpha - 1] \subseteq \mathbb{N}$ and the indexing set $\mathbb{S} \subseteq \mathbb{N}$, define the next generating functions for $\widehat{S}$ and $\widetilde{S}$ by the following pair of equations.

$$\widehat{S}(\alpha; \ x; \ z) := \frac{z}{x} \ \widetilde{S}(z) e^{(1-\alpha)z} \left(\frac{\alpha z e^{\alpha z}}{e^{\alpha z} - 1} \right)^x \tag{6.2}$$

$$\widetilde{S}(z) := \sum_{n=0}^{\infty} \frac{(-1)^n (s - s_0 + 1)^n \, [n \in \mathbb{S}]_{\delta}}{n!} \ z^n \tag{6.3}$$

The results of equations (6.4) and (6.5) define the α-factorial polynomial expansions for the binomial coefficients [A007318; and A000984] [24, (5.1); §5] involving the coefficient extractions corresponding to an arbitrary indexing set $\mathbb{S}$ with closed-form exponential generating function defined by equation (6.3). For the case where $\alpha \mid s - s_0$, the next forms give alternate symbolic expressions that can be used to rephrase the classical result given by the binomial theorem.

$$\binom{\frac{s-s_0}{\alpha}}{x} = \frac{(x+1)}{\alpha^x} \sum_{k=0}^{x} \frac{(-1)^k \ \sigma_k^{\alpha}(x+1)}{(x-k)!} \ (s-s_0)^{x-k} \tag{6.4}$$

$$\sum_{n \in \mathbb{S}} [(s-s_0)^n] \binom{\frac{s-s_0}{\alpha}}{x} = \frac{(-1)^x (x+1)}{\alpha^x} \ [z^{x+1}] \widehat{S}(\alpha; \ x+1; \ z) \tag{6.5}$$

The form of equation (6.4) allows for the binomial coefficient expansions to be expressed in terms of the various formulations of the α-factorial polynomial identities in §5. In particular the identities of equations (5.22) and (5.23) may be applied to the binomial coefficient forms. A number of the enumerations from §3.3 are also related to these expansions.

Given the expansion identities and applications discussed in §4, it is also interesting to consider that the Newton series expansion for the digamma function can be evaluated symbolically in terms of the binomial coefficients through the next identity.

$$\psi^{(0)}(s+1) = -\gamma_{\mathrm{E}} - \sum_{k=1}^{\infty} \binom{s}{k} \frac{(-1)^k}{k} \quad := \quad \sum_{k=1}^{\infty} \sum_{j=0}^{k} c_{k,j} s^j$$

The coefficients $c_{k,j}$ defined implicitly by this expansion may of course then be considered in terms of the symbolic binomial forms in s given by the results of this section.

6.1.2 Remarks on Special Cases of the Indexing Sets $\mathbf{S} \subseteq \mathbf{N}$

The full form of the generating function in equation (6.2) can be explicitly formulated for the set $\mathbb{S}$ by determining a specific closed-form for the generating function $\widetilde{S}(z)$ corresponding to the definition given for that particular indexing set. For example, let the formal expansion variable for the α-factorial function be denoted by s and consider the following useful special cases of the series indexing set $\mathbb{S} \subseteq \mathbb{N}$:

$$\begin{aligned} \mathbb{S} := \langle 1 \rangle \quad &\implies \widetilde{S}(z) = \sum_{n=0}^{\infty} \frac{(-1)^n s^n}{n!}\, z^n = e^{-sz} \\ \mathbb{S} := \langle qn+r \rangle \quad &\implies \widetilde{S}(z) = \sum_{n=0}^{\infty} \frac{(-1)^{qn+r} s^{qn+r}}{(qn+r)!}\, z^n. \end{aligned}$$

More generally, suppose $\mathbb{G} = \langle g_n \rangle$ is a sequence defined over the natural numbers, $G(z)$ denotes the ordinary generating function for the set $\mathbb{G}$, $\widehat{G}(z)$ denotes the exponential generating function for $\mathbb{G}$, $\mathbb{A} := \langle qn+r \rangle$ is a prescribed arithmetic progression over the natural numbers, and $\omega_q := e^{2\pi\imath/q}$ denotes the *primitive q^{th} root of unity*. The desired form of the generating function $\widetilde{S}(z)$ from equation (6.3) can then expressed by the generating function $\widetilde{S}_G(z)$ through the following equations [34, §1.2.9] [62, §2.4].

$$\begin{aligned} \widetilde{S}_G(z) := & \sum_{n=0}^{\infty} \frac{(-s)^n\, g_n}{n!} [n \in \mathbb{A}]_{\delta}\, z^n \\ = & \frac{1}{q} \sum_{k=0}^{q-1} \frac{1}{2\pi} \int_{-\pi}^{\pi} \omega_q^{-kr}\; G(-\omega_q^k\; e^{-\imath t} sz) e^{e^{\imath t}} dt \\ = & \frac{1}{q} \sum_{k=0}^{q-1} \omega_q^{-kr}\; \widehat{G}(-\omega_q^k\; sz) \end{aligned}$$

The second equation results from the fact that given the ordinary generating function for a sequence, the exponential generating function for that sequence may be obtained by the transformation identity [24, p. 566; cf. p. 373]

$$\widehat{G}(z) = \frac{1}{2\pi} \int_{-\pi}^{+\pi} G(ze^{-\imath t}) e^{e^{\imath t}} dt, \tag{6.6}$$

where the original generating function, G, is treated as an analytic function over its domain. The following examples serve to illustrate the utility in the form of these special case variations on the set $\mathbb{S}$.

