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Riordan paths and derangements

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Abstract

Riordan paths are Motzkin paths without horizontal steps on the x-axis. We establish a correspondence between Riordan paths and $(321, 3\bar{1}42)$-avoiding derangements. We also present a combinatorial proof of a recurrence relation for the Riordan numbers in the spirit of the Foata–Zeilberger proof of a recurrence relation on the Schröder numbers.

Keywords: Riordan number; Riordan path; $(321,3\bar{1}42)$-avoiding derangement

1. Introduction

The *Riordan numbers* have many combinatorial interpretations, see [1] and The On-Line Encyclopedia of Integer Sequences [8, A005043]. For example, the nth Riordan number r_n equals the number of plane trees with n edges in which no vertex has outdegree one, which are called *short bushes*. Let $\mathscr{B}_n$ denote the set of short bushes with n edges (see Fig. 1). The first few Riordan numbers are 1, 0, 1, 1, 3, 6, 15, 36, 91, 232. In general, r_n is given by the formula

$$r_n = \frac{1}{n+1}\sum_{k=1}^{n-1}\binom{n+1}{k}\binom{n-k-1}{k-1}, \tag{1.1}$$

see [8, A005043].

The first result of this paper was motivated by the question of finding a combinatorial interpretation of the Riordan numbers in terms of permutations with forbidden patterns. In this aspect, we find that the Riordan numbers are closely related to the Motzkin numbers. The authors have obtained a combinatorial proof of the fact that permutations avoiding the patterns $(321, 3\bar{1}42)$ are counted by the Motzkin numbers. In this paper, we show that the Riordan number r_n equals the number of derangements on $[n]=\{1, 2, \ldots, n\}$ that avoid the patterns $(321, 3\bar{1}42)$. Thus, the Riordan numbers can be considered as a derangement analogue of the Motzkin numbers.

The second result of this paper is a combinatorial proof of a recurrence relation on the Riordan numbers in the spirit of the Foata–Zeilberger proof of a recurrence on the Schröder numbers [6], see also [10–12].

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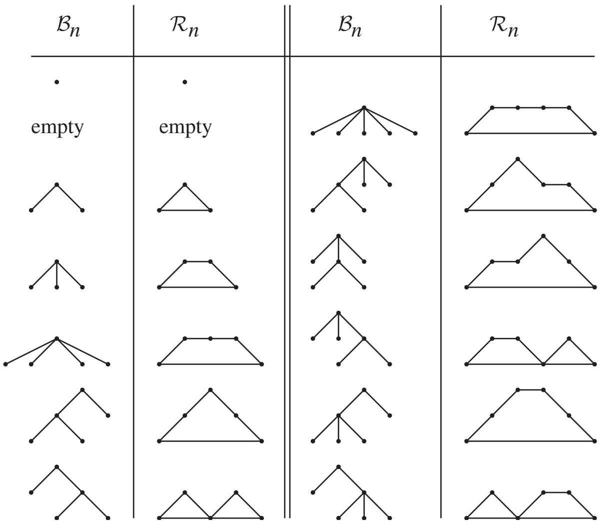

Fig. 1. Short bushes and Riordan paths.

2. Riordan paths

In this section, we give a brief review of the Riordan numbers and the Riordan paths. We first give a combinatorial derivation of formula (1.1) by using the decomposition algorithm obtained in [2]. Let F_n be the set of labelled plane trees with n edges in which no vertex has outdegree one. Moreover, let $F_{n,k}$ be the set of trees in F_n with k internal vertices. Suppose that the set of children of each internal vertex forms a block. Using the decomposition algorithm in [2], we obtain a bijection between $F_{n,k}$ and the set of forests with k small plane trees with $n+k$ vertices such that the roots of the small trees belong to $\{1, 2, \ldots, n+1\}$, and each small tree contains at least two children. Recall that a small tree is a tree containing only the root and at least one child. So $|F_{n,k}|$ can be computed as follows: we have $\binom{n+1}{k}$ choices for the roots, and the remaining n different labels are partitioned into k blocks with each block containing at least two elements. Thus, we have

$$|F_{n,k}| = \binom{n+1}{k}\binom{n-k-1}{k-1} n!,$$

which implies formula (1.1) because of the relation $|F_n| = (n+1)! r_n$.

Recall that a *Motzkin path* of length n is a lattice path in the plane from $(0, 0)$ to $(n, 0)$, consisting of up steps $U = (1, 1)$, down steps $D = (1, -1)$, and horizontal steps $H = (1, 0)$, and never going below the x-axis [1,5,9]. The *height* of any step is defined to be the y-coordinate of its starting point. A 2-*Motzkin path* is a Motzkin path where the horizontal steps can be of two kinds: straight or wavy. Motzkin paths are counted by the Motzkin numbers [8, A001006] and 2-Motzkin paths are counted by the Catalan numbers [8, A000108]; see, for example, [4,5].

