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Research Article

DEGENERATED BERNOULLI NUMBERS AND POLYNOMIALS

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ABSTRACT
The degenerate Bernoulli numbers $\beta_m(\lambda)$ can be defined by means of the exponential generating function $t\left[(1+\lambda t)^{\frac{1}{\lambda}}-1\right]^{-1}$ L.Carlitz proved an analogue of the Staudt- clausen theorem for these numbers and he showed that $\beta_m(\lambda)$ is polynomials in λ of degree ≤ m. As further applications we derive several identities, recurrences, and congruences involving the Bernoulli numbers, degenerate Bernoulli numbers and polynomials.

Key Words: *Bernoulli Polynomial, Bernoulli Number, Degenerate Bernoulli Polynomial, Degenerate Bernoulli Number*
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INTRODUCTION

Carlitz (1956) defined the degenerate Bernoulli numbers $\beta_m(\lambda)$ by means of the generating function

$$\frac{t}{\left[(1+\lambda t)^{\frac{1}{\lambda}}-1\right]}=\sum_{m=0}^{\infty}\beta_m(\lambda)\frac{t^m}{m!} \tag{1.1}$$

We have $\beta_m(0)=\beta_m$, the ordinary Bernoulli number In (Carlitz, 1956, 1979) Carlitz proved many properties of $\beta_m(\lambda)$, including an analogue of the staudt-clausentheorem. He also pointed out that $\beta_m(\lambda)$ is a polynomials in λ with degree $\le m$. we have

$\beta_0(\lambda)=1$

$\beta_1(\lambda)=\frac{-1}{2}+\frac{\lambda}{2}$

$\beta_2(\lambda)=\frac{1}{6}-\frac{\lambda^2}{6}$

$\beta_3(\lambda)=\frac{-\lambda}{4}+\frac{\lambda^3}{4}$

$\beta_4(\lambda)=\frac{-1}{30}+\frac{2}{3}\lambda^2-\frac{19}{30}\lambda^4$

$\beta_5(\lambda)=\frac{1}{4}\lambda-\frac{5}{2}\lambda^3+\frac{9}{4}\lambda^5$ And so on.

Carlitz (1956, 1979) also defined the degenerate Bernoulli polynomials $\beta_m(\lambda,x)$ for $\lambda\neq 0$ by means of the generating function.

$$\frac{t}{\left[(1+\lambda t)^{u}-1\right]}(1+\lambda t)^{\mu x}=\sum_{m=0}^{\infty}\beta_m(\lambda,x)\frac{t^m}{m!} \tag{1.2}$$

Where λμ=1.These are polynomials in λ and *x* with rational coefficients. We often write $\beta_m(\lambda)\, for \beta_m(\lambda,0)$, and refer to the polynomial $\beta_m(\lambda)$ as a degenerate Bernoulli number. The first few are

$$\beta_0(\lambda,x)=1$$

$$\beta_1(\lambda,x)=x-\frac{1}{2}+\frac{1}{2}\lambda$$

$$\beta_2(\lambda,x)=x^2-x+\frac{1}{6}-\frac{1}{6}\lambda^2$$

$$\beta_3(\lambda,x)=x^3-\frac{3}{2}x^2+\frac{1}{2}x-\frac{3}{2}\lambda x^2+\frac{3}{2}\lambda x+\frac{1}{4}\lambda^3-\frac{1}{4}\lambda$$ And so on.

One combinatorial significance these polynomials have found is in expressing sums of generalized falling factorials $\left(\frac{i}{\lambda}\right)_m$: specifically, we have

$$\sum_{i=0}^{a-1}\left(\frac{i}{\lambda}\right)_m=\frac{1}{m+1}\left[\beta_{m+1}(\lambda,a)-\beta_{m+1}(\lambda)\right] \tag{1.3}$$

For all integers *a>0* and *m≥0* [2 Eq. (5.4)], where

$$\left(\frac{i}{\lambda}\right)_m=i(i-\lambda)(i-2\lambda)......(i-(m-1)\lambda).$$

The Bernoulli polynomials $\beta_m(x)$ may be defined by the generating function,

$$\frac{t}{(e^t-1)}e^{xt}=\sum_{m=0}^{\infty}\beta_m(x)\frac{t^m}{m!} \tag{1.4}$$

And their values at *x*=0 are called the Bernoulli numbers and denoted β_m.Since $(1+\lambda t)^{\mu}\to e^t as\quad \lambda\to 0$ it is evident that $\beta_m(0,x)=\beta_m(x)$ letting $\lambda\to 0$ in (1.3) yields the familiar identity

$$\sum_{i=0}^{a-1}i^m=\frac{1}{m+1}\left[\beta_{m+1}(a)-\beta_{m+1}\right] \tag{1.5}$$

Expressing power sums in terms of Bernoulli polynomials.