As an example, define the exponential generating function corresponding to the complete expansion forms of the *quadruple-factorial* function for $\alpha := 4$ in the formal variable s by equation (6.7).

$$\widehat{S}(x;\ z) = \frac{z}{x}\ e^{-(3+s)z} \left(\frac{4ze^{4z}}{e^{4z}-1}\right)^x \tag{6.7}$$

The full expansion forms for $(s-1)!_{(4)}$ and the corresponding range of offset values corresponding each distinct polynomial expansion of the function are summarized in Table 6.1 where the form of the generating function $\widehat{S}(z)$ is defined by equation (6.7) [A034176; A000407; A007696; and A047053].

Table 6.1: Full-Indexed Quadruple-Factorial Expansions

x	$x!\ (-1)^{x+1}\ [z^x]\widehat{S}(x;\ z)$	$s \mapsto 4x$	$s \mapsto 4x-1$	$s \mapsto 4x-2$	$s \mapsto 4x-3$
1	1	1	1	1	1
2	$(s-1)$	7	6	5	4
3	$(s-1)(s-5)$	77	60	45	32
4	$(s-1)(s-5)(s-9)$	1155	840	585	384

Next, consider the expansions of the function $(s-1)!_{(4)}$ corresponding to the distinct even-indexed and odd-indexed coefficient sets enumerated by the respective generating functions in equations (6.8) and (6.9).

$$\widehat{S}_{2n}(x;\ z) := \sum_{k\geq 0}\left(\sum_{j\geq 0}\begin{bmatrix}x\\2j+1\end{bmatrix}_4 \frac{(-s)^{2j}}{x!}\right) z^k = \frac{z}{x}\ \cosh(sz)\ e^{-3z}\left(\frac{4ze^{4z}}{e^{4z}-1}\right)^x \tag{6.8}$$

$$\widehat{S}_{2n+1}(x;\ z) := \sum_{k\geq 0}\left(\sum_{j\geq 0}\begin{bmatrix}x\\2j+2\end{bmatrix}_4 \frac{(-s)^{2j+1}}{x!}\right) z^k = -\frac{z}{x}\ \sinh(sz)\ e^{-3z}\left(\frac{4ze^{4z}}{e^{4z}-1}\right)^x \tag{6.9}$$

These particular forms of the generating function from equation (6.3) follow from the cases of the series variable transformations that result from the function $\frac{1}{2}\left(e^{-sz} \pm e^{sz}\right)$ [24, cf. §7.2]. A comparison of the even-indexed, odd-indexed, and complete polynomial expansions of the quadruple-factorial function is provided in Table 6.2.

Table 6.2: Alternately-Indexed Quadruple-Factorial Expansions

x	$x!\ (-1)^{x+1}[z^x]\widehat{S}(x;\ z)$	$x!\ (-1)^{x+1}[z^x]\widehat{S}_{2n}(x;\ z)$	$x!\ (-1)^{x+1}[z^x]\widehat{S}_{2n+1}(x;\ z)$
2	$s-1$	-1	s
3	s^2-6s+5	s^2+5	$-6s$
4	$s^3-15s^2+59s-45$	$-15s^2-45$	s^3+59s
5	$s^4-28s^3+254s^2-812s+585$	s^4+254s^2+585	$-28s^3-812s$

6.1.3 Arithmetic Progressions of Expansion Coefficients

For the indexing set $\mathbb{S} := \mathbb{N}$, the α-factorial expansion polynomials are described succinctly by the recurrence relation in equation (6.10) for the polynomials $\bar{p}_n^{\alpha}(s)$ and by the resulting closed-form recurrence solution in equation (6.11).

$$\bar{p}_n^{\alpha}(s) = (s - 1 + (1-n)\ \alpha)\ \bar{p}_{n-1}^{\alpha}(s) + [n = 0]_{\delta} \tag{6.10}$$

$$= (-\alpha)^n\ \Gamma\left(n + \frac{1-s}{\alpha}\right) \Gamma\left(\frac{1-s}{\alpha}\right)^{-1} \tag{6.11}$$

In general, the expansion polynomials corresponding to the indexing set $\mathbb{S}$ are of an arbitrary finite integral degree where the form of the resulting expansion sequences cannot be determined recursively in such simple terms as that of (6.10). That being said, for the particular form of the indexing set defined by the arithmetic progression $\mathbb{A} = \langle qn + r \rangle$, a similar technique to that of §6.1.2 can be applied in formulating the expansion polynomials corresponding to the generalized analog of equation (6.11). The polynomial expansions corresponding to this choice of $\mathbb{S} := \mathbb{A}$ can be expressed through the form of equation (6.12) [34, §1.2.9].

$$p_n^{\alpha}(s) = \frac{1}{q} \sum_{k=0}^{q-1} e^{-2\pi\imath kr/q}\ \bar{p}_n^{\alpha}\left(e^{2\pi\imath k/q}\ s\right) \tag{6.12}$$

As an example, consider the indices corresponding to the arithmetic progression defined by the set $\mathbb{S} := \langle 3n \rangle$ consisting of all non-negative integer multiples of 3 [A008585]. Let ω denote the *primitive cube root of unity*. The following identities give the exact form of the polynomials defined by equation (6.12) for this specific progression as well as the exponential generating function corresponding to the expansions over these indices.