The Riordan number r_n counts Motzkin paths of length n with no horizontal steps of height 0 [8, A005043]. This fact follows from a bijection of Deutsch and Shapiro between plane trees and 2-Motzkin paths [4]. For any short bush T, let the leftmost and rightmost edges of a vertex correspond to up and down steps, respectively, and let the remaining edges correspond to horizontal steps. Then we obtain a Motzkin path without horizontal steps on the x-axis by traversing T in preorder.

A Motzkin path of length n without horizontal steps on the x-axis will be called a *Riordan path* of length n, and let $\mathcal{R}_n$ be the set of Riordan paths of length n. Fig. 1 is an illustration of the correspondence between short bushes and Riordan paths.

The Riordan numbers r_n are related to the Catalan numbers $c_n = (1/(n+1))\binom{2n}{n}$ by the relation

$$c_n = \sum_{k=0}^{n} \binom{n}{k} r_k, \tag{2.1}$$

which leads to the following formula:

$$r_n = \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} c_k. \tag{2.2}$$

The above formula (2.2) has been derived by Bernhart [1] using a difference operator. Here we present a combinatorial interpretation of (2.1).

Combinatorial Proof of (2.1). Let $P = p_1 p_2 \cdots p_{2n}$ be a Dyck path of length $2n$. We divide the path P into n segments $Q_1 Q_2 \cdots Q_n$ such that $Q_i = p_{2i-1} p_{2i}$. For each Q_i, there are four possible combinations: UU, UD, DU, and DD. Suppose we use the four kinds of steps of a 2-Motzkin path to encode UU, UD, DU, and DD, that is, UU is represented by an up step, UD is represented by a wavy horizontal step, DU is represented by a straight horizontal step, and DD is represented by a down step. Then we get a 2-Motzkin path M without straight horizontal steps on the x-axis. Suppose that M contains $n - k$ wavy horizontal steps. Note that if we remove all the wavy horizontal steps, we are led to a Riordan path of length k. Conversely, given a Riordan path of length k, we can reconstruct $\binom{n}{k}$ 2-Motzkin paths without straight horizontal steps on the x-axis by inserting $n - k$ wavy horizontal steps. □

The above proof implies the following interpretation of the Catalan number.

Corollary 2.1. *The number of* 2*-Motzkin paths of length n without straight horizontal steps on the x-axis equals the Catalan number* c_n.

3. Riordan paths and derangements

In this section, we give a correspondence between Riordan paths and derangements with forbidden patterns $(321, 3\bar{1}42)$. This is motivated by the recent work of the authors [3] on the bijection ϕ between Motzkin paths of length n and $S_n(321, 3\bar{1}42)$, where S_n denotes the set of permutations on $[n]$, and $S_n(321, 3\bar{1}42)$ denotes the set of permutations avoiding the patterns $(321, 3\bar{1}42)$. We say that a permutation $\pi = \pi_1 \pi_2 \cdots \pi_n$ avoids the pattern 321 if it does not contain any subsequence $\pi_i \pi_j \pi_k$ such that $\pi_i > \pi_j > \pi_k$ for $1 \leqslant i < j < k \leqslant n$. Moreover, we say that π avoids the pattern $3\bar{1}42$ if any subsequence $\pi_i \pi_j \pi_k$ $(i < j < k)$ of pattern 231, namely, $\pi_j > \pi_i > \pi_k$, can be extended to a subsequence of pattern 3142; in other words, there exists $i < m < j$ such that $\pi_j > \pi_i > \pi_k > \pi_m$.

It was shown by Gire [7] that $|S_n(321, 3\bar{1}42)|$ equals the Motzkin number m_n (see [8, A001006]). The authors [3] established a correspondence between Motzkin paths of length n and reduced decompositions of permutations in $S_n(321, 3\bar{1}42)$. In order to make a connection between Riordan paths and permutations with forbidden patterns, we are led to the consideration of further restrictions on $S_n(321, 3\bar{1}42)$ so that we may get a subset of permutations $S_n(321, 3\bar{1}42)$ that are in one-to-one correspondence with Riordan paths of length n with m horizontal steps on the x-axis.

We now recall the definition of ϕ which is given in terms of reduced decompositions of permutations in S_n.