A Recurrence Relation of β_m

In (Howard, 1996), For any positive integer m and any positive integer n ≥1,we have

$$\beta_m=\frac{1}{n(1-n^m)}\sum_{k=0}^{m-1}n^k\binom{m}{k}\beta_k\sum_{j=1}^{n-1}j^{m-k} \tag{2.1}$$

Proof: Let n be any positive integer greater than 1.Noticing that $\frac{(1-e^{nx})}{(1-e^x)}$ is the sum of finite geometric series, we have

$$\frac{(1-e^{nx})}{(1-e^{x})}=\sum_{j=0}^{n-1}e^{jx}=\sum_{j=0}^{n-1}\sum_{m=0}^{\infty}\frac{j^m x^m}{m!}=\sum_{m=0}^{\infty}\sum_{j=0}^{n-1}\frac{j^m x^m}{m!}$$

Multiplying both sides by $\frac{x}{(1-e^{nx})}$, we obtain

$$\frac{x}{(1-e^{nx})}=\frac{1}{n}\left(\frac{nx}{1-e^{nx})}\right)=\sum_{m=0}^{\infty}\sum_{j=0}^{n-1}\frac{j^m x^m}{m!}$$

From the definition of the Bernoulli numbers, we now have

$$\sum_{m=0}^{\infty}\beta_m\frac{x^m}{m!}=\left(\frac{1}{n}\sum_{m=0}^{\infty}\frac{\beta_m.n^m.x^m}{m!}\right)\sum_{m=0}^{\infty}\sum_{j=0}^{n-1}\frac{j^m x^m}{m!}$$

By the Cauchy product rule, we get

$$\sum_{m=0}^{\infty}\beta_m\frac{x^m}{m!}=\frac{1}{n}\sum_{m=0}^{\infty}\sum_{k=0}^{m}\left(\frac{\beta_k n^k x^k}{k!}\sum_{j=0}^{n-1}\frac{j^{m-k}.x^{m-k}}{(m-k)!}\right)$$

$$\sum_{m=0}^{\infty}\beta_m\frac{x^m}{m!}=\sum_{m=0}^{\infty}\left(\frac{1}{n}\sum_{k=0}^{m}\left(\frac{\beta_k n^k}{k!(m-k)!}\sum_{j=0}^{n-1}j^{m-k}\right)x^m\right)$$

Because a power series expansion is unique, we have

$$\beta_m=\frac{1}{n}\sum_{k=0}^{m}\left(n^k\binom{m}{k}\beta_k\sum_{j=0}^{n-1}j^{m-k}\right)$$

$$\beta_m=\frac{1}{n}\sum_{k=0}^{m-1}\left(n^k\binom{m}{k}\beta_k\sum_{j=0}^{n-1}j^{m-k}\right)+\frac{1}{n}\left(n^m.\beta_m\sum_{j=0}^{n-1}1\right)$$

$$\beta_m=\frac{1}{n}\sum_{k=0}^{m-1}\left(n^k\binom{m}{k}\beta_k\sum_{j=0}^{n-1}j^{m-k}\right)+\left(n^m.\beta_m\right)$$

Therefore

$$\beta_m=\frac{1}{n(1-n^m)}\sum_{k=0}^{m-1}\left(n^k\binom{m}{k}\beta_k\sum_{j=0}^{n-1}j^{m-k}\right)\text{For all } m\geq 1$$

A Recurrence Relation of $\beta_m(\lambda,x)$

In this section, we derive the following recurrence relation for $\beta_m(\lambda,x)$

$$\beta_m(\lambda,x)=\sum_{k=0}^{m}\binom{m}{k}\beta_k(\lambda)\left(\frac{x}{\lambda}\right)_{m-k} \tag{3.1}$$

Proof:-We know that the degenerate Bernoulli polynomial

$$\frac{t}{[(1+\lambda t)^{\mu}-1]}(1+\lambda t)^{\mu x}=\sum_{m=0}^{\infty}\beta_m(\lambda,x)\frac{t^m}{m!}$$

