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A NOTE ON THE NUMBER OF DERANGEMENTS*

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Abstract

We give an interesting expression of the number $d(n)$ of derangements as a polynomial with the redundant variable x.

A derangement is the permutation of $\{1, 2, \cdots, n\}$ that there is no i satisfying $\sigma(i) = i$, it is well-known that the number $d(n)$ of derangements equals:

$$d(n) = \sum_{k=0}^{n} (-1)^k \frac{n!}{k!} \tag{1}$$

and satisfies the recursive relations [1, p.180]

$$d(0) = 1,\ d(n) = nd(n-1) + (-1)^n,\ n = 1, 2, ..., \tag{2}$$

hence

$$d(0) = 1,\ d(1) = 0,\ d(2) = 1,\ d(3) = 2,\ d(4) = 9, ...$$

Consider the following polynomials of the variable x:

$$d(0, x) = 1,\ d(n, x) = \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} (x+k)^k (x+k+1)^{n-k},\ n = 1, 2, ... \tag{3}$$

We find that:

$$d(1, x) = -(x+1) + (x+1) = 0,$$

$$d(2, x) = (x+1)^2 - 2(x+1)(x+2) + (x+2)^2 = 1,$$

$$\begin{aligned} d(3, x) &= -(x+1)^3 + 3(x+1)(x+2)^2 - 3(x+2)^2(x+3) + (x+3)^3 \\ &= 6x^2 + 24x + 26 - 6(x+2)^2 = 2, \end{aligned}$$

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and

$$\begin{aligned} d(4,x) &= (x+1)^4 - 4(x+1)(x+2)^3 + 6(x+2)^2(x+3)^2 \\ &\quad -4(x+3)^3(x+4) + (x+4)^4 \\ &= -2x^4 - 8x^3 + 30x^2 + 180x + 225 + 2(x+3)^3(x^2-2x-12) = 9. \end{aligned}$$

It seems that $d(n,x)$ is independent of the variable x and $d(n,x)$ is the number of derangements.

THEOREM 1. Suppose x is an arbitrary variable, the number $d(n)$ of derangements has the following form:

$$d(n) = \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k}(x+k)^k(x+k+1)^{n-k},\ d(0) = 1. \tag{4}$$

PROOF. Note that

$$\sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} = 0.$$

Then the order of the polynomial $d(n,x)$ of (3) is no more than $n-1$. Let m be the order of $d(n,x)$, it is clear that the order of $d(n-1,x+1)$ is $m-1$, we will prove that $d(n,x)$ is free of x and satisfies the recurrence relation (2). In view of the Abel identity [1 p.128]

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x(x-kt)^{k-1}(y+kt)^{n-k},\ \text{for any } x, y, t, \tag{5}$$

the right of (5) is

$$\sum_{k=0}^{n} \binom{n}{k} (x-kt)^k(y+kt)^{n-k} + nt\sum_{k=0}^{n-1} \binom{n-1}{k} [x-(k+1)t]^k[y+(k+1)t]^{n-k-1},$$

hence we have

$$\begin{aligned} &\sum_{k=0}^{n} \binom{n}{k} (x-kt)^k(y+kt)^{n-k} \\ = \quad &-nt\sum_{k=0}^{n-1} \binom{n-1}{k} [x-(k+1)t]^k[y+(k+1)t]^{n-k-1} + (x+y)^n, \end{aligned}$$

if $t=-1, y=-x-1$, the above equality implies

$$d(n,x) = nd(n-1,x+1) + (-1)^n. \tag{6}$$

However $d(n,x)$ is a polynomial of x with order m and the order of polynomial $d(n-1,x+1)$ is $m-1$, by the arbitrariness of the variable x, we know x of $d(n,x)$ is redundant and (6) implies $d(n,x)$ satisfies the recurrence relation (2). The proof is complete.

We have

$$d(n)=\sum_{k=0}^{n-1}\binom{n}{k}(-1)^k(n-k)^k(n-k-1)^{n-k},\ d(0)=1. \tag{7}$$

This equality is the special form only when $x=-n$ in Theorem 1, it appears in [1, p.201] and shows that $d(n)$ is also the permanent of the matrix $(J-I)$ where J is the $n\times n$ matrix whose entries are 1 and I is the $n\times n$ unit matrix.

Theorem 1 can also be proved by the probabilistic method (see [2]). Moreover we have the following analog of the convolution of $d(n)$:

THEOREM 2. Suppose x, y are arbitrary variables, $d(n)$ is the number of derangements, then

$$\sum_{k=0}^{n}\binom{n}{k}d(k)d(n-k)$$
$$=\sum_{k_1+k_2+k_3=n}\binom{n}{k_1\ k_2\ k_3}(-1)^{k_3}(x+k_1)^{k_1}(y+k_2)^{k_2}(x+y+k_1+k_2+2)^{k_3}.$$

PROOF. Expanding the last term in the right of the above equality as

$$(x+y+k_1+k_2+2)^{k_3}=[(x+k_1+1)+(y+k_2+1)]^{n-k_1-k_2},$$

and applying Theorem 1 then we can obtain its validity.

References

[1] L. Comtet, Advanced Combinatorics, D. Reidel Publishing Co., Boston, 1974.

[2] P. Sun and T. M. Wang, A probabilistic interpretation to umbral calculus, Journal of Mathematical Research & Exposition., 3(2004), 391–399.