$$p_n^{\alpha}(s) = \frac{(-1)^n \alpha^n}{3} \left(\frac{\Gamma\left(n + \frac{1-s}{\alpha}\right)}{\Gamma\left(\frac{1-s}{\alpha}\right)} + \frac{\Gamma\left(n + \frac{1-\omega s}{\alpha}\right)}{\Gamma\left(\frac{1-\omega s}{\alpha}\right)} + \frac{\Gamma\left(n + \frac{1-\omega^2 s}{\alpha}\right)}{\Gamma\left(\frac{1-\omega^2 s}{\alpha}\right)} \right)$$

$$\sum_{n=0}^{\infty} p_n^{\alpha}(s) \frac{z^n}{n!} = \frac{(1+\alpha z)^{-\frac{1-\omega^2 s}{\alpha}}}{3} \left((1+\alpha z)^{\frac{\imath\sqrt{3}s}{\alpha}} + (1+\alpha z)^{\frac{\left(1-\omega^2\right)s}{\alpha}} + 1 \right)$$

While the form of equation (6.12) works well for computing the expansion polynomial forms for specific given cases of an indexing arithmetic progression, a more useful and general result can be derived by considering the exponential generating function of the form in equation (6.11). Define the exponential generating function and analogous property corresponding to the expansions of equations (6.10) and (6.11) as follows:

$$\widehat{p}(z) := \sum_{n=0}^{\infty} \frac{\bar{p}_n^{\alpha}(s)}{n!}\ z^n = (1+\alpha z)^{\frac{(s-1)}{\alpha}} = \sum_{k=0}^{\infty} \frac{\operatorname{Log}(1+\alpha z)^k}{(1+\alpha z)^{1/\alpha}\ \alpha^k\ k!}\ s^k$$

$$S_{\mathbb{A}}(s;\ z) := \sum_{k \in \mathbb{A}} [s^k] \widehat{p}(z)\ k!\ s^k = \frac{\alpha^{q-r}\ s^r\ \operatorname{Log}(1+\alpha z)^r}{(1+\alpha z)^{1/\alpha}\ (\alpha^q - s^q\ \operatorname{Log}(1+\alpha z)^q)}.$$

In order to compute the expansion polynomial forms defined by equation (6.1) for the arbitrary progression $\mathbb{S} = \mathbb{A}$, it is necessary to obtain the exponential generating series for the function $S_{\mathbb{A}}(s;\ z)$ in terms of the formal parameter s. It follows from the last equations and the transformation result of (6.6) that the identity in the next equation holds for the original sequence of expansion polynomials, $p_n^{\alpha}(s)$.

$$\frac{p_n^{\alpha}(s)}{n!} = \sum_{k=0}^{n} [s^k z^n] S_{\mathbb{A}}(s;\ z)\ \frac{s^k}{k!} = [z^n] \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} S_{\mathbb{A}}(se^{-\imath t};\ z)\ e^{e^{\imath t}}\ dt \right)$$

Additional enumerations for the indexing set $\mathbb{S} := \langle qn + r \rangle$ are based on the results of equations (6.11) and (6.12). The derivation of these results provides additional insight in to the expansion polynomial forms of (6.1) assumed over the arithmetic progression of indices where in these cases the expansion variable s is treated as a *non-formal* series index.

First, observe that (6.12) results from the form of (6.11) as the defined sum of polynomial terms of the general form $e_k\ \bar{p}_n^{\alpha}(w_k s)$ [34]. It follows from this observation that

$$\sum_{n=0}^{\infty} e_k\ \bar{p}_n^{\alpha}(w_k s)\ \frac{z^n}{n!} = e_k\ (1 + \alpha z)^{(sb+c)/\alpha}$$

where $b := w_k$, $c := -1$, and the coefficients e_k are complex-valued scalars. For the α-factorial function, the polynomial expansion parameter s is defined numerically by a specific subset of values over the natural numbers corresponding to each α [cf. §3.1; Table 6.1]. Assume that the input is defined as the arbitrary index $s \in \mathbb{N}$ and form the generating series over the index through the following equation.

$$\bar{p}(t;\ z) := \sum_{s=0}^{\infty} (1 + \alpha z)^{(sb+c)/\alpha} t^s = \frac{(1 + \alpha z)^{b/\alpha}}{1 - t(1 + \alpha z)^{c/\alpha}}$$

Next, let the sequence of polynomials, $\widehat{p}_k$, given in terms of the implicit variable t be defined as in the next equation.

$$\widehat{p}_k := \frac{p_k}{k!} = [\alpha^0 z^k] \bar{p}(t;\ z) = [z^k] \left(\frac{e^{bz}}{1 - e^{cz} t} \right)$$

The identities for the coefficients in the series expansion of the function $\bar{p}(t;\ z)$ can be expressed from the form of the last equation as follows:

$$\begin{aligned} [z^n] \bar{p}(t;\ z) &= \sum_{k=1}^{n} \begin{bmatrix} n \\ k \end{bmatrix} \frac{k!}{n!}\ (-1)^{n+k}\ \alpha^{n-k}\ \widehat{p}_k = \sum_{k=1}^{n} (-1)^{n+k} \alpha^{n-k}\ \sigma_{n-k}(n)\, k\ \widehat{p}_k \\ &= \sum_{k=0}^{\infty} \left(\sum_{j=0}^{n} \frac{(b + ck)^j\ \alpha^{n-j}\ B_{n-j}^{(n)}}{n\ (j-1)!\ (n-j)!}\ [n \geq 1]_{\delta} + [n = 0]_{\delta} \right) t^k. \end{aligned}$$

The given coefficient properties then allow the expansion polynomial terms in equation (6.12) to be enumerated by a series in t over the polynomial domain of $s \in \mathbb{N}$ and in terms of the discrete sequence index n. The resulting series form in equation (6.13) follows from the last coefficient identity where $\Phi(z;\ s;\ a) := \sum_{n=0}^{\infty} (n+a)^{-s}\ z^n$ denotes the *Lerch transcendent* function.