Definition 3.1. For any $1 \leqslant i \leqslant n - 1$, define the map $s_i \colon S_n \to S_n$ such that s_i acts on a permutation by interchanging the elements in positions i and $i + 1$. We call s_i the simple transposition, and write the action of s_i on the right of the permutation, denoted by πs_i. Therefore we have $\pi(s_i s_j) = (\pi s_i) s_j$.

The canonical reduced decomposition of $\pi \in S_n$ has the following form:

$$\pi = (1\ 2\ \cdots\ n)\sigma = (1\ 2\ \cdots\ n)\sigma_1 \sigma_2 \cdots \sigma_k, \tag{3.1}$$

where

$$\sigma_i = s_{h_i} s_{h_i - 1} \cdots s_{t_i}, \quad h_i \geqslant t_i \ \ (1 \leqslant i \leqslant k) \quad \text{and}$$
$$1 \leqslant h_1 < h_2 < \cdots < h_k \leqslant n - 1.$$

We call h_i the *head* and t_i the *tail* of σ_i. For short, we say that π has the canonical reduced decomposition $\sigma_1 \sigma_2 \cdots \sigma_k$.

For example, $\pi = 315264$ has the canonical reduced decomposition $(s_2 s_1)(s_4 s_3)(s_5)$. It is shown in [3] that permutations in $S_n(321, 3\bar{1}42)$ can be characterized by their reduced decompositions.

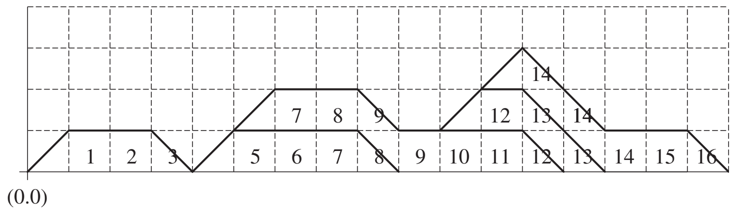

Fig. 2. The $(x + y)$-labeling and strip decomposition.

Theorem 3.2. *Let* π *be a permutation in* S_n *with the reduced decomposition as given in* (3.1). *Then* $\pi \in S_n(321, 3\bar{1}42)$ *if and only if*

$$t_{i+1} \geqslant t_i + 2, \quad 1 \leqslant i \leqslant k - 1. \tag{3.2}$$

We now give a brief description of the bijection ϕ between Motzkin paths of length n and $S_n(321, 3\bar{1}42)$ by the *strip decomposition* of Motzkin paths [3]. This bijection involves a labelling of the cells in the region of a Motzkin path. The region of a Motzkin path is meant to be the area surrounded by the path and the x-axis. Furthermore, the region of a Motzkin path is subdivided into cells which are either unit squares or triangles with unit bottom sides. A triangular cell contains either an up step or a down step. We will not label triangular cells containing up steps. The other types of cells, either square or triangular, have bottom sides, say, with points (i, j) and $(i + 1, j)$. We will label these cells with s_{i+j} or simply $i + j$. We call this labelling the $(x + y)$-labelling.

We now define the strip decomposition of a Motzkin path. Suppose $P_{n,k}$ is a Motzkin path of length n that contains k up steps. If $k = 0$, then the strip decomposition of $P_{n,0}$ is simply the empty set. For any $P_{n,k} \in M_n$, let $A \to B$ be the last up step and $E \to F$ the last down step on $P_{n,k}$. Then we define the strip of $P_{n,k}$ as the path from B to F along the path $P_{n,k}$. Now we move the points from B to E one layer lower, namely, subtract the y-coordinate by 1, and denote the adjusted points by $B', \ldots, E'$. We now form a new Motzkin path by using the path $P_{n,k}$ up to the point A, then joining the point A to B' and following the adjusted segment until we reach the point E', then continuing with the points on the x-axis to reach the destination $(n, 0)$. Denote this Motzkin path by $P_{n,k-1}$, which may end with some horizontal steps.

From the strip of $P_{n,k}$, we may define the value h_k as the label of the cell containing the step $E \to F$. Clearly, we have $h_k \leqslant n - 1$. The value t_k is defined as the label of the cell containing the step starting from the point B.

Iterating the above procedure, we get a set of parameters $\{(h_i, t_i) | 1 \leqslant i \leqslant k\}$ satisfying condition (3.2). For each step in the above procedure, we obtain a product of transpositions $\sigma_i = s_{h_i} s_{h_i - 1} \cdots s_{t_i}$. Finally, we get the corresponding canonical reduced decomposition $\sigma = \sigma_1 \sigma_2 \cdots \sigma_k$ and the corresponding permutation $\pi = (1\ 2\ \cdots\ n)\sigma$, see Fig. 2. We then obtain the following property of the bijection ϕ.

