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# An umbral setting for cumulants and factorial moments

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**Abstract**

We provide an algebraic setting for cumulants and factorial moments via the classical umbral calculus. Our main tools are the compositional inverse of the unity umbra, this being related to logarithmic power series, and a new umbra here introduced, the singleton umbra. We develop formulae that express cumulants, factorial moments and central moments as umbral functions.


**Résumé**

Nous etudions les cumulants et les moments factoriels par le calcul ombral classique. Les outils principaux sont l'inverse compositionnel de l'ombre unité, lié à la série formelle logarithmique, et un nouvel ombre, ici présenté, l'ombre singleton. De diverses formules sont données exprimant les cumulants, les moments factoriels et les moments centraux par des fonctions ombral.


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## 1. Introduction

The main purpose of this paper is to show how the classical umbral calculus provides a precision algebraic tool for handling cumulants and factorial moments. The classical umbral calculus consists of a symbolic technique to deal with sequences of numbers $a_n$ indexed by nonnegative integers $n = 0, 1, 2, 3, \ldots$: the subscripts are treated as if they were powers. This technique has been used extensively since the nineteenth century, despite wide-spread scepticism on the part of the mathematical community, which criticized its lack of rigorous foundations. To the best of our knowledge, the umbral method was first proposed by Rev. John Blissard in a series of papers from 1861 (cf. [5] for the full list of papers). It is however impossible to give full credit to Blissard for the original idea, since Blissard's calculus has its mathematical roots in symbolic differentiation. In the thirties, Bell [1] reviewed the whole subject in several papers, restoring the purport of Blissard's idea. In [2] he tried to give a rigorous foundation to the mystery at the center of the umbral calculus, but his attempt did not gain a following. In fact, in the first modern textbook of combinatorics [11], Riordan often employed this symbolic method without giving any formal justification. Gian-Carlo Rota was the first to disclose the "umbral magic art" of shifting from $a^n$ to $a_n$, bringing to light the underlying linear functional (cf. [14]). This idea led Rota and his collaborators to conceive a beautiful theory (cf. [9,13]) which has led to a large variety of applications (see [4] for a list of papers updated to 2000). Some years later, Roman and Rota [12] gave rigorous form to the umbral tricks in the setting of Hopf algebra (see also [8]). In 1994, however, Rota himself wrote (cf. [18]):

> "...Although the notation of Hopf algebra satisfied the most ardent advocate of spic-and-span rigor, the translation of "classical" umbral calculus into the newly found rigorous language made the method altogether unwieldy and unmanageable. Not only was the eerie feeling of witchcraft lost in the translation, but, after such a translation, the use of calculus to simplify computation and sharpen our intuition was lost by the wayside ..."

Then, in the paper [18] *The Classical Umbral Calculus* (1994) Rota and Taylor try to restore the feeling intended by the founders of the umbral calculus, introducing notation indispensable to avoiding the misunderstandings of the past, yet keeping such new notation to a minimum. In this new setting, the basic device is to represent a unital sequence of numbers by a symbol $\alpha$, named umbra, that is, to associate the sequence $1, a_1, a_2, \ldots$ to the sequence $1, \alpha, \alpha^2, \ldots$ of powers of $\alpha$ through an operator $E$ that resembles the expectation operator of random variables (r.v.'s). This new way of dealing with sequences of numbers has been applied to combinatorial and algebraic subjects (cf. [17,23,7]), wavelet theory (cf. [19]) and difference equations (cf. [24]). It has also led to a finely adapted language for r.v. theory, as shown in [16,5].

The present work is inspired by this last point of view. As a matter of fact, an umbra carries the structure of a random variable (r.v.), while making no reference to a probability space, bringing us thus closer to statistical methods. The use of symbolic methods in statistics is not, however, a novelty. For instance, Stuart and Ord [22] resort to such a technique in handling moments about a point. Furthermore, in the umbral calculus, questions concerning convergence of series are not materiel, as we show below when dealing with cumulants.

Among the sequences of numbers related to r.v.'s, cumulants play a central role, characterizing all r.v.'s occurring in the classical stochastic processes. For instance, a r.v. having Poisson distribution of parameter $x$ is the unique probability distribution for which all its cumulants are equal to $x$. It seems therefore that a r.v. is better described by its cumulants than by its moments. Moreover, due to their properties of additivity and invariance under translation, the cumulants are not necessarily connected with the moments of any probability distribution. We can define cumulants $\kappa_j$ of any sequence $a_n, n = 1, 2, 3, \ldots$ by

$$\sum_{n=0}^{\infty} \frac{a_n t^n}{n!} = \exp\left\{\sum_{j=1}^{\infty} \frac{\kappa_j t^j}{j!}\right\}$$

in complete disregard to questions of whether any series converges. Using this approach, many difficulties related to the "problem of cumulants" are resolved. (The "problem of cumulants" is to characterize those sequences that are cumulants of probability distributions.) The simplest example is that the second cumulant of a probability distribution must always be nonnegative, and is zero only if all of the higher cumulants are zero. Cumulants are subject to no such constraints when they are analyzed from an algebraic point of view. What is more, in statistics they do not play any dual role with respect to factorial moments. The algebraic setting here proposed brings to the light their close relationship, through an umbral analogy, with the well known complementary notions of compound and randomized Poisson r.v.'s (cf. [6]).

Umbral notations are introduced by means of r.v. semantics. Our intention is thus to make the reader comfortable with the umbral system of calculation, in a way that requires no prior knowledge. We skip some technical proofs of formal matters, for which the reader is referred via citations.

The novelties of this paper are the following. In Section 2, we introduce new operations among umbrae such as disjoint sum and disjoint difference, which permit the umbral representation of r.v. mixtures. In Section 3, we introduce a new umbra, the *singleton umbra*, which plays a role dual to the Bell umbra, introduced in [5]. Their relationship is encoded by the compositional inverse of the unity umbra, via the Lagrange inversion formula. The singleton umbra is the keystone of the umbral presentation of cumulants. So, in Section 4, we give a new and intrinsic definition of cumulant umbra unlike the recursive definition given by Rota and Shen in [16]. Starting from this definition, we simplify many results proved in [16]. We also state a very simple inversion theorem which permits us to generate an umbra from its cumulant. In Section 5, we give the definition of the factorial umbra of an umbra $\alpha$ and show that its moments are the factorial moments of the umbra $\alpha$. We also provide an inversion theorem which permits us to generate an umbra from its factorial umbra. Such inversion theorems state a new and very simple umbral relationship between cumulants and factorial moments. In the last two sections, we also give various umbral formulae for cumulants and factorial moments that parallel those known in statistics but simplify the proofs as well as the forms. This happens for instance for the equations expressing cumulants in terms of moments (and vice versa) and also for their recursive formulas.

