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NOTE

A New Decomposition of Derangements

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We give a new decomposition of derangements, which gives a direct interpretation of a formula for their generating function. This decomposition also works for counting derangements by number of excedances. © 2001 Academic Press

1. INTRODUCTION

A permutation π of $[n]=\{1, 2, ..., n\}$ is a *derangement*, if $\pi(i)\neq i$, for all $i\in[n]$. A value $i\in[n]$ is an *excedance* of π if $i<\pi(i)$. The number of excedances in π is denoted by $\operatorname{exc}\pi$. Let $\mathscr{D}_n$ be the set of derangements of $[n]$, and $d_n(x)$ the polynomial

$$d_n(x)=\sum_{\pi\in\mathscr{D}_n} x^{\operatorname{exc}\pi}.$$

For example, $d_0(x)=1$, $d_1(x)=0$, $d_2(x)=x$, $d_3(x)=x+x^2$, $d_4(x)=x+7x^2+x^3$. The generating function of $d_n(x)$ can be written as [2, 5]

$$\sum_{n\geqslant 0} d_n(x)\frac{t^n}{n!}=\frac{1}{1-\sum_{n\geqslant 2}(x+x^2+\cdots+x^{n-1})\,t^n/n!}. \tag{1}$$

Of course (1) can be proved by various methods, but, as pointed out by Gessel [4], it seems difficult to directly interpret (1) (even in the $x=1$

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case !) in terms of derangements. In [4] Gessel gave a direct proof of (1) in a different model with $x=1$. His proof is actually based on a factorization of some *D-permutations*, and cannot be generalized in a straightforward way to prove (1). Our purpose is to give a decomposition of derangements which interprets (1) directly.

A sequence $\sigma = s_1 s_2 \cdots s_k$ of k distinct integers $s_1, ..., s_k$ is called a *cycle* of length k if $s_1 = \min\{s_1, ..., s_k\}$. A cycle σ is called *unimodal* (resp. *prime*), if there exists i, $2 \leqslant i \leqslant k$, such that $s_1 < \cdots < s_{i-1} < s_i$ and $s_i > s_{i+1} > \cdots > s_k$ if $i<k$ (resp. in addition, $s_{i-1} < s_k$). Hence each unimodal (resp. prime) cycle is of length $\geqslant 2$. Considering that s_1 is the smallest in our case, this definition is consistent with the usual definition of "unimodal". Clearly each cycle $\sigma = s_1 \cdots s_k$ can be identified with the cyclic permutation σ' of the set $\{s_1, ..., s_k\}$ by $\sigma'(s_i) = s_{i+1}$ for $i \in [k]$, with $s_{k+1} = s_1$. We let $\operatorname{exc} \sigma$ denote the number of excedances of the associated cyclic permutation σ'.

Let $(l_1, ..., l_m)$ be a composition of n. A *P-decomposition* of type $(l_1, ..., l_m)$ of $[n]$ is a sequence of prime cycles $\tau = (\tau_1, \tau_2, ..., \tau_m)$ such that τ_i is of length l_i and the underlying sets of τ_i, $i \in [m]$, form a partition of $[n]$.

Define the *excedance of* τ as the total number of excedances in its prime cycles, i.e., $\operatorname{exc} \tau = \operatorname{exc} \tau_1 + \cdots + \operatorname{exc} \tau_m$, and weight τ by $x^{\operatorname{exc} \tau}$. It turns out that the right-hand side of (1) is the excedance generating function of P-decompositions. Indeed, since the weight of prime cycles on any l-set is $x + x^2 + \cdots + x^{l-1}$, the generating function of P-decompositions of type $(l_1, ..., l_m)$ is given by

$$\binom{l_1 + \cdots + l_m}{l_1, ..., l_m} \prod_{i=1}^{m} (x + \cdots + x^{l_i - 1}) \frac{t^{l_1 + \cdots + l_m}}{(l_1 + \cdots + l_m)!}.$$

Summing on $l_1, ..., l_m \geqslant 2$ and $m \geqslant 0$, we obtain the right hand side of (1).

In the next section we give an algorithm (or bijection), which maps each derangement into a P-decomposition with the same number of excedances, and thus prove (1). In Section 3 we will apply a similar decomposition to give a direct interpretation of a generating function of *Eulerian polynomials*. Finally, in Section 4 we indicate how to extend our algorithm to deal with similar problems in multipermutations.