Theorem 3.3. *The bijection* ϕ *is a correspondence between Motzkin paths of length n with m horizontal steps on the x-axis and permutations in* $S_n(321, 3\bar{1}42)$ *that have m fixed points.*

Proof. For any Motzkin path P of length n with m horizontal steps on the x-axis, label its steps with $0, 1, 2, \ldots, n - 1$ from left to right. Suppose that the m horizontal steps on the x-axis are labelled by $x_1, x_2, \ldots, x_m$, where $0 \leqslant x_1 < x_2 < \cdots < x_m \leqslant n - 1$. By the strip decomposition and the $(x + y)$-labelling, $s_{x_1}, s_{x_2}, \ldots, s_{x_m}$ do not occur in the corresponding canonical reduced decomposition with respect to the bijection ϕ. Note that a horizontal step on the x-axis is followed by an up step or a horizontal step on the x-axis (except that it is the last step). Thus $x_1 + 1, x_2 + 1, \ldots, x_m + 1$ are fixed points of the corresponding permutation in $S_n(321, 3\bar{1}42)$ by applying Theorem 3.2. □

Corollary 3.4. *For any Motzkin path P of length n, let* $\pi \in S_n(321, 3\bar{1}42)$ *be its corresponding permutation with respect to the bijection* ϕ. *Suppose that* π *has the canonical reduced decomposition of form* (3.1), *then*:

(1) $t_1 - 1$ *is the number of initial horizontal steps on the x-axis at the beginning of the Motzkin path P.*
(2) $n - 1 - h_k$ *is the number of final horizontal steps on the x-axis at the end of the Motzkin path P.*
(3) $\sum_i (t_{i+1} - h_i - 2)$ *equals the number of horizontal steps of the Motzkin path P on the x-axis that are neither initial nor final steps, where the summation is over all i such that* $h_i + 1 < t_{i+1}$.

Recall that a permutation $\pi = \pi_1\pi_2 \cdots \pi_n$ is said to be a *derangement* if π does not have any fixed points, that is, $\pi_i \neq i$ for all $i \in [n]$. Let $D_n(321, 3\bar{1}42)$ denote $(321, 3\bar{1}42)$-avoiding derangements in S_n. Then we have the following correspondence.

Corollary 3.5. *The bijection ϕ is a correspondence between Riordan paths of length n and $D_n(321, 3\bar{1}42)$.*

For example, for the Riordan path in Fig. 2, we have

$$P_{17,5} = UHHDUUHHDHUUDDHHD.$$

From the strip decomposition, we get the parameter set

$$\{(3,1),\ (8,5),\ (12,7),\ (13,12),\ (16,14)\}.$$

The canonical reduced decomposition is given below:

$$(s_3s_2s_1)(s_8s_7s_6s_5)(s_{12}s_{11}s_{10}s_9s_8s_7)(s_{13}s_{12})(s_{16}s_{15}s_{14}). \tag{3.3}$$

The corresponding permutation is

$$4\ 1\ 2\ 3\ 9\ 5\ 13\ 6\ 7\ 8\ 10\ 14\ 11\ 17\ 12\ 15\ 16.$$

Corollary 3.6. *Let P be a Riordan path of length n. Then the area of P minus the sum of heights of the up steps is equal to the inversion number of the permutation $\phi(P) \in D_n(321, 3\bar{1}42)$.*

Corollary 3.7. *Let $\sigma = \sigma_1 \cdots \sigma_k$ be the canonical reduced decomposition of $\pi \in S_n$, where $\sigma_i = s_{h_i}s_{h_i-1}\cdots s_{t_i}$ for $1 \leqslant i \leqslant k$. Then $\pi \in D_n(321, 3\bar{1}42)$ if and only if $t_1 = 1$, $h_k = n-1$, and*

$$h_i + 2 \geqslant t_{i+1} \geqslant t_i + 2, \quad 1 \leqslant i \leqslant k-1.$$

4. A recurrence relation

In this section, we give a combinatorial proof of the following recurrence relation on the Riordan numbers:

Theorem 4.1. *For $n \geqslant 2$, we have*

$$(n+1)r_n = (n-1)(2r_{n-1} + 3r_{n-2}) \tag{4.1}$$

with initial values $r_0 = 1$, $r_1 = 0$, and $r_2 = 1$.

Proof. We proceed to establish the following bijection:

$$\psi\colon\ [3(n-1)] \times \mathscr{R}_{n-2} \bigcup [2(n-1)] \times \mathscr{R}_{n-1} \Longrightarrow [n+1] \times \mathscr{R}_n \tag{4.2}$$

which yields the identity (4.1).