$$[z^n]\bar{p}(t;\ z) = \sum_{j=0}^{n} \frac{c^j \alpha^{n-j}\ B_{n-j}^{(n)}}{n\ (j-1)!\ (n-j)!}\ \Phi\left(t;\ -j;\ \frac{b}{c}\right)\ [n \geq 1]_\delta + \frac{[n=0]_\delta}{1-t} \tag{6.13}$$

It is then possible to extend the enumeration result defined over the polynomial index $s \in \mathbb{N}$ to the polynomial expansions originally defined by equation (6.1) for the particular indexing arithmetic progression as follows:

$$\sum_{s=0}^{\infty} p_n^{\alpha}(s) t^s = \sum_{k=0}^{q-1} \sum_{j=0}^{n} \frac{(-1)^j\ \omega_q^{-kr}\ \alpha^{n-j}\ B_{n-j}^{(n)}}{q\ n\ (j-1)!\ (n-j)!}\ \Phi(t;\ -j;\ -\omega_q^k)\, [n \geq 1]_\delta + \frac{[n=0]_\delta}{1-t}. \tag{6.14}$$

The generating series of equation (6.14) results from the initial conditions of the recurrence in (6.10) and by combining the forms of (6.12) and (6.13) where $\omega_q := e^{2\pi\imath/q}$ again denotes the primitive q^{th} root of unity.

Consider the generating function in (3.32) and the next series expansion result given by equation (6.15).

$$\sum_{k=1}^{\infty} \frac{(k-1+z)\alpha^{N_c} z^{N_c+k-1} e^{(\alpha N_c+1)z}}{(k-1)!\,(e^{\alpha z}-1)^{N_c}}\ s^{k-1} = (s+1)e^{sz}\left(\frac{\alpha^{N_c} z^{N_c+1}\ e^{(\alpha N_c+1)z}}{(e^{\alpha z}-1)^{N_c}}\right) \tag{6.15}$$

Let the shorthand notation $\omega := e^{2\pi\imath/q}$ denote the primitive q^{th} root of unity. For the indexing set $\mathbb{S} := \mathbb{A}$, the following interesting expansion form is based on the result of (6.15) for $q, n \in \mathbb{N}$ and where $q \geq 2$ and $n \geq 1$ [34, §1.2.9].

$$[z^n]\left(\sum_{k=0}^{q-1} \left(1+s\omega^k\right) \omega^{-kr} e^{sz\omega^k}\right) = \frac{(\omega^{qr}-1)\left((s+\omega)\ \omega^n - (s+1)\ \omega^r\right)}{\omega^{(q-1)r}\left(\omega^n - \omega^r\right)\left(\omega^r - \omega^{n+1}\right)} \cdot \frac{s^n}{n!}$$

The exponential generating function defined next in equation (6.16) results from the expansion identity of the last equation.

$$\widehat{P}_n(z) := \frac{\alpha^n z^{n+1} e^{(\alpha n+1)z}}{q\ \left(e^{\alpha z}-1\right)^n} \sum_{n=0}^{\infty} \frac{(\omega^{qr}-1)\left((s+\omega)\ \omega^n - (s+1)\ \omega^r\right)}{\omega^{(q-1)r}\left(\omega^n - \omega^r\right)\left(\omega^r - \omega^{n+1}\right)} \cdot \frac{(sz)^n}{n!} \tag{6.16}$$

Finally, for positive integer n and the given indexing progression $\mathbb{S}$, the expansion polynomials of equation (6.1) are then expressed in terms of equation (6.16) by the series coefficients

$$\frac{p_{n+1}^{\alpha}(s)}{n!} = [z^n]\widehat{P}_n(z).$$

6.1.4 An Exponential Generating Series for $\mathbf{S} \subseteq \mathbf{N}$

For the more general indexing set $\mathbb{S} \subseteq \mathbb{N}$, define the polynomial form $b_n(s)$ and the corresponding exponential generating functions for the sequence by equation (6.17).

$$b_j(s) := \sum_{k \in \mathbb{S}} \binom{j}{k} s^k \implies \widehat{B}(z) := \sum_{j=0}^{\infty} \frac{b_j(s)}{j!}\ z^j \implies \widehat{B}'(z) = \sum_{j=0}^{\infty} \frac{b_{j+1}(s)}{j!}\ z^j \tag{6.17}$$

The identity of equation (3.12) yields the next pair of identities for the general expansion polynomial form in (6.1) as follows:

$$\begin{aligned} p_n^{\alpha}(s) &= \sum_{j=0}^{n-2} \begin{bmatrix} n-1 \\ j+1 \end{bmatrix} (-1)^{n+1} \alpha^{n-2-j}\ b_{j+1}(-s) \\ \frac{(-1)^{n+1}}{(n-2)!}\ p_n^{\alpha}(-s) &= [z^n] \left(z^2 \widehat{B}'(z) \left(\frac{\alpha z e^{\alpha z}}{e^{\alpha z}-1} \right)^{n-1} \right). \end{aligned} \tag{6.18}$$

The generating functions enumerating the sequence of the $b_j(s)$ polynomials have a special form for the case when the indexing set is defined as the arithmetic progression $\mathbb{S} := \langle qn+r \rangle$. The generating function for this special case is given as equation (6.19).

$$B(z) := \sum_{j=0}^{\infty} \sum_{k=0}^{j} \binom{j}{qk+r} s^{qk+r} z^n = \frac{(-sz)^r (z-1)^{q-1-r}}{-(1-z)^q + (sz)^q} \tag{6.19}$$

The exponential generating function corresponding to the special case and in equation (6.18) then follows from the transformation given in (6.6). Additional identities for (6.18) result from expanding the right hand side generating series in terms of the generalized Bernoulli polynomials, Stirling polynomials, alternate α-factorial expansion forms, and the related results of §5.