2. UNIMODAL AND PRIME DECOMPOSITIONS

Given a derangement π of $[n]$, we first factorize it into cycles of length $\geqslant 2$,

$$\pi = (C_1, ..., C_k),$$

sorted in the *decreasing* order of their minima. For each cycle $\sigma = s_1 s_2 \cdots s_k$ we define the following U-algorithm to decompose it into a sequence of unimodal cycles. For the algorithm we set $s_{k+1} = s_1$.

U-ALGORITHM.

1. If σ is unimodal then $U(\sigma) = (\sigma)$.
2. Otherwise, let i be the largest integer such that $s_{i-1} > s_i < s_{i+1}$, let j be the unique integer greater than i such that $s_j > s_i > s_{j+1}$, and set $U(\sigma) = (U(\sigma_1), \sigma_2)$, where $\sigma_1 = s_1 \cdots s_{i-1} s_{j+1} \cdots s_k$ and $\sigma_2 = s_i s_{i+1} \cdots s_j$, which is unimodal.

EXAMPLE 2.1. Let $\sigma = 1\ 8\ 4\ 7\ 12\ 14\ 11\ 9\ 13\ 10\ 6\ 3\ 5\ 2$. The U-algorithm runs as

$$\begin{aligned} \sigma &\to (U(1\ 8\ 4\ 7\ 12\ 14\ 11\ 9\ 13\ 10\ 6\ 2), 3\ 5) \\ &\to (U(1\ 8\ 4\ 7\ 12\ 14\ 11\ 6\ 2), 9\ 13\ 10, 3\ 5) \\ &\to (U(1\ 8\ \ 2), 4\ 7\ 12\ 14\ 11\ 6, 9\ 13\ 10, 3\ 5) \\ &\to (1\ 8\ 2, 4\ 7\ 12\ 14\ 11\ 6, 9\ 13\ 10, 3\ 5). \end{aligned}$$

We extend U to π by applying U to each of its cycles to obtain

$$U(\pi) = (U(C_1), U(C_2), ..., U(C_r)) = (u_1, ..., u_m),$$

which is called the *unimodal decomposition* of π.

Note that the first cycle C_1 of π corresponds to the segment $(u_1, ..., u_i)$, where i is the smallest integer satisfying $\min(u_1) > \min(u_{i+1})$, and the second to a segment of $(u_{i+1}, ..., u_m)$ in the same manner, etc., so that the underlying set of each cycle can be read off from the *unimodal decomposition* of π. The following result characterizes all the sequences of unimodal cycles obtained by the U-algorithm.

LEMMA 2.2. *A sequence of disjoint unimodal cycles, $u = (u_1, ..., u_m)$, is a unimodal decomposition of a derangement in $\mathscr{D}_n$ if and only if the underlying sets of u_i, $i \in [m]$, form a partition of $[n]$ and* $\max(u_{i-1}) > \min(u_i)$ *for each $i = 2, ..., m$.*

Proof. Clearly it suffices to show the "if" part. Without loss of generality we may assume that $\min(u_1) < \min(u_i)$, for each $i = 2, ..., m$. We build π step by step. Let $\pi^{(1)} = u_1$. For $i > 1$, assume that $\pi^{(i-1)}$ has been built and that $\pi^{(i-1)} = s_1 s_2 \cdots s_l$, where $s_1, ..., s_l$ is an appropriate rearrangement of elements in $u_1, u_2, ..., u_{i-1}$. Let $u_i = r_1 r_2 \cdots r_a$. Since $\max(u_{i-1}) > \min(u_i)$, there is an integer j such that $s_j > \min(u_i)$, let j_0 be the largest such integer and set $\pi^{(i)} = s_1 s_2 \cdots s_{j_0} r_1 r_2 \cdots r_a s_{j_0+1} \cdots s_l$. Let $\pi = \pi^{(m)}$. Clearly $U(\pi) = u$. ∎

For each unimodal cycle $\sigma = s_1 s_2 \cdots s_k$ we define the following V-algorithm to decompose it into a sequence of prime cycles.

V-ALGORITHM.