By (1.1) we get

$$\sum_{m=0}^{\infty}\beta_m(\lambda)\frac{t^m}{m!}(1+\lambda t)^{\mu x}=\sum_{m=0}^{\infty}\beta_m(\lambda,x)\frac{t^m}{m!}$$

By the Binomial expansion

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$$\sum_{m=0}^{\infty}\beta_m(\lambda)\frac{t^m}{m!}\sum_{m=0}^{\infty}\left(\frac{x}{\lambda}\right)_m\frac{t^m}{m!}=\sum_{m=0}^{\infty}\beta_m(\lambda,x)\frac{t^m}{m!}$$

By the Cauchy product rule

$$\beta_m(\lambda,x)=\sum_{k=0}^{m}\binom{m}{k}\beta_k(\lambda)\left(\frac{x}{\lambda}\right)_{m-k}$$

Where $\left(\frac{x}{\lambda}\right)_m = [x(x-\lambda)(x-2\lambda).....(x-(m-1)\lambda]$

This is new recurrence relation.

Properties of Degenerate Bernoulli Polynomial

In this section, some of well-known properties of Degenerate Bernoulli polynomials are derived from the generating function (1.2)

Property 1:

$$\beta_m(\lambda,x+y)=\sum_{k=0}^{m}\binom{m}{k}\beta_k(\lambda,x)\left(\frac{y}{\lambda}\right)_{m-k} \tag{4.1}$$

Proof: Now put $x \to x+y$ in (1.2)

$$\frac{t}{[(1+\lambda t)^{\mu}-1]}(1+\lambda t)^{\mu(x+y)}=\sum_{m=0}^{\infty}\beta_m(\lambda,x+y)\frac{t^m}{m!}$$

$$\frac{t}{[(1+\lambda t)^{\mu}-1]}(1+\lambda t)^{\mu x}(1+\lambda t)^{\mu y}=\sum_{m=0}^{\infty}\beta_m(\lambda,x+y)\frac{t^m}{m!}$$

By the equation (1.2)

$$\sum_{m=0}^{\infty}\beta_m(\lambda,x)\frac{t^m}{m!}(1+\lambda t)^{\mu y}=\sum_{m=0}^{\infty}\beta_m(\lambda,x+y)\frac{t^m}{m!}$$

By the help of Binomial expansion

$$(1+\lambda t)^{\mu y}=\sum_{m=0}^{\infty}\left(\frac{y}{\lambda}\right)_m\frac{t^m}{m!}$$

Therefore

$$\sum_{m=0}^{\infty}\beta_m(\lambda,x)\frac{t^m}{m!}\sum_{m=0}^{\infty}\left(\frac{y}{\lambda}\right)_m\frac{t^m}{m!}=\sum_{m=0}^{\infty}\beta_m(\lambda,x+y)\frac{t^m}{m!}$$

By the Cauchy product rule

$$\left(\sum_{n=0}^{\infty}a_n\frac{t^n}{n!}\right).\left(\sum_{n=0}^{\infty}b_n\frac{t^n}{n!}\right)=\sum_{n=0}^{\infty}c_n\frac{t^n}{n!}$$

Where $c_n=\sum_{k=0}^{n}\binom{n}{k}a_k.b_{n-k}$

$$\beta_m(\lambda,x+y)=\sum_{k=0}^{m}\binom{m}{k}\beta_k(\lambda,x)\left(\frac{y}{\lambda}\right)_{m-k}$$

Here *y*=1 then,

$$\beta_m(\lambda, x+1) = \sum_{k=0}^{m} \binom{m}{k} \beta_k(\lambda, x) \left(\frac{1}{\lambda}\right)_{m-k} \tag{4.2}$$

Property 2:

$$\beta_m^{'}(\lambda, x) = \lambda^{-1} \beta_m(\lambda, x) \tag{4.3}$$

Proof: By the generating function of degenerate Bernoulli polynomials

$$\frac{t.(1+\lambda t)^{\mu x}}{[(1+\lambda t)^{\mu} - 1]} = \sum_{m=0}^{\infty} \beta_m(\lambda, x) \frac{t^m}{m!}$$