Finally, we would like to remark on the preliminary nature of this paper. Despite the fact that we have defined all basic tools necessary to develop an umbral theory of cumulants and factorial moments, several applications remain to be developed. The reader interested is referred to the section "Problem eleven: cumulants" of the Gian-Carlo Rota's Fubini lecture *Twelve problems in probability no one likes to bring up* (cf. [15]).

## 2. Umbrae and random variables

In the following, we recall the terminology, notations and some basic definitions of the classical umbral calculus, as it has been introduced by Rota and Taylor in [18] and further developed in [5]. Fundamental is the idea of associating a sequence of numbers $1, a_2, a_3, \ldots$ to an indeterminate $\alpha$ which is then said to "represent" the sequence. This device is familiar in probability theory, where $a_i$ represents the $i$-th moment of a r.v. $X$. In this case, the sequence $1, a_1, a_2, \ldots$ results from applying the expectation operator $E$ to the sequence $1, X, X^2, \ldots$ consisting of powers of the r.v. $X$.

More formally, an umbral calculus consists of the following data:

(a) a set $A = \{\alpha, \beta, \ldots\}$, called the *alphabet*, whose elements are named *umbrae*;
(b) a commutative integral domain $R$ whose quotient field is of characteristic zero;
(c) a linear functional $E$, called *evaluation*, defined on the polynomial ring $R[A]$ and taking values in $R$ such that
  (i) $E[1] = 1$;
  (ii) $E[\alpha^i \beta^j \cdots \gamma^k] = E[\alpha^i]E[\beta^j] \cdots E[\gamma^k]$ for any set of distinct umbrae in $A$ and for $i, j, \ldots, k$ nonnegative integers (*uncorrelation property*);
(d) an element $\epsilon \in A$, called *augmentation* [12], such that $E[\epsilon^n] = \delta_{0,n}$, for any nonnegative integer $n$, where

$$\delta_{i,j} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases} \qquad i, j \in N;$$

(e) an element $u \in A$, called *unity* umbra [5], such that $E[u^n] = 1$, for any nonnegative integer $n$.

A sequence $a_0 = 1, a_1, a_2, \ldots$ in $R$ is umbrally represented by an umbra $\alpha$ when

$$E[\alpha^i] = a_i, \qquad \text{for } i = 0, 1, 2, \ldots.$$

The elements $a_i$ are called *moments* of the umbra $\alpha$ in analogy with r.v. theory. The umbra $\epsilon$ can be view as a r.v. which takes the value 0 with probability 1 and the umbra $u$ as a r.v. which takes the value 1 with probability 1. Note that the uncorrelation property among umbrae parallels the analogous property for r.v.'s, and that $E[\alpha^{n+k}] \neq E[\alpha^n]E[\alpha^k]$.

**Example 2.1** (*Bell Umbra*). The *Bell umbra* $\beta$ is the umbra such that

$$E[(\beta)_n] = 1 \qquad n = 0, 1, 2, \ldots$$

where $(\beta)_0 = 1$ and $(\beta)_n = \beta(\beta - 1) \cdots (\beta - n + 1)$ is the lower factorial. It follows that $E[\beta^n] = B_n$ where $B_n$ is the $n$-th Bell number (cf. [5]), i.e. the number of partitions of a finite nonempty set with $n$ elements, or the $n$-th coefficient in the Taylor series expansion of the function $\exp(e^t - 1)$. So $\beta$ is the umbral counterpart of a Poisson r.v. with parameter 1.

We call *factorial moments* of an umbra $\alpha$ the elements

$$a_{(n)} = \begin{cases} 1, & n = 0 \\ E[(\alpha)_n], & n > 0 \end{cases}$$

where $(\alpha)_n = \alpha(\alpha - 1) \cdots (\alpha - n + 1)$ is the lower factorial. So the definition of $\beta$ in Example 2.1 could be reformulated as follows: the Bell scalar umbra is the umbra whose factorial moments are $b_{(n)} = 1$ for any nonnegative integer $n$.

### 2.1. Similar umbrae and dot-product

The notion of similarity among umbrae comes in handy in order to manipulate sequences such

$$\sum_{i=0}^{n} \binom{n}{i} a_i a_{n-i}, \qquad n \in N \tag{1}$$

as moments of umbrae. The sequence (1) cannot be represented by using only the umbra $\alpha$ with moments $a_0 = 1, a_1, a_2, \ldots$. Indeed, $\alpha$ being correlated to itself, the product $a_i a_{n-i}$ cannot be written as $E[\alpha^i \alpha^{n-i}]$. So we need two distinct umbrae having the same sequence of moments, as happens for r.v.'s equal in distribution. Therefore, if we choose an umbra $\alpha'$ uncorrelated with $\alpha$ but with the same sequence of moments, it is

$$\sum_{i=0}^{n} \binom{n}{i} a_i a_{n-i} = E\left[\sum_{i=0}^{n} \binom{n}{i} \alpha^i (\alpha')^{n-i}\right] = E[(\alpha + \alpha')^n]. \tag{2}$$

Then the sequence (1) represents the moments of the umbra $(\alpha + \alpha')$. A way to formalize this matter is to define two equivalence relations among umbrae.

Two umbrae $\alpha$ and $\gamma$ are *umbrally equivalent* when

$$E[\alpha] = E[\gamma],$$

in symbols $\alpha \simeq \beta$. They are *similar* when

$$\alpha^n \simeq \gamma^n, \qquad n = 0, 1, 2, \ldots$$

in symbols $\alpha \equiv \gamma$. We note that equality implies similarity, which implies umbral equivalence. The converses are false. We shall denote by the symbol $n.\alpha$ the *dot-product* of $n$ and $\alpha$, an auxiliary umbra (cf. [18]) similar to the sum $\alpha' + \alpha'' + \cdots + \alpha'''$ where $\alpha', \alpha'', \ldots, \alpha'''$ are a set of $n$ distinct umbrae each similar to the umbra $\alpha$. So the sequence in (2) is umbrally represented by the umbra $2.\alpha$. We assume that $0.\alpha$ is an umbra similar to the augmentation $\epsilon$.