6.1.5 Recurrence Relations and Ordinary Differential Equations for $\mathbf{S} \subseteq \mathbf{N}$

Let $\mathbb{S} \subseteq \mathbb{N}$ denote an arbitrary indexing set for the polynomial expansions considered by equation (6.1) and let the closed-form exponential generating function $\widetilde{S}$ be defined by the next equation.

$$\widetilde{S}(z) := \sum_{n \in \mathbb{S}} \frac{(-1)^n\ s^n}{n!}\ z^n$$

The α-factorial expansion polynomials for general the indexing set $\mathbb{S}$ can be defined through the following identities where the form of the series coefficients in equation (6.20) follows from equations (6.4) and (5.9) and from the generating function in equation (5.3).

$$\begin{aligned} \frac{(-1)^n}{n!} p_{n+1}^{\alpha}(s) &= [z^n] \widetilde{S}(z) S_{\alpha}(n+1;\ z) \\ p_{n+1}^{\alpha}(s) &= \sum_{k=0}^{x} [z^k] \widetilde{S}(z) \frac{1}{(x+1)^2} \left(k\ \sigma_{x-k}^{\alpha}(x) + (\alpha x + 1 - \alpha)\ \sigma_{x-1-k}^{\alpha}(x) \right) \end{aligned} \tag{6.20}$$

For $x \in \mathbb{N}$, the expansion polynomials satisfy the recurrence relation in (6.22) where the generating function $R(x;\ z)$ is defined by the first equation (6.21).

$$R(x;\ z) = \alpha^{x-1} z^{x+1} e^{(1+(x-1)\alpha)z} \left(e^{\alpha z} - 1\right)^{1-x} \left(\widetilde{S}''(z) + \widetilde{S}'(z)\right) \tag{6.21}$$

$$[z^x] R(x;\ z) = (x-1)x(x+1)\ p_{x+1}^{\alpha}(s) - (x-1)x(\alpha x + 1 - \alpha)\ p_x^{\alpha}(s) \tag{6.22}$$

Let the next pair of ordinary generating functions be defined as follows:

$$R(z) := \sum_{x=0}^{\infty} [z^x] R(x;\ z)\ z^x \quad \text{ and } \quad P(z) := \sum_{x=0}^{\infty} p_x^{\alpha}(s)\ z^x.$$

It follows from the given recurrence relations for the expansion polynomials in (6.1) that the generating function defined by the form of $P(z)$ satisfies the ordinary differential of equation (6.23).

$$P^{(3)}(z) + \left(\frac{\alpha+1}{\alpha z - 1} + \frac{3}{z}\right) P''(z) + \frac{2(\alpha+1)}{z(\alpha z - 1)} P'(z) + \frac{p_0^{\alpha}(s)}{z^3(\alpha z - 1)} = \frac{R(z)}{z^3(1 - \alpha z)} \tag{6.23}$$

6.1.6 Application to Wilson's Theorem and Related Identities

The classical result of *Wilson's theorem* can be rephrased in terms of the α-factorial expansions considered by this section. Let the positive integer p assume the form $p := ak + b$ for $a, b \in \mathbb{N}$. Wilson's theorem states the result of the next equation [24, (4.49)].

$$(ak + b - 1)! = -1 \pmod{p} \quad \Longleftrightarrow \quad p \text{ is prime} \tag{6.24}$$

The following derivation gives the symbolic expansion of the factorial function from the original theorem of equation (6.24) in terms of the coefficients of the α-factorial triangle in (3.1) and the previous results of the section.

$$\begin{aligned}(as + b - 1)! &= \prod_{m=0}^{n-1} (as - am + b - 1) = \sum_{j=0}^{n} \begin{bmatrix} n+1 \\ j+1 \end{bmatrix}_a (-1)^{n+j}\ (as+b)^j \\ &= (-1)^n\ \Gamma(n+2)\ [z^{n+1}] \left(\frac{z}{(n+1)} e^{-(as+b)z} e^{(1-a)z} \left(\frac{aze^{az}}{e^{az} - 1}\right)^{n+1}\right)\end{aligned}$$

It follows that for $n := \lceil (as + b - 1)/a \rceil \longmapsto s$, the form of the input factorial function to Wilson's theorem may be expanded as in equation (6.25).

$$(as + b - 1)! = (-1)^s\ \Gamma(s+1)\ [z^s] \left(e^{-(as+b)z} e^{(1-a)z} \left(\frac{aze^{az}}{e^{az} - 1}\right)^{s+1}\right) \tag{6.25}$$

Additional results related to Wilson's theorem can be handled using related techniques for the α-factorial function expansions. Consider the result of *Clement's theorem* as stated by equation (6.26) concerning the modular properties of *twin prime* clusters [18].

$$4\left((p-1)! + 1\right) + p \equiv 0 \pmod{p(p+2)} \quad \Longleftrightarrow \quad p,\ p+2 \text{ prime} \tag{6.26}$$

Let the function $a : \mathbb{N} \to \mathbb{Z}$ be defined such that $a(p)$ assumes a positive integral value if and only if p and $p+2$ are both prime and is zero-valued otherwise. It follows from this definition and from equation (6.2) for $\widetilde{S}(z) := e^{-p}$ that the next generating function identities for the theorem statement of (6.26) hold if and only if the variable index p corresponds to a twin prime pair of the form $(p,\ p+2)$.

$$\frac{a(p)(p^2+2p)}{p!} = [z^p t^p]\left(\frac{(z+4)te^z}{1-t} - 4z\operatorname{Log}\left(1 - \frac{e^z zt}{e^z-1}\right)\right)$$
$$\frac{a(p)(p^2+2p)}{p!} = [z^p t^p]\left(\frac{(p+4)te^z}{1-t} + \frac{4z^2te^{pz}}{p(e^z-1-zt)}\right)$$