1. If σ is prime then $V(\sigma) = (\sigma)$.
2. Otherwise, let j be the smallest integer such that $s_j > s_i > s_{j+1} > s_{i-1}$ for some integer i greater than 1 and set $V(\sigma) = (V(\sigma_1), \sigma_2)$, where $\sigma_1 = s_1 \cdots s_{i-1} s_{j+1} \cdots s_k$ and $\sigma_2 = s_i s_{i+1} \cdots s_j$, which is prime.

We extend V-algorithm to $U(\pi)$ by applying V to each of its components to obtain

$$V \circ U(\pi) = (V(u_1), V(u_2), ..., V(u_m)) = (\tau_1, ..., \tau_m),$$

which is called the *prime decomposition* of π.

The structure of the unimodal decomposition of π can be easily obtained from its prime decomposition. The first unimodal cycle in $U(\pi)$ corresponds to the segment $(\tau_1, ..., \tau_i)$, where i is the smallest integer satisfying $\max(\tau_i) > \min(\tau_{i+1})$, and the second to a segment of $(\tau_{i+1}, ..., \tau_m)$ in the same manner, etc.

EXAMPLE 2.3. Let σ be the same as the preceding example, whose unimodal decomposition is $U(\sigma) = (1\ 8\ 2, 4\ 7\ 12\ 14\ 11\ 6, 9\ 13\ 10, 3\ 5)$. Note that only the second cycle in $U(\sigma)$ is not prime. The V-algorithm applied to the second cycle runs as

$$4\ 7\ 12\ 14\ 11\ 6 \rightarrow (V(4\ 7\ 11\ 6), 12\ 14) \rightarrow (4\ 6, 7\ 11, 12\ 14).$$

Therefore $V \circ U(\sigma) = (1\ 8\ 2, 4\ 6, 7\ 11, 12\ 14, 9\ 13\ 10, 3\ 5)$.

It is clear that the composition $V \circ U$ maps any derangement of $[n]$ into a P-decomposition of $[n]$. The following result shows that this mapping is bijective.

THEOREM 2.4. *Any P-decomposition of* $[n]$ *is the prime decomposition of a unique derangement in* $\mathscr{D}_n$.

Proof. Let $\tau = (\tau_1, \tau_2, ..., \tau_m)$ be a P-decomposition of $[n]$. We first construct a sequence of unimodal cycles as follows: starting from the right, if there is any pair of adjacent τ_i and τ_{i+1} such that $\max(\tau_i) < \min(\tau_{i+1})$, then we insert the elements of τ_{i+1} in τ_i just before the maximum of τ_i and obtain a new cycle $\tau_i^* \tau_{i+1}$. Repeat this process with $(\tau_1, ..., \tau_i^* \tau_{i+1}, ..., \tau_m)$, until there are no more such pairs. By Lemma 2.2, the resulting sequence σ is a unimodal decomposition of some $\pi \in \mathscr{D}_n$, i.e., $U(\pi) = \sigma$. It follows that $V \circ U(\pi) = V(\sigma) = \tau$.

From the U-algorithm it is clear that the number of excedances in a cycle is the same as the sum of excedances in each unimodal component. Also the prime decomposition has the same property. Thus we have proved (1).

3. APPLICATION TO EULERIAN POLYNOMIALS

If instead of derangements we let $A_n(x)$ denote the sum of $x^{\operatorname{exc} \pi}$ for all permutations π of $[n]$, then the polynomials $xA_n(x)$ are the well-known *Eulerian polynomials* and have several other combinatorial interpretations in addition to counting permutations by number of excedances [6]. By virtue of classical theory of generating functions we see immediately that $A_n(x)$ are related to $d_n(x)$ by

$$\sum_{n \geqslant 0} A_n(x) \frac{t^n}{n!} = e^t \sum_{n \geqslant 0} d_n(x) \frac{t^n}{n!}.$$

Hence it follows from (1) that

$$\sum_{n \geqslant 0} A_n(x) \frac{t^n}{n!} = \frac{1}{1 - \sum_{n \geqslant 1} (x-1)^{n-1} t^n/n!}. \tag{2}$$

A similar proof can be given for (2), but in this case a *weight-preserving sign-reversing* involution is needed.