We shall hereafter consider the dot product of $n$ and $\alpha$ as an umbra in its own right, where we *saturate* the alphabet $A$ with sufficiently many umbrae similar to any expression whatever. For a formal definition of a saturated umbral calculus see [18]. It can be shown that saturated umbral calculi exist and that every umbral calculus can be embedded in a saturated umbral calculus.

The following statements are easily proven:

**Proposition 2.2.** (i) *If* $n.\alpha \equiv n.\beta$ *for some integer* $n \neq 0$ *then* $\alpha \equiv \beta$*;*
(ii) *if* $c \in R$ *then* $n.(c\alpha) \equiv c(n.\alpha)$ *for any nonnegative integer* $n$*;*
(iii) $n.(m.\alpha) \equiv (nm).\alpha \equiv m.(n.\alpha)$ *for any two nonnegative integers* $n, m$*;*
(iv) $(n+m).\alpha \equiv n.\alpha + m.\alpha'$ *for any two nonnegative integers* $n, m$ *and any two distinct umbrae* $\alpha \equiv \alpha'$*;*
(v) $(n.\alpha + n.\beta) \equiv n.(\alpha + \beta)$ *for any nonnegative integer* $n$ *and any two distinct umbrae* $\alpha$ *and* $\beta$.

Two umbrae $\alpha$ and $\gamma$ are said to be *inverse* to each other when $\alpha + \gamma \equiv \varepsilon$. We denote the inverse of the umbra $\alpha$ by $-1.\alpha'$, with $\alpha \equiv \alpha'$. Recall that, in dealing with a saturated umbral calculus, the inverse of an umbra is not unique, but any two umbrae inverse to any given umbra are similar.

**Example 2.3** (*Uniform Umbra*). The *Bernoulli umbra* (cf. [18]) represents the sequence of Bernoulli numbers $B_n$ such that

$$\sum_{k\geq 0} \binom{n}{k} B_k = B_n.$$

The inverse of the Bernoulli umbra is the umbral counterpart of the uniform r.v. over the interval [0, 1] (cf. [23]).

### 2.2. Generating functions

The formal power series in $R[A][[t]]$

$$u + \sum_{n\geq 1} \alpha^n \frac{t^n}{n!} \tag{3}$$

is the *generating function* (g.f.) of the umbra $\alpha$, and it is denoted by $e^{\alpha t}$. The notion of umbrally equivalence and similarity can be extended coefficientwise to formal power series $R[A][[t]]$ (see [24] for a formal construction). So

$$\alpha \equiv \beta \Leftrightarrow e^{\alpha t} \simeq e^{\beta t}.$$

Moreover, any exponential formal power series[1] in $R[[t]]$ like

$$f(t) = 1 + \sum_{n\geq 1} a_n \frac{t^n}{n!}$$

can be umbrally represented by a formal power series (3) in $R[A][[t]]$. In fact, if the sequence $1, a_1, a_2, \ldots$ is umbrally represented by $\alpha$ then

$$f(t) = E[e^{\alpha t}] \qquad \text{i.e.} \qquad f(t) \simeq e^{\alpha t},$$

assuming that we naturally extend $E$ to be linear. We will say that $f(t)$ is umbrally represented by $\alpha$. Henceforth, when no confusion is possible, we will just say that $f(t)$

[1] Observe that with this approach we disregard questions of whether any series converges.

is the g.f. of $\alpha$. For example the g.f. of the augmentation umbra $\epsilon$ is 1, while the g.f. of the unity umbra $u$ is $e^x$.

Recall that for a r.v. $X$, when the moment generating function (m.g.f.) exists, $E[\exp(tX)] = f(t)$, it admits an exponential expansion in terms of the moments, which are completely determined by the related distribution function (and vice versa). In this case the m.g.f. encodes all the information of $X$ and the notion of equivalence in distribution among r.v.'s corresponds to the notion of similarity of umbrae.

The first advantage of the umbral notation introduced for g.f.'s is the representation of operations among g.f.'s by operations among umbrae. For example the product of exponential g.f.'s is umbrally represented by a sum of the corresponding umbrae:

$$g(t)\, f(t) \simeq e^{(\alpha+\gamma)t} \qquad \text{with } f(t) \simeq e^{\alpha t}, g(t) \simeq e^{\gamma t}. \tag{4}$$

Via (4), the g.f. of $n.\alpha$ is $f(t)^n$. If $\alpha$ is an umbra with g.f. $f(t)$, the inverse umbra $-1.\alpha'$ has g.f. $[f(t)]^{-1}$. The sum of exponential g.f.'s is umbrally represented by a disjoint sum of umbrae. The *disjoint sum* (respectively *disjoint difference*) of $\alpha$ and $\gamma$ is the umbra $\eta$ (respectively $\iota$) with moments

$$\eta^n \simeq \begin{cases} u, & n = 0 \\ \alpha^n + \gamma^n, & n > 0 \end{cases} \qquad \left( \text{respectively} \qquad \iota^n \simeq \begin{cases} u, & n = 0 \\ \alpha^n - \gamma^n, & n > 0 \end{cases} \right),$$

in symbols $\eta \equiv \alpha \dot{+} \gamma$ (respectively $\iota \equiv \alpha \dot{-} \gamma$). By the definition, it follows

$$f(t) \pm [g(t) - 1] \simeq e^{(\alpha \dot{\pm} \gamma)t}.$$

**Example 2.4** (*Unbiased Estimators*). Consider a disjoint sum of $n$ times the umbra $\alpha$. We will denote this umbra by $\dot{+}_n\alpha$. Its g.f. is $1+n[f(t)-1]$. The umbra $\dot{+}_n\alpha$ has the following probabilistic counterpart. Let $\{X_i\}_{i=1}^n$ be a random sample of independent and identically distributed (i.i.d.) r.v's. As is well-known, the power sum symmetric function

$$S_r = \sum_{i=1}^{n} X_i^r$$

gives unbiased estimators $S_r/n$ of the moments of $X_i$. But $E[(\dot{+}_n\alpha)^r] = n a_r$, hence the umbral corresponding of the power sum symmetric functions sequence $S_r$ is the umbra $\dot{+}_n\alpha$.