These new expansions allow for the modular residues required for primality by the theorems in equations (6.24) and (6.26) to be considered analytically in terms of the enumerations of §5, including the full arsenal of modular properties documented in the referenced literature for the generalized (higher-order) Bernoulli polynomials. An abundance of properties giving the generalized Bernoulli polynomials, Nörlund polynomials, and classical Stirling number triangles in terms of equivalent forms modulo primes, prime powers, and other bases are derived in the referenced works of Adelberg and a lengthy list of other authors [7; 2; 4; 6; 5; 14; 22; 27; 56; 61; 65; 63] [3, §6, §8, and §9] [28, §3.2]. Additionally, the determination of the largest power of an arbitrary prime dividing the factorial function and the applications of these bounds is discussed in the references by Adelberg and the works of several other authors [4, cf. §2, (3), (4), and (5)] [19] [24, §4.4]. Results concerning the residues and properties of the expansions identified for the theorems of this section are another topic considered by the author's continued research and exploration of these forms.

6.2 An Extension of Stirling's Approximation and Asymptotics Related to the Generalized j-Factorial Functions

6.2.1 Preliminaries

The primary values of the α-factorial functions considered by this section are defined by the function in equation (6.27).

$$\mathrm{MF}_\alpha(x) := \prod_{m=0}^{x-1}(1+m\ \alpha) = x!\ [z^x](1-\alpha z)^{-1/\alpha} = \left[\begin{matrix} x+1 \\ 1 \end{matrix}\right]_\alpha \tag{6.27}$$

A table of the $\mathrm{MF}_\alpha(x)$ function values corresponding to the first several α parameters is provided in Table 6.3 [A00142; A001147; A007559; A007696; A008548; and A008542].

Table 6.3: The $\mathbf{MF_a(x)}$ Function Values

x	$\mathrm{MF}_1(x)$	$\mathrm{MF}_2(x)$	$\mathrm{MF}_3(x)$	$\mathrm{MF}_4(x)$	$\mathrm{MF}_5(x)$	$\mathrm{MF}_6(x)$	$\mathrm{MF}_7(x)$
1	1	1	1	1	1	1	1
2	2	3	4	5	6	7	8
3	6	15	28	45	66	91	120
4	24	105	280	585	1056	1729	2640
5	120	945	3640	9945	22176	43225	76560
6	720	10395	58240	208845	576576	1339975	2756160
7	5040	135135	1106560	5221125	17873856	49579075	118514880
8	40320	2027025	24344320	151412625	643458816	2131900225	5925744000

The asymptotic formula of *Stirling's approximation* for the single factorial function is a well-known classical result that is restated for reference and clarity as equation (6.28).

$$x! \sim \sqrt{2\pi x}\ \left(\frac{x}{e}\right)^x := \mathrm{SA}(x) \tag{6.28}$$

Stirling's original result is discussed in his book *Methodus Differentialis*. Works that are related to the methods employed in generalizing Stirling's formula include several items in the references by Bühring and others [12; 59] [24, §9.6]. A historical summary of approaches to and proofs concerning Stirling's formula and the closely-related *Stirling series* is also detailed in the work of Davis [20] and Dominici [21].

The α-factorial function approximated by the asymptotic result derived in the next section is defined by the function in equation (6.27). While this function offers the most intuitive set of values for the integer-valued α-factorial functions, the expansions corresponding to each individual α-factorial triangle in (3.1) may differ by an offset, or shift, in the expansion variable [cf. §3.1; Table 6.1]. Specifically, for $s_0 \in [-1,\ \alpha+2]$, the alternate function values may be computed from a specific offset as the product $\prod_{m=1}^{x-1}(1+s_0+m\ \alpha)$. The ratio of the shifted α-factorial function to the prescribed factorial function form in (6.27) is denoted as the function $R(x)$ given by the next equation.

$$R(x) := \prod_{m=1}^{x-1}\left(1+\frac{s_0}{1+m\alpha}\right) = \frac{\Gamma(1+\frac{1}{\alpha})\ \Gamma(x+\frac{s_\circ+1}{\alpha})}{\Gamma(1+\frac{s_\circ+1}{\alpha})\ \Gamma(x+\frac{1}{\alpha})}$$

It follows from the given product that the values of $R(x)$ can be approximated by Stirling's approximation formula in (6.28) in the form of the next equation.

$$R(x) \sim \frac{e^{-\frac{s_\circ}{\alpha}}\left(\frac{1}{\alpha}+x\right)^{-\frac{1}{\alpha}-x}\Gamma\left(1+\frac{1}{\alpha}\right)\left(\frac{s_\circ+1}{\alpha}+x\right)^{\frac{s_\circ}{\alpha}+\frac{1}{\alpha}+x}\sqrt{\frac{\alpha x+s_\circ+1}{\alpha}}}{\sqrt{\frac{\alpha x+1}{a}}\Gamma\left(1+\frac{s_\circ+1}{\alpha}\right)} \tag{6.29}$$

While the consideration is unnecessary in the original form for the single factorial function where $\alpha := 1$, in the general case for increasing $\alpha \geq 2$, the ratio of the functions $(s-1)!_{(\alpha)}/(s+s_0-1)!_{(\alpha)}$ may in fact differ significantly from unity for the approximation in the limit as $s \to \infty$. The approximation for $R(x)$ considered by equation (6.29) allows for the shifted α-factorial function variants to be approximated by the results derived next in §6.2.2 for the function offset defined by equation (6.27).