A sequence $\sigma = a_1 a_2 \cdots a_k$ of k distinct integers $a_1, a_2, ..., a_k$ is called *unimodal* if $k = 1$ or $k \geqslant 2$ and there exists an integer i, $1 \leqslant i \leqslant k$, such that $a_1 < a_2 < \cdots < a_i$ and $a_i > a_{i+1} > \cdots > a_k$ if $i < k$. This is the usual definition of "unimodal". We define the weight of the unimodal sequence σ by $x^{i-1}(-1)^{k-i}$, i.e., an ascent is given x and a descent -1.

A *U-decomposition* (resp. *I-decomposition*) of $[n]$ is a sequence of unimodal (resp. increasing) sequences $(\tau_1, \tau_2, ..., \tau_m)$ such that the underlying sets of τ_i, $i \in [m]$, form a partition of $[n]$ (resp. in addition, for $i > 1$, if τ_i is a singleton then it is greater than the last entry of τ_{i-1}). Hence the right side of (2) is the generating function of U-decompositions.

We now set up a weight-preserving sign-reversing involution on the U-decompositions to reduce the above generating function to that of I-decompositions. Given a U-decomposition $\pi = (\pi_1, \pi_2, ..., \pi_l)$, we call an integer k *attachable*, if k forms a singleton, i.e., $\pi_i = k$ for some $i > 1$, and k is smaller than the last entry of π_{i-1}; *detachable*, if there exists π_j whose last entry is k and whose penultimate entry is greater than k. The involution is then defined by detaching or attaching the smallest attachable or detachable integer (if any). It is clear that π is fixed if and only if π is an I-decomposition.

On the other hand, given a permutation π of $[n]$, we can factorize it into ordered cycles $\pi = (s_1, ..., s_r, c_1, ..., c_t)$, where $s_1, ..., s_r$ are the singletons ordered in increasing order and $c_1, ..., c_t$ the cycles of length $\geqslant 2$ ordered in decreasing order of their minima. Applying $V \circ U$ algorithm to each cycle c_i we obtain

$$\pi = (s_1, ..., s_r, V \circ U(c_1), ..., V \circ U(c_t)) = (\pi_1, ..., \pi_m),$$

where each π_i is a prime or singleton cycle. Since each prime cycle $a_1 \cdots a_{k-1} a_k \cdots a_l$ with $a_1 < \cdots < a_{k-1} < a_l < a_{l-1} < \cdots < a_k$ is in one-to-one correspondence with a sequence of increasing segments, $(a_1 a_2 \cdots a_{k-1} a_l, a_{l-1}, a_{l-2}, ..., a_k)$, which has no attachable or detachable element, we see that π is in one-to-one correspondence with an I-decomposition of $[n]$. Note that the singletons in π correspond to the singletons to the left of the first increasing sequence of length greater than one in an I-decomposition.

Therefore both sides of (2) are the generating functions of I-decompositions.

4. REMARKS

Our decompositions work also for permutations of a multiset $\{1^{n_1}, 2^{n_2}, ..., m^{n_m}\}$. More precisely, let $w = w_1 w_2 \cdots w_n$ be such a permutation and $\delta(w) = p_1 p_2 \cdots p_n$ the nondecreasing rearrangement of the letters in w, where $n = n_1 + \cdots + n_m$. Then w is a *multiderangement* if $p_i \neq w_i$ for each $i = 1, ..., n$, while the statistic of *excedance* of w is defined by $\operatorname{exc} w = \#\{i \colon w_i > p_i\}$. Let $\mathscr{R}(\mathbf{n})$ be the set of all such permutations and define

$$d_{\mathbf{n}}(x) = \sum_{w \in \mathscr{R}(\mathbf{n})} x^{\operatorname{exc} w}.$$

Using Foata's factorization of multipermutations (see [3]) we can factorize each multiderangement as a product of cycles of length at least 2, combining with our two decompositions we get the following result,

$$\sum_{n_1, ..., n_m \geqslant 0} d_{\mathbf{n}}(x)\, x_1^{n_1} \cdots x_m^{n_m} = \frac{1}{1 - x\, e_2 - (x + x^2)\, e_3 - \cdots - (x + x^2 + \cdots + x^{m-1})\, e_m},$$

where e_i $(2 \leqslant i \leqslant m)$ is the i-th elementary symmetric function of $x_1, ..., x_m$. The above result seems to be first proved by Askey and Ismail [1] using MacMahon's Master Theorem.

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