### 2.3. Auxiliary umbrae

In the following, suppose $\alpha$ an umbra with g.f. $f(t)$ and $\gamma$ an umbra with g.f. $g(t)$. The introduction of the g.f. device leads to the definition of new auxiliary umbrae useful for the development of the system of calculation. For this purpose, we should replace $R$ by a polynomial ring having coefficients in $R$ and any desired number of indeterminates. In this paper, we deal with $R[x, y]$. This allows us to define the dot-product of $x$ and $\alpha$ via a g.f., i.e. $x.\alpha$ is the auxiliary umbra having generating function

$$e^{(x.\alpha)} \simeq f(t)^x.$$

Proposition 2.2 still holds, replacing $n$ with $x$ and $m$ with $y$. Then, an umbra is said to be *scalar* if the moments are elements of $R$ while it is said to be *polynomial* if the moments are polynomials.

**Example 2.5** (*Bell Polynomial Umbra*). The *Bell polynomial umbra* $\phi$ is the umbra having factorial moments equal to $x^n$ (cf. [5]). This umbra has g.f. $\exp[x(\mathrm{e}^t - 1)]$ so that $\phi \equiv x.\beta$, where $\beta$ is the Bell umbra. It turns out that the Bell polynomial umbra $x.\beta$ is the umbral counterpart of a Poisson r.v. with parameter $x$.

**Example 2.6** (*Moments about a Point*). The moments $E[(X - a)^n]$ about a point $a \in \mathbf{R}$ of a r.v. $X$ are easily represented by umbrae through the following definition: the umbra $\alpha^a$ having moments about a point $a \in R$ is defined as

$$\alpha^a \equiv \alpha - a.u. \tag{5}$$

If $a, b \in R$ and $b - a = c$, then

$$\alpha^a \equiv \alpha - (b + c).u \equiv \alpha^b + c.u,$$

is the umbral version of the equations giving the moments about $a$ in terms of the moments about $b$ (cf. [22] for another symbolic expression).

The dot-product $\gamma.\alpha$ of two umbrae is the auxiliary umbra having g.f.

$$\mathrm{e}^{(\gamma.\alpha)t} \simeq [f(t)]^{\gamma} \simeq \mathrm{e}^{\gamma \log f(t)} \simeq g[\log f(t)].$$

The moments of the dot-product $\gamma.\alpha$ are (cf. [5])

$$E[(\gamma.\alpha)^n] = \sum_{i=0}^{n} g_{(i)} B_{n,i}(a_1, a_2, \ldots, a_{n-i+1}) \qquad n = 0, 1, 2, \ldots \tag{6}$$

where $g_{(i)}$ are the factorial moments of the umbra $\gamma$, $B_{n,i}$ are the (partial) Bell exponential polynomials (cf. [11]) and $a_i$ are the moments of the umbra $\alpha$. Observe that $E[\gamma.\alpha] = g_1 a_1 = E[\gamma]\,E[\alpha.]$. The following properties hold (cf. [5]):

**Proposition 2.7.** (a) *If $\eta.\alpha \equiv \eta.\gamma$ then $\alpha \equiv \gamma$;*
(b) *if $c \in R$ then $\eta.(c\alpha) \equiv c(\eta.\alpha)$ for any two distinct umbrae $\alpha$ and $\eta$;*
(c) *if $\gamma \equiv \gamma'$ then $(\alpha + \eta).\gamma \equiv \alpha.\gamma + \eta.\gamma'$;*
(d) $\eta.(\gamma.\alpha) \equiv (\eta.\gamma).\alpha$.

Observe that from property (b) it follows that

$$\alpha.x \equiv \alpha.(xu) \equiv x(\alpha.u) \equiv x\alpha. \tag{7}$$

**Remark 1.** The auxiliary umbra $\gamma.\alpha$ is the umbral version of a random sum. Indeed the m.g.f. $g[\log f(t)]$ corresponds to a r.v. $S_N = X_1 + X_2 + \cdots + X_N$ where $N$ is a discrete r.v. having m.g.f. $g(t)$ and $X_i$ are i.i.d. r.v.'s having m.g.f. $f(t)$. The right-distributive property of the dot-product $\gamma.\alpha$ runs in parallel with the probability theory because a random sum $S_{N+M}$ is similar to $S_N + S_M$, where $N$ and $M$ are independent discrete r.v.'s. The left-distributive property of the dot-product $\gamma.\alpha$ does not hold as well as it does in r.v. theory. In fact, let $Z = X + Y$ be a r.v. with $X$ and $Y$ independent r.v.'s. As it is easy to verify, a random sum $S_N = Z_1 + Z_2 + \cdots + Z_N$, with $Z_i$ i.i.d. r.v.'s equal in distribution to $Z$, is not equal in distribution to $S_N^X + S_N^Y$ with $S_N^X = X_1 + X_2 + \cdots + X_N$, $X_i$ being i.i.d. r.v.'s equal in distribution to $X$ and with $S_N^Y = Y_1 + Y_2 + \cdots + Y_N$, $Y_i$ being i.i.d. r.v.'s equal in distribution to $Y$.

**Example 2.8** (*Randomized Poisson r.v.*). Let us consider the Bell polynomial umbra $x.\beta$. If in the place of $x$ we put a generic umbra $\alpha$, we get the auxiliary umbra $\alpha.\beta$ whose factorial moments are

$$(\alpha.\beta)_n \simeq \alpha^n \qquad n = 0, 1, 2, \ldots$$

and moments given by the exponential umbral polynomials (cf. [5])

$$(\alpha.\beta)^n \simeq \Phi_n(\alpha) \simeq \sum_{i=0}^{n} S(n,i)\alpha^i \quad n = 0, 1, 2, \ldots. \tag{8}$$

Its g.f. is $f[e^t - 1]$. The umbra $\alpha.\beta$ represents a random sum of independent Poisson r.v.'s with parameter 1 indexed by an integer r.v. $Y$, i.e. a randomized Poisson r.v. with parameter $Y$.