6.2.2 Derivation of the Generalized Approximation Formula

Suppose that $\langle h_\alpha \rangle$ is a sequence of positive real numbers, $a_1 \neq b_1$, and c is a fixed positive real scalar. For sufficiently large α, equation (6.30) restates the content of Lemma 4.8 in Lando's book concerning the asymptotic approximations of certain hypergeometric sequences [36, §4.2].

$$\frac{h_{\alpha+1}}{h_\alpha} = A \; \frac{\alpha^k + a_1 \alpha^{k-1} + \cdots + a_k}{\alpha^k + b_1 \alpha^{k-1} + \cdots + b_k} \implies h_\alpha \sim c \; A^\alpha \; \alpha^{a_1 - b_1} \tag{6.30}$$

The lemma is the inspiration for generalizing Stirling's approximation for the single factorial function to approximate the rates of growth of the α-factorial function cases defined by equation (6.27) for the case of integer $\alpha \geq 2$.

The forms of §6.1 can be reformulated as a sequence over the integer index α as parametrized by the formal expansion variable s (also x). From the results of equations (6.4) and (6.5) the required coefficient form in the lemma is

$$(a_1 - b_1) \; s^{x-1} = x \; \left(\sigma_1^{\alpha+1}(x+1) - \sigma_1^{\alpha}(x+1)\right) \; s^{x-1} = \frac{1}{2} \; x(x-1) \; s^{x-1}.$$

It follows from equation (6.30) that the form of the x^{th} distinct expansion of the α-factorial function, $(s-1)!_{(\alpha)}$, in the formal input s is approximated by the form

$$a_s(x) \sim c_x s^{x(x-1)/2}$$

where the constant c_x is strictly a function of the expansion index x and is entirely independent of the fixed s. To study the form assumed by the constant, consider the following forms:

$$\begin{aligned}
\bar{c}_x &:= (s - 1 - \alpha(x-2)) \; \bar{c}_{x-1} + [x = 1]_\delta \\
\widehat{c}_x &:= \frac{\bar{c}_x}{\sqrt{s}^{(x^2-x)}} = \frac{(-\alpha)^{x-1} \; s^{(x-x^2)/2} \; \Gamma\left(x - 1 + \frac{1-s}{\alpha}\right)}{\Gamma\left(\frac{1-s}{\alpha}\right)} \\
&= \sum_{k=1}^{x-1} \sum_{j=0}^{k} \begin{bmatrix} x-1 \\ k \end{bmatrix} \binom{k}{j} \; (-1)^{x-1+j} \; \alpha^{x-1-k} \; s^{(x-x^2)/2+j}.
\end{aligned}$$

For $r := s^x$ and $b := \alpha \; (s/r)^{1/2}$, the expression for the coefficient results in the next equation of (6.31).

$$\begin{aligned}
\sum_{x=0}^{\infty} \widehat{c}_x \; t^x &= \frac{(s-1)}{\alpha^2} \; b^2 t^2 \; e^{\frac{(s-1)}{\alpha} bt} \; \left(\frac{bt}{e^{bt} - 1}\right)^{x-1} \\
&\implies \widehat{c}_x = \alpha^{x-2} \; (s-1) \; s^{(x-x^2)/2} \; B_{x-2}^{(x-1)}\left(\frac{s-1}{\alpha}\right)
\end{aligned} \tag{6.31}$$

For each distinct expansion indexed by x, the formal expansion parameter s corresponding to the form of equation (6.27) is given by the particular value $s := (\alpha x + 2 - \alpha)$. The generalized Bernoulli polynomial identity in (6.31) has the special case expansion form given by equation (6.32) [4, cf. (16)] [5, *Nörlund Numbers*] [57].

$$B_{x-2}^{(x-1)}\left(x-1+\frac{1}{\alpha}\right)=\frac{\Gamma\left(x-1+\frac{1}{\alpha}\right)}{\Gamma\left(1+\frac{1}{\alpha}\right)}\approx\frac{\mathrm{SA}\left(x-2+\frac{1}{\alpha}\right)}{\Gamma\left(1+\frac{1}{\alpha}\right)} \tag{6.32}$$

As discussed by Graham et. al. [24, §9.6], Stirling's approximation also holds for the non-integer inputs that result from the Bernoulli polynomial expansion of the last equation. The approximate form of $\widehat{c}_x$ then follows from the Bernoulli polynomial approximation in (6.32) and from the form of equation (6.31).

For $a_s(x) \mapsto \mathrm{MF}_\alpha(x)$, Stirling's approximation formula is generalized to the more general α-factorial function where $\alpha \geq 2$ as the result expressed in equation (6.33). The denominator factor of $\Gamma(1/\alpha)$ in the approximation is a constant dependent on the prescribed parameter α for the functions approximated by the result.

$$\widetilde{M}_x^\alpha=\frac{\sqrt{2\pi}\ \alpha^{x+\frac{1}{2}}e^{-\frac{1}{\alpha}-x+2}\left(x-2+\frac{1}{\alpha}\right)^{\frac{1}{\alpha}+x}(\alpha x-\alpha+1)}{(\alpha x-2\alpha+1)^{3/2}\ \Gamma\left(\frac{1}{\alpha}\right)} \tag{6.33}$$

The entire form of the α-factorial function cases defined by (6.27) for the positive integer $\alpha \geq 1$ is approximated by the identity of equation (6.34).

$$\mathrm{MF}_\alpha(x)\sim\mathrm{SA}(x)\left[\alpha=1\right]_\delta+\widetilde{M}_x^\alpha\left[\alpha\geq 2\right]_\delta. \tag{6.34}$$