Differentiate above equation with respect to *x*

$$\frac{\mu t.(1+\lambda t)^{\mu x}}{[(1+\lambda t)^{\mu} - 1]} = \sum_{m=0}^{\infty} \frac{d}{dx} \beta_m(\lambda, x) \frac{t^m}{m!}$$

$$\mu.\sum_{m=0}^{\infty} \beta_m(\lambda, x) \frac{.t^m}{m!} = \sum_{m=0}^{\infty} \beta_m^{'}(\lambda, x) \frac{t^m}{m!}$$

Equating the coefficients

$$\beta_m^{'}(\lambda, x) = \mu.\beta_m(\lambda, x)$$

$$\beta_m^{'}(\lambda, x) = \frac{1}{\lambda} \beta_m(\lambda, x) \text{ Where } \mu\lambda = 1$$

$$\beta_m^{'}(\lambda, x) = \lambda^{-1} \beta_m(\lambda, x)$$

Property 3:

$$\beta_m(\lambda, 1-x) = (-1)^m \beta_m(\lambda, x) \tag{4.4}$$

Proof:- By equation (1.2)

$$\frac{t.(1+\lambda t)^{\mu x}}{[(1+\lambda t)^{\mu} - 1]} = \sum_{m=0}^{\infty} \beta_m(\lambda, x) \frac{t^m}{m!}$$

$x \to 1-x$ in above equation

Put

$$\frac{t.(1+\lambda t)^{\mu(1-x)}}{[(1+\lambda t)^{\mu} - 1]} = \sum_{m=0}^{\infty} \beta_m(\lambda, 1-x) \frac{t^m}{m!}$$

$$\frac{t.(1+\lambda t)^{\mu}(1+\lambda t)^{-\mu x}}{[(1+\lambda t)^{\mu} - 1]} = \sum_{m=0}^{\infty} \beta_m(\lambda, 1-x) \frac{t^m}{m!}$$

$$\frac{t.(1+\lambda t)^{\mu}(1+\lambda t)^{-\mu x}}{(1+\lambda t)^{\mu}[1-(1+\lambda t)^{-\mu}]} = \sum_{m=0}^{\infty} \beta_m(\lambda, 1-x) \frac{t^m}{m!}$$

$$\frac{t.(1+\lambda t)^{-\mu x}}{[1-(1+\lambda t)^{-\mu}]} = \sum_{m=0}^{\infty} \beta_m(\lambda, 1-x) \frac{t^m}{m!}$$

$$-\frac{t.(1+\lambda t)^{-\mu x}}{[(1+\lambda t)^{-\mu} - 1]} = \sum_{m=0}^{\infty} \beta_m(\lambda, 1-x) \frac{t^m}{m!}$$

$$\sum_{m=0}^{\infty} (-1)^m \beta_m(\lambda, x)\frac{t^m}{m!} = \sum_{m=0}^{\infty} \beta_m(\lambda, 1-x)\frac{t^m}{m!}$$

Equating the coefficients

$$\beta_m(\lambda, 1-x) = (-1)^m \beta_m(\lambda, x)$$

Property 4:

$$\beta_n(\lambda, x+1) - \beta_n(\lambda, x) = n\left(\frac{x}{\lambda}\right)_{n-1} \tag{4.5}$$

Proof: By equation (1.2)

$$\frac{t.(1+\lambda t)^{\mu x}}{[(1+\lambda t)^{\mu} - 1]} = \sum_{n=0}^{\infty} \beta_n(\lambda, x)\frac{t^n}{n!} \tag{4.6}$$

Put $x \to x+1$ in above equation

$$\frac{t.(1+\lambda t)^{\mu(x+1)}}{[(1+\lambda t)^{\mu} - 1]} = \sum_{n=0}^{\infty} \beta_n(\lambda, x+1)\frac{t^n}{n!} \tag{4.7}$$

Subtracting (4.7)-(4.6)

$$\frac{t.(1+\lambda t)^{\mu(x+1)}}{[(1+\lambda t)^{\mu} - 1]} - \frac{t.(1+\lambda t)^{\mu x}}{[(1+\lambda t)^{\mu} - 1]} = \sum_{n=0}^{\infty} \beta_n(\lambda, x+1)\frac{t^n}{n!} - \sum_{n=0}^{\infty} \beta_n(\lambda, x)\frac{t^n}{n!}$$

$$t.(1+\lambda t)^{\mu x} = \sum_{n=0}^{\infty} [\beta_n(\lambda, x+1) - \beta_n(\lambda, x)]\frac{t^n}{n!}$$