As suggested in [13], there is a connection between compound Poisson processes and polynomial sequence of binomial type, i.e. sequence $\{p_n(x)\}$ of polynomials with degree $n$ satisfying the identities

$$p_n(x+y) = \sum_{i=0}^{n} \binom{n}{i} p_i(x) p_{n-i}(y)$$

for any $n$ (cf. for instance [9]). Two different approaches can be found in [3] and in [21]. A natural device to make clear this connection is the *$\alpha$-partition* umbra $\beta.\alpha$, introduced in [5]. Its g.f. is $\exp[f(t) - 1]$ and it suggests interpreting a partition umbra as a compound Poisson r.v. with parameter 1. As is well-known, a compound Poisson r.v. with parameter 1 is introduced as a random sum $S_N = X_1 + X_2 + \cdots + X_N$ where $N$ has a Poisson distribution with parameter 1. The umbra $\beta.\alpha$ fits perfectly this probabilistic notion, taking into consideration that the Bell scalar umbra $\beta$ plays the role of a Poisson r.v. with parameter 1. What is more, since a Poisson r.v. with parameter $x$ is umbrally represented by the Bell polynomial umbra $x.\beta$, a compound Poisson r.v. with parameter $x$ is represented by the *polynomial $\alpha$-partition* umbra $x.\psi \equiv x.\beta.\alpha$ with g.f. $\exp[x(f(t) - 1)]$. The name "partition umbra" has a probabilistic ground. Indeed the parameter of a Poisson r.v. is usually denoted by $x = \lambda t$, with $t$ representing a time interval, so that when this interval is partitioned into non-overlapping ones, their contributions are stochastically independent and add to $S_N$. This last circumstance is umbrally expressed by the relation

$$(x+y).\beta.\alpha \equiv x.\beta.\alpha + y.\beta.\alpha \tag{9}$$

giving the binomial property for the polynomial sequence represented by $x.\beta.\alpha$. In terms of g.f.'s, the formula (9) means that

$$h_{x+y}(t) = h_x(t) h_y(t) \tag{10}$$

where $h_x(t)$ is the g.f. of $x.\beta.\alpha$. Vice versa every g.f. $h_x(t)$ satisfying the equality (10) is the g.f. of a polynomial $\alpha$-partition umbra. The $\alpha$-partition umbra represents the sequence of *partition polynomials* $Y_n = Y_n(a_1, a_2, \ldots, a_n)$ (or complete Bell exponential polynomials [11]), i.e.

$$E[(\beta.\alpha)^n] = \sum_{i=0}^{n} B_{n,i}(a_1, a_2, \ldots, a_{n-i+1}) = Y_n(a_1, a_2, \ldots, a_n), \tag{11}$$

where $a_i$ are the moments of the umbra $\alpha$. Moreover every $\alpha$-partition umbra satisfies the relation

$$(\beta.\alpha)^n \simeq \alpha'(\beta.\alpha + \alpha')^{n-1} \qquad \alpha \equiv \alpha',\ n = 0, 1, 2, \ldots \tag{12}$$

and conversely (see [5] for a proof). The previous property will allow a useful umbral characterization of the cumulant umbra (see Corollary 4.12 in Section 4). The umbra $\beta.\alpha$ plays a central role also in the umbral representation of the composition of exponential g.f.'s. Indeed, the *composition umbra* of $\alpha$ and $\gamma$ is the umbra $\tau \equiv \gamma.\beta.\alpha$. The umbra $\tau$ has g.f. $g[f(t) - 1]$ and moments

$$E[\tau^n] = \sum_{i=0}^{n} g_i B_{n,i}(a_1, a_2, \ldots, a_{n-i+1}) \tag{13}$$

with $g_i$ and $a_i$ moments of the umbra $\gamma$ and $\alpha$ respectively. We denote by $\alpha^{\langle -1 \rangle}$ the compositional inverse of $\alpha$, i.e. the umbra having g.f. $f^{-1}(t)$ such that $f^{-1}[f(t) - 1] = f[f^{-1}(t) - 1] = 1 + t$. For an intrinsic umbral expression of the compositional inverse umbra see [5], where there is also stated an umbral version of the Lagrange inversion formula.

**Example 2.9** (*Randomized Compound Poisson r.v.*). As already underlined in Example 2.8, the umbra $\gamma.\beta$ represents a randomized Poisson r.v. Hence it is natural to look at the composition umbra as a compound randomized Poisson r.v., i.e. a random sum indexed by a randomized Poisson r.v. Moreover, since $(\gamma.\beta).\alpha \equiv \gamma.(\beta.\alpha)$ (cf. statement (d) of Proposition 2.7), the previous relation reveals another aspect of this r.v.: the umbra $\gamma.(\beta.\alpha)$ generalizes the concept of a random sum of i.i.d. compound Poisson r.v.'s with parameter 1 indexed by an integer r.v. $X$, i.e. a randomized compound Poisson r.v. with random parameter $X$.

Finally, the symbol $\alpha^{.n}$ denotes an auxiliary umbra similar to the product $\alpha'\alpha'' \cdots \alpha'''$ where $\alpha', \alpha'', \ldots, \alpha'''$ are a set of $n$ distinct umbrae each similar to the umbra $\alpha$. We assume that $\alpha^{.0}$ is an umbra similar to the unity umbra $u$. The moments of $\alpha^{.n}$ are:

$$E[(\alpha^{.n})^k] = E[(\alpha^k)^{.n}] = a_k^n, \qquad k = 0, 1, 2, \ldots, \tag{14}$$

i.e. the $n$-th power of the moments of the umbra $\alpha$. Thanks to this notation in [5], the umbral expression of the Bell exponential polynomials was given as follows:

$$B_{n,i}(a_1, a_2, \ldots, a_{n-i+1}) \simeq \binom{n}{i} \alpha^{.i} (i.\overline{\alpha})^{n-i} \tag{15}$$

whenever $a_1 \neq 0$ and where $\overline{\alpha}$ is the umbra with moments

$$E[\overline{\alpha}^n] = \frac{a_{n+1}}{a_1(n+1)}, \qquad n = 1, 2, \ldots. \tag{16}$$

Table 1
Duality between the singleton umbra and the Bell umbra

| Umbra | Generating function |
|---|---|
| $\chi$ | $(1+t) = 1 + \sum_{n=1}^{\infty} \left[\sum_{k=1}^{n} s(n,k)\right] \frac{t^n}{n!}$ |
| $x.\chi$ | $(1+t)^x = 1 + \sum_{n=1}^{\infty} \left[\sum_{k=1}^{n} s(n,k) x^k\right] \frac{t^n}{n!}$ |
| $\alpha.\chi$ | $f[\log(1+t)]$ |
| $\chi.\alpha$ | $1 + \log[f(t)]$ |
| $\beta$ | $e^{e^{t-1}} = 1 + \sum_{n=1}^{\infty} \left[\sum_{k=1}^{n} S(n,k)\right] \frac{t^n}{n!}$ |
| $x.\beta$ | $e^{x(e^{t-1})} = 1 + \sum_{n=1}^{\infty} \left[\sum_{k=1}^{n} S(n,k) x^k\right] \frac{t^n}{n!}$ |
| $\alpha.\beta$ | $f(e^t - 1)$ |
| $\beta.\alpha$ | $e^{f(t)-1}$ |

**Example 2.10** (*The Central Umbra*). We call *central umbra* the umbra $\alpha^{a_1}$ having moments about $a_1 = E[\alpha]$. From (5), the classical relation between moments and central moments of a r.v. has the following umbral expression:

$$(\alpha^{a_1})^n \simeq \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} (\alpha')^k \alpha^{\cdot(n-k)}, \qquad \alpha \equiv \alpha' \; n = 1, 2, \ldots$$

where $E[(a_1.u)^k] = a_1^k = E[\alpha^{\cdot k}]$ from (14).