6.2.3 Relative Error in the Generalized Approximations

Let the relative error in the approximations of equations (6.28) and (6.33) be defined in respective order by the functions $\varepsilon_{\mathrm{SA}}$ and $\varepsilon_{\mathrm{MF}}$ such that

$$\varepsilon_{\mathrm{SA}}(x):=\left|\frac{SA(x)}{x!}-\frac{x!}{\mathrm{SA}(x)}\right|\quad\text{ and }\quad\varepsilon_{\mathrm{MF}}(\alpha;\ x):=\left|\frac{\widetilde{M}_x^\alpha}{\mathrm{MF}_\alpha(x)}-\frac{\mathrm{MF}_\alpha(x)}{\widetilde{M}_x^\alpha}\right|.$$

The relative error functions computed for each of the approximation formulas results appear to be comparable and are summarized for the first several inputs of x in Table 6.4. It stands to reason that if the empirical observation holds, the error in computing the α-factorial functions from the approximate result in (6.33) may be estimated from some known variant of the Stirling series coefficients, such as in the forms of established error terms considered classically in the asymptotic series for the single factorial function.

Table 6.4: Relative Error in the Approximation Results

x	$\varepsilon_{\mathrm{SA}}(x)$	$\varepsilon_{\mathrm{MF}}(2;\ x)$	$\varepsilon_{\mathrm{MF}}(3;\ x)$	$\varepsilon_{\mathrm{MF}}(4;\ x)$	$\varepsilon_{\mathrm{MF}}(5;\ x)$	$\varepsilon_{\mathrm{MF}}(6;\ x)$
1	0.1623010	1.9027600	1.7507200	1.6964300	1.7025100	1.7427600
2	0.0827049	0.3080580	0.4368270	0.5517920	0.6558050	0.7510340
3	0.0553629	0.1096830	0.1230140	0.1309570	0.1362300	0.1399840
4	0.0415843	0.0663379	0.0710266	0.0736272	0.0752806	0.0764245
5	0.0332909	0.0474968	0.0498588	0.0511299	0.0519240	0.0524673
6	0.0277531	0.0369790	0.0383966	0.0391469	0.0396113	0.0399271
7	0.0237940	0.0302711	0.0312150	0.0317094	0.0320135	0.0322196
8	0.0208229	0.0256216	0.0262948	0.0266449	0.0268594	0.0270043
9	0.0185112	0.0222096	0.0227137	0.0229745	0.0231339	0.0232413
10	0.0166613	0.0195991	0.0199908	0.0201925	0.0203155	0.0203984

The application of *Euler's summation formula* as it applies to the original form of Stirling's series for $\mathrm{Log}(n!)$ is an important topic of interest considered in *Concrete Mathematics* [24, §9.5 and §9.6]. The discussion of Stirling's approximation for $n!$ in the reference leads to a similar derivation of the more general identity for the approximation of equation (6.27). The result given by equation (6.35) follows from the instructive treatment and the remarks provided in the reference [24, §9.6: pp. 479–481] where $\varphi_{m,n} \in (0,\ 1) \subseteq \mathbb{R}$, B_k is an element of the classical sequence of Bernoulli numbers [A000367; and A002445], and $\alpha \in [2,\ \infty) \subseteq \mathbb{N}$ [31, §3] [44, §5].

$$\begin{aligned}\mathrm{Log}(\mathrm{MF}_\alpha(x)) = \sum_{1 \leq k < x} \mathrm{Log}(1+\alpha k) &= (1+x)\,\mathrm{Log}(1+x) + (1-x) - \mathrm{Log}(4) + \sigma_C \\ &+ \sum_{k=1}^{m} \frac{\alpha^{2k-1}\ B_{2k}}{2k(2k-1)(1+\alpha x)^{2k-1}} + \frac{\alpha^{2m+1}\ B_{2m+2}\ \varphi_{m,n}}{(2m+2)(2m+1)(1+\alpha x)^{2m+1}}\end{aligned} \tag{6.35}$$

The constant σ_C from the last equation may be treated as a function of α and serves as a generalized analog to the form of *Stirling's constant*, $\sigma = \frac{1}{2}\ln(2\pi)$, derived explicitly for the single factorial case in the reference [24, §9.6]. The form of (6.35) is intended to suffice as an incomplete start to rigorously determining the unknown error term bounds for the approximation in equation (6.33).

6.2.4 Example: Eulerian Triangle Row Sums

The row sums of the Eulerian number triangles provide an immediate application of the combined generalized approximation result in equation (6.34) for an arbitrary positive integer-valued α. The row sums of the first and "second-order" Eulerian number triangles [A066094; and A008517] can be expressed in terms of products of the α-factorial function expansions for the respective cases where $\alpha = 1$ and $\alpha = 2$ [24, cf. (6.42) and §6.2]. The given properties of the well-known Eulerian number triangles suggest the generalization defined by the recurrence relation of the next equation [24, cf. (6.35) and (6.41)].

$$\left\langle \begin{matrix} n \\ k \end{matrix} \right\rangle_{(m)} := (k+1)\left\langle \begin{matrix} n-1 \\ k \end{matrix} \right\rangle_{(m)} + (mn-k-m+1)\left\langle \begin{matrix} n-1 \\ k-1 \end{matrix} \right\rangle_{(m)} + [n=k=0]_\delta \tag{6.36}$$

The row sums formed by the generalized m^{th}-*order Eulerian triangles* from the last equation may then be expressed through the following identity:

$$\sum_{k=0}^{n-1} \left\langle \begin{matrix} n \\ k \end{matrix} \right\rangle_{(m)} = \prod_{j=1}^{n-1} (1 + m\ j) = \mathrm{MF}_m(n).$$

For sufficiently large row index n, the corresponding row sums of the triangle in (6.36) may then be approximated by the full form of equation (6.34) for each parameter $m \in [1,\ \infty) \subseteq \mathbb{N}$.

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