By the Binomial expansion

$$t.\sum_{n=0}^{\infty} \left[\frac{x}{\lambda}\right]_n \frac{t^n}{n!} = \sum_{n=0}^{\infty} [\beta_n(\lambda, x+1) - \beta_n(\lambda, x)]\frac{t^n}{n!}$$

$$\sum_{n=0}^{\infty} \left[\frac{x}{\lambda}\right]_n \frac{t^{n+1}}{n!} = \sum_{n=0}^{\infty} [\beta_n(\lambda, x+1) - \beta_n(\lambda, x)]\frac{t^n}{n!}$$

$n \to n-1$ In L.H.S.

$$\sum_{n=0}^{\infty} \left[\frac{x}{\lambda}\right]_{n-1} \frac{t^n}{(n-1)!} = \sum_{n=0}^{\infty} [\beta_n(\lambda, x+1) - \beta_n(\lambda, x)]\frac{t^n}{n!}$$

Multiply and divide by n!

$$\sum_{n=0}^{\infty} \left[\frac{x}{\lambda}\right]_{n-1} \frac{t^n}{n!}.\frac{n!}{(n-1)!} = \sum_{n=0}^{\infty} [\beta_n(\lambda, x+1) - \beta_n(\lambda, x)]\frac{t^n}{n!}$$

$$\sum_{n=0}^{\infty} \left[\frac{x}{\lambda}\right]_{n-1} \frac{t^n}{n!}.\frac{n(n-1)!}{(n-1)!} = \sum_{n=0}^{\infty} [\beta_n(\lambda, x+1) - \beta_n(\lambda, x)]\frac{.t^n}{n!}$$

Equating the coefficients

$$\beta_n(\lambda, x+1) - \beta_n(\lambda, x) = n\left(\frac{x}{\lambda}\right)_{n-1}$$

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Property 4:

$$\left(\frac{x}{\lambda}\right)_m = \frac{1}{m+1}.\sum_{k=0}^{m}\binom{m+1}{k}\beta_k(\lambda,x)\left(\frac{1}{\lambda}\right)_{m+1-k} \tag{4.8}$$

Proof: By equation (4.2)

$$\beta_m(\lambda,x+1) = \sum_{k=0}^{m}\binom{m}{k}\beta_k(\lambda,x)\left(\frac{1}{\lambda}\right)_{m-k}$$

$$\beta_m(\lambda,x+1) = \sum_{k=0}^{m-1}\binom{m}{k}\beta_k(\lambda,x)\left(\frac{1}{\lambda}\right)_{m-k} + \binom{m}{m}\beta_m(\lambda,x)\left(\frac{1}{\lambda}\right)_{m-m}$$

$$\beta_m(\lambda,x+1) - \beta_m(\lambda,x) = \sum_{k=0}^{m-1}\binom{m}{k}\beta_k(\lambda,x)\left(\frac{1}{\lambda}\right)_{m-k}$$

But we know that by (4.5)

$$\beta_n(\lambda,x+1) - \beta_n(\lambda,x) = n\left(\frac{x}{\lambda}\right)_{n-1}$$

$$m\left(\frac{x}{\lambda}\right)_{m-1} = \sum_{k=0}^{m-1}\binom{m}{k}\beta_k(\lambda,x)\left(\frac{1}{\lambda}\right)_{m-k} \tag{4.9}$$

$$m \to m+1$$

Now put

$$(m+1)\left(\frac{x}{\lambda}\right)_m = \sum_{k=0}^{m}\binom{m+1}{k}\beta_k(\lambda,x)\left(\frac{1}{\lambda}\right)_{m+1-k}$$

$$\left(\frac{x}{\lambda}\right)_m = \frac{1}{m+1}.\sum_{k=0}^{m}\binom{m+1}{k}\beta_k(\lambda,x)\left(\frac{1}{\lambda}\right)_{m+1-k}$$

Now put *x=0* in equation (4.9), then

$$\sum_{k=0}^{m-1}\binom{m}{k}\beta_k(\lambda,0)\left(\frac{1}{\lambda}\right)_{m+1-k} = 0 \tag{4.10}$$

By definition

$$\beta_m(\lambda,0) = \beta_m(\lambda)$$

Therefore

$$\sum_{k=0}^{m-1}\binom{m}{k}\beta_k(\lambda)\left(\frac{1}{\lambda}\right)_{m+1-k} = 0 \tag{4.11}$$

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