## 3. The singleton umbra

The singleton umbra plays a role dual to the Bell umbra, even if it has no probabilistic counterpart. The singleton umbra turns out to be an effective symbolic tool for umbral representation of some well-known r.v.'s, as well as of cumulants and factorial moments.

**Definition 3.1** (*The Singleton Umbra*). An umbra $\chi$ is said to be a singleton umbra if

$$\chi^n \simeq \delta_{1,n} \qquad n = 1, 2, \ldots.$$

The g.f. of the singleton umbra $\chi$ is $1 + t$.

**Example 3.2** (*Gamma r.v.*). The m.g.f. of a Gamma r.v. with parameters $a$ and $c$ is

$$M(t) = \frac{1}{(1-ct)^a}.$$

This g.f. is umbrally represented by the inverse of $-c(a.\chi)$ (see (ii) of Proposition 2.2 replacing $n$ by $a \in R$).

Table 1 highlights the duality between the singleton umbra $\chi$ and the Bell umbra $\beta$.

The connection between the singleton umbra $\chi$ and the Bell umbra $\beta$ is made clear in the following proposition.

**Proposition 3.3.** *Let $\chi$ be the singleton umbra, $\beta$ the Bell umbra and $u^{\langle -1\rangle}$ the compositional inverse of the unity umbra $u$. It follows that*

$$\chi \equiv u^{\langle -1\rangle}.\beta \equiv \beta.u^{\langle -1\rangle}, \tag{17}$$

$$\beta.\chi \equiv u \equiv \chi.\beta. \tag{18}$$

**Proof.** The g.f. of $u^{\langle -1\rangle}.\beta.u$ is $1+t$, since $u^{\langle -1\rangle}$ and $u$ are compositional inverses. So equivalence (17) follows by property (a) of Proposition 2.7, that is,

$$u^{\langle -1\rangle}.\beta.u \equiv \chi \equiv \chi.u.$$

Equivalence (18) follows via g.f.'s in Table 1. □

Distributive properties of the singleton umbra with respect to the sum and the disjoint sum of umbrae are given in the following.

**Proposition 3.4.**

$$\chi.(\alpha+\gamma) \equiv \chi.\alpha\dot{+}\chi.\gamma \tag{19}$$

$$(\alpha\dot{+}\gamma).\chi \equiv \alpha.\chi\dot{+}\gamma.\chi. \tag{20}$$

**Proof.** Let $f(t)$ be the g.f. of $\alpha$ and $g(t)$ the g.f. of $\gamma$. Equivalence (19) follows by observing that the g.f. of $\chi.(\alpha+\beta)$ is $1+\log[f(t)g(t)] = 1+\log[f(t)]+\log[g(t)]$, so it is the g.f. of $\chi.\alpha\dot{+}\chi.\beta$. Equivalence (20) follows by observing that the g.f. of $(\alpha\dot{+}\beta).\chi$ is $f[\log(1+t)]+g[\log(1+t)]-1$, that is, the g.f. of $\alpha.\chi\dot{+}\beta.\chi$. □

The notion of mixture of r.v.'s has an umbral counterpart in the disjoint sum $\dot{+}$. Indeed let $\{\alpha_i\}_{i=1}^n$ be $n$ umbrae and $\{p_i\}_{i=1}^n \in R$ be $n$ weights such that

$$\sum_{i=1}^n p_i = 1.$$

The mixed umbra $\gamma$ of $\{\alpha_i\}_{i=1}^n$ is the following weighted disjoint sum of $\{\alpha_i\}_{i=1}^n$:

$$\gamma \equiv \chi.p_1.\beta.\alpha_1\dot{+}\chi.p_2.\beta.\alpha_2\dot{+}\cdots\dot{+}\chi.p_n.\beta.\alpha_n \tag{21}$$

where $\beta$ is the Bell umbra and $\chi$ is the singleton umbra. From (19) equivalence (21) can be rewritten as

$$\gamma \equiv \chi.(p_1.\beta.\alpha_1 + p_2.\beta.\alpha_2 + \cdots + p_n.\beta.\alpha_n).$$

Since the g.f. of $\sum_{i=1}^n p_i.\beta.\alpha_i$ is $\exp(\sum_{i=1}^n p_i[f_i(t)-1])$, where $f_i(t)$ is the g.f. of $\alpha_i$, from Table 1 it follows that the g.f. of $\gamma$ is $\sum_{i=1}^n p_i f_i(t)$.

**Example 3.5** (*Bernoulli Umbral r.v.*). Let us consider a Bernoulli r.v. $X$ of parameter $p$. Its m.g.f. is $g(t) = q + pe^t$ with $q = 1-p$. The Bernoulli umbral r.v. is a mixture of the umbra $\varepsilon$ and the unity umbra $u$:

$$\xi \equiv \chi.q.\beta.\varepsilon\dot{+}\chi.p.\beta.u.$$

Recalling that $\chi.q.\beta.\varepsilon \equiv \varepsilon$, we have

$$\xi \equiv \chi.p.\beta.$$

Indeed,

$$E[e^{\xi t}] = 1 + \log[e^{p(e^t-1)}] = q + pe^t.$$

**Example 3.6** (*Binomial Umbral r.v.*). As is well-known, any binomial r.v. $Y$ with parameters $n \in \mathbf{N}$, $p \in [0,1]$, is the sum of $n$ i.i.d. Bernoulli r.v.'s having parameter $p$. Then the binomial umbral r.v. is

$$n.\xi \equiv n.\chi.p.\beta.$$

The parallelism is evident if we recall that the m.g.f. of a binomial r.v. $Y$ is $f(t) = (q + pe^t)^n$.