We begin with an interpretation of $[3(n-1)] \times \mathscr{R}_{n-2}$ as the multi-set of Riordan paths of length $n-2$ in which exactly one step is labelled by one of the labels a, b, and c, plus three copies of the set of Riordan paths of length $n-2$ without labels. Similarly, $[2(n-1)] \times \mathscr{R}_{n-1}$ can be represented by the set of labelled Riordan paths of length $n-1$ in which exactly one step is labelled by either 1 or 2. The set $[n+1] \times \mathscr{R}_n$ can be represented by the set of Riordan paths of length n for which at most one step is labelled by the symbol $*$.

For example, since $\mathscr{R}_4 = \{UUDD, UDUD, UHHD\}$, $[5] \times \mathscr{R}_4$ consists of the following labelled paths:

$$\begin{array}{lllll} UUDD & U^*UDD & UU^*DD & UUD^*D & UUDD^* \\ UDUD & U^*DUD & UD^*UD & UDU^*D & UDUD^* \\ UHHD & U^*HHD & UH^*HD & UHH^*D & UHHD^*. \end{array}$$

We now give a construction of the map ψ.

(1) For the three copies of the paths in $\mathscr{R}_{n-2}$ without labels, we, respectively, add UD, U^*D, and UD^* to the beginning of the paths. In this way, we obtain all the paths beginning with UD in $[n+1] \times \mathscr{R}_n$. For example, for $n = 4$, the three copies of UD are mapped to $UDUD$, U^*DUD, and UD^*UD, respectively.

(2) For the paths having a step p_i of height k labelled by a in $\mathscr{R}_{n-2}$: If $k = 0$, namely, $p_i = U$, we add an up step to the beginning of the path and insert a down step following the corresponding down step of p_i, namely, the first down step after p_i that touches the x-axis. This gives all the Riordan paths of length n without labels such that there are no horizontal steps of height 1 before the path returns to the x-axis. Otherwise, let p_j be the last up step of height $k-1$ before the step p_i, then we add an up step after p_j and a down step before p_i and label p_j with $*$. Hence, we have all the Riordan paths of length n which contain the consecutive steps U^*U. For example, U^aD and UD^a are mapped to $UUDD$ and U^*UDD, respectively.

(3) For the paths having a step p_i labelled by b (or c) in $\mathscr{R}_{n-2}$, we add U^*D (or UD^*) after p_i. In this way, we get all Riordan paths of length n containing the consecutive steps U^*D (or UD^*) which are not at the beginning of the Riordan paths. For example, U^bD and UD^b (or U^cD and UD^c) are mapped to UU^*DD and UDU^*D (or UUD^*D and $UDUD^*$), respectively.

(4) For the paths having a step p_i of height k labelled by 1 in $\mathscr{R}_{n-1}$: If $p_i = D$ and $k = 1$, then we change the corresponding up step (that is, the nearest up step before p_i that touches the x-axis) to an H step, and add an up step to the beginning of the path. So we obtain all the Riordan paths of length n without labels such that there is at least one horizontal step of height 1 before the path returns to the x-axis. Otherwise, we add a horizontal step after p_i, and label the new horizontal step with $*$. This yields all the Riordan paths of length n containing H^*. For example, U^1HD, UH^1D, and UHD^1 are mapped to UH^*HD, UHH^*D and $UHHD$, respectively.

(5) For the paths having a step p_i labelled by 2 in $\mathscr{R}_{n-1}$: If p_i is an up step (or a down step), then we label p_i with $*$ and add a horizontal step H after p_i (before p_i). Thus, we obtain all the Riordan paths of length n containing the consecutive steps U^*H (or HD^*). If $p_i = H$, then its height is nonzero. In this case, we may assume that p_j is the first down step after p_i. Then we replace p_i by U, and add a down step before p_j and label p_j with $*$. So we obtain all the Riordan paths of length n containing consecutive steps DD^*. For example, U^2HD, UH^2D, and UHD^2 are mapped to U^*HHD, $UUDD^*$, $UHHD^*$, respectively.

In summary, we obtain all the Riordan paths in $[n+1] \times \mathscr{R}_n$. It can be seen that the above procedure is reversible. Hence ψ is a bijection. □

Note that relation (4.1) is derived from the generating function of Bernhart [1]. Our proof is in the spirit of the Foata–Zeilberger proof of a recurrence relation on the Schröder numbers [6], and Sulanke's proofs of the recurrences for Schröder paths, parallelogram polyominoes, and Motzkin paths [10–12].

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