## 4. The cumulant umbra

For a r.v. having moments $a_1, a_2, \ldots, a_n$ and cumulants $\kappa_1, \kappa_2, \ldots, \kappa_n$,

$$a_n = \sum_{\pi} c_\pi \kappa_\pi \qquad \text{and} \qquad \kappa_n = \sum_{\pi} d_\pi a_\pi \tag{22}$$

the sums here are taken over the partitions $\pi = [j_1^{m_1}, j_2^{m_2}, \ldots, j_k^{m_k}]$ of the integer $n$, and

$$\begin{aligned} c_\pi &= \frac{n!}{(j_1!)^{m_1}(j_2!)^{m_2}\cdots(j_k!)^{m_k}} \frac{1}{m_1!m_2!\cdots m_k!} \\ d_\pi &= c_\pi(-1)^{\nu_\pi - 1}(\nu_\pi - 1)! \qquad \text{and} \qquad \nu_\pi = m_1 + m_2 + \cdots + m_k \\ a_\pi &= \prod_{j\in\pi} a_j \qquad \text{and} \qquad \kappa_\pi = \prod_{j\in\pi} \kappa_j. \end{aligned}$$

A different formulation of (22) was given by Rota and Shen in [16] by Moebius' inversion formula. In this section we show how the umbral calculus simplifies the above expressions, as well as the recursive formulae for moments in terms of cumulants.

Let $\alpha$ be an umbra with g.f. $f(t)$.

**Definition 4.1.** The cumulant of an umbra $\alpha$ is the umbra $\kappa_\alpha$ defined by

$$\kappa_\alpha \equiv \chi.\alpha$$

where $\chi$ is the singleton umbra.

Definition 4.1 gives the umbral version of the second equality in (22). Moreover the first moment of the cumulant umbra $\kappa_\alpha$ is $a_1$, i.e. the first moment of the umbra $\alpha$, being $E[\kappa_\alpha] = E[\chi]E[\alpha] = E[\alpha] = a_1$.

**Example 4.2** (*Cumulant of the Umbra* $\varepsilon$). Since $\varepsilon \equiv \chi.\varepsilon$, the umbra $\varepsilon$ is the cumulant umbra of itself, i.e. $\kappa_\varepsilon \equiv \varepsilon$.

**Example 4.3** (*Cumulant of the Umbra* $u$). Since $\chi \equiv \chi.u$, the umbra $\chi$ is the cumulant umbra of the umbra $u$, i.e. $\kappa_u \equiv \chi$.

**Example 4.4** (*Cumulant of the Bell Umbra*). Since $u \equiv \chi.\beta$ (see (18)), the umbra $u$ is the cumulant umbra of the Bell umbra $\beta$, i.e. $\kappa_\beta \equiv u$. From Example 2.1, a Poisson r.v. of parameter 1 has cumulants equal to 1.

**Proposition 4.5.** *The cumulant umbra $\kappa_\alpha$ has g.f.*

$$k(t) = 1 + \log[f(t)]. \tag{23}$$

**Proof.** See Table 1. $\square$

**Example 4.6** (*Cumulant of the Singleton Umbra*). Since $1 + \log(1 + t)$ is the g.f. of the umbra $u^{\langle -1\rangle}$, this umbra is the cumulant umbra of the umbra $\chi$, i.e. $\kappa_\chi \equiv u^{\langle -1\rangle}$.

**Example 4.7** (*Cumulant of the Bernoulli Umbral r.v.*). From Example 3.5, the cumulant umbra of the Bernoulli umbral r.v. is $\chi.(\chi.p.\beta) \equiv u^{\langle -1\rangle}.p.\beta$.

**Example 4.8** (*Cumulant of the Binomial Umbral r.v.*). From Example 3.6, the cumulant umbra of the Binomial umbral r.v. is $\chi.(n.\chi.p.\beta)$, i.e.

$$\chi.(\chi'.p.\beta' + \chi''.p.\beta'' + \cdots + \chi'''.p.\beta'''),$$

where $\chi', \chi'', \ldots, \chi'''$ are a set of $n$ distinct umbrae each similar to the singleton umbra $\chi$, while $\beta', \beta'', \ldots, \beta'''$ are a set of $n$ distinct umbrae each similar to the Bell umbra $\beta$. From (19) and recalling Examples 2.4 and 4.6, it results that

$$\chi.n.\chi.p.\beta \equiv \chi.\chi'.p.\beta' \dot{+} \chi.\chi''.p.\beta'' \dot{+} \cdots \dot{+} \chi.\chi'''.p.\beta''' \equiv \dot{+}_n u^{\langle -1\rangle}.p.\beta.$$

This parallels the analogous result in probability theory.

From (23), the moments of the cumulant umbra $\kappa_\alpha$ are

$$(k_a)_n = E[\kappa_\alpha^n] = \left[\frac{\mathrm{d}^n}{\mathrm{d}t^n} \log\{f(t)\}\right]_{t=0}$$

which is equivalent to the definition of the $n$-th cumulant of a r.v. $X$ having m.g.f. $f(t)$.

To state the explicit version of the second equality in (22)

$$k_n = \sum_{i=1}^{n} (-1)^{i-1}(i-1)! B_{n,i}(a_1, a_2, \ldots, a_{n-i+1}) \tag{24}$$

giving cumulants in terms of moments usually requires laborious computations (cf. for example [10]). The umbral definition of cumulants allows a simple proof of (24). Indeed, since $\chi \equiv u^{\langle -1\rangle}.\beta$, the cumulant umbra of $\alpha$ is the umbral composition of $u^{\langle -1\rangle}$ and $\alpha$:

$$\kappa_\alpha \equiv u^{\langle -1\rangle}.\beta.\alpha \tag{25}$$

and then its moments are given by (13). Note that Eq. (25) is the umbral version of the second equality in (22). Finally, equality (24) follows recalling that the moments of $u^{\langle -1\rangle}$ are the coefficient of the exponential expansion

$$1 + \log(1 + t) = 1 + \sum_{i=1}^{\infty} (-1)^{i-1}(i-1)! \frac{t^i}{i!}.$$

Similarly, the three main algebraic properties of cumulants can be easily recovered from next theorem.

**Theorem 4.9.** *It is*

(a) *(the additivity property)*

$$\chi.(\alpha+\gamma) \equiv \chi.\alpha \dot{+} \chi.\gamma, \tag{26}$$

*i.e. the cumulant umbra of a sum of two umbrae is equal to the disjoint sum of the two corresponding cumulant umbrae;*

(b) *(the semi-invariance under translation property) for any $c \in R$*

$$\chi.(\alpha + c.u) \equiv \chi.\alpha \dot{+} \chi.c;$$

(c) *(the homogeneity property) for any $c \in R$*

$$\chi.(c\alpha) \equiv c(\chi.\alpha).$$

**Proof.** Property (a) follows from (19). Property (b) follows from (26), setting $\gamma \equiv c.u$ for any $c \in R$. Finally, property (c) follows from (b) of Proposition 2.7. □

**Example 4.10** (*Cumulant of the Central Umbra*). The sequence of cumulants related to the central umbra $\alpha^{a_1}$ is the same as the sequence for $\alpha$, except that the first cumulant is equal to 0. Indeed, by the additivity property of the cumulant umbra it is

$$\chi.(\alpha - a_1.u) \equiv \chi.\alpha \dot{-} \chi.a_1.$$

The results follows from (7).

The umbral version of the first equality in (22) is given in the following theorem.

**Theorem 4.11** (*Inversion Theorem*). *Let $\kappa_\alpha$ be the cumulant umbra of $\alpha$, then*

$$\alpha \equiv \beta.\kappa_\alpha$$

*where $\beta$ is the Bell umbra.*

**Proof.** We have

$$\beta.\kappa_\alpha \equiv \beta.\chi.\alpha \equiv u.\alpha \equiv \alpha. \quad \square$$

The inversion theorem enables one to calculate the moments of the umbra $\alpha$ from its cumulants. Recalling (11) it is

$$a_n = Y_n[(k_a)_1, (k_a)_2, \ldots, (k_a)_n] \tag{27}$$

with $a_n$ the $n$-th moment of the umbra $\alpha$ and $(k_a)_n$ the $n$-th moment of the umbra $\kappa_\alpha$. Eq. (27) is the explicit version of the first equality in (22).

**Remark 2.** The complete Bell polynomials in (11) are a polynomial sequence of binomial type. Since, from the inversion theorem, any umbra $\alpha$ is the partition umbra of its cumulant $\kappa_\alpha$, it is possible to prove a more general result: every polynomial sequence of binomial type is completely determined by its sequence of formal cumulants. Indeed, in [5] it is proved that any polynomial sequence of binomial type represents the moments of a polynomial umbra $x.\alpha$ and vice versa. So from the inversion theorem any polynomial sequence of binomial type represents the moments of a polynomial umbra $x.\beta.\kappa_\alpha$.

The next corollary follows from (12) and from the inversion theorem.

**Corollary 4.12.** *If $\kappa_\alpha$ is the cumulant umbra of $\alpha$, then*

$$\alpha^n \simeq \kappa_\alpha (\kappa_\alpha + \alpha)^{n-1} \tag{28}$$

*for any nonnegative integer $n$.*

Equivalences (28) were assumed by Shen and Rota in [16] as definition of the cumulant umbra. In terms of moments, equivalences (28) give

$$a_n = \sum_{j=0}^{n-1} \binom{n-1}{j} a_j (k_a)_{n-j}$$

that is widely used in statistical settings [20].

**Example 4.13** (*Lévy Process*). Let $(X_t, t \geq 0)$ be a real-valued Lévy process, i.e. a process starting from 0 and with stationary and independent increments. According to the Lévy–Khintchine formula (cf. [6]), if we assume that $X_t$ has a convergent m.g.f. in some neighbourhood of 0, it is

$$E[\mathrm{e}^{\theta X_t}] = \mathrm{e}^{tk(\theta)} \tag{29}$$

where $k(\theta)$ is the cumulant g.f. of $X_1$. The inversion theorem gives the umbral version of Eq. (29):

$$t.\alpha \equiv t.\beta.\kappa_\alpha.$$

### 4.1. *Cumulants of a Poisson r.v.'s*

From Example 2.9, the umbra $\gamma.\beta.\alpha$ corresponds to a compound randomized Poisson r.v., i.e. a random sum $S_N = X_1 + \cdots + X_N$ with $N$ a randomized Poisson r.v. of parameter a r.v. $Y$. In particular $\alpha$ corresponds to $X$ and $\gamma$ corresponds to $Y$. Since $\chi.(\gamma.\beta.\alpha) \equiv \kappa_\gamma.\beta.\alpha$ the cumulant umbra of the composition of $\alpha$ and $\gamma$ is the composition of $\alpha$ and $\kappa_\gamma$. Then from (13), the cumulants of a compound randomized Poisson r.v. are given by

$$\sum_{i=1}^{n} k_i B_{n,i}(a_1, a_2, \ldots, a_{n-i+1}) \tag{30}$$

where $a_i$ are the moments of the r.v. $X$ and $k_i$ are the cumulants of the r.v. $Y$. Now set $\gamma \equiv x.u$ in $\gamma.\beta.\alpha$. This means to consider a r.v. $Y$ such that $P(Y = x) = 1$. Then, the random sum $S_N$ becomes a compound Poisson r.v. of parameter $x$ corresponding to the polynomial $\alpha$-partition umbra $x.\beta.\alpha$, with $\alpha$ the umbral counterpart of $X$ and cumulants

$$\sum_{i=1}^{n} k_i B_{n,i}(a_1, a_2, \ldots, a_{n-i+1}) \simeq x a_n. \tag{31}$$

Indeed (31) follows from (30) since the moments $k_i$ of $\chi.x$ are equal to 0, except for the first, which is equal to $x$. If $x = 1$ the cumulant of the $\alpha$-partition umbra is $\alpha$, so the moments of $X$ are the cumulants of the corresponding compound Poisson r.v. Now, in $x.\beta.\alpha$ take $\alpha \equiv u$. From (31), the cumulants of the Bell polynomial umbra $x.\beta$ are equal to $x$, as is a Poisson r.v. of parameter $x$.

Finally, in $\gamma.\beta.\alpha$ set $\alpha \equiv u$. The cumulant umbra of $\gamma.\beta$ is $\kappa_\gamma.\beta$ with $\kappa_\gamma$ the cumulant umbra of $\gamma$. Its probabilistic counterpart is a randomized Poisson r.v. of parameter a r.v. $Y$, corresponding to the umbra $\gamma$. From (8) the cumulants of a randomized Poisson r.v. of parameter a r.v. $Y$ are the moments of $\kappa_\gamma.\beta$, i.e.

$$\sum_{i=0}^{n} S(n,i) k_i$$

with $k_i$ the cumulants of the r.v. $Y$.

## 5. The factorial umbra

The factorial moments of a r.v. do not play a very prominent role in statistics, but they provide very concise formulae for the moments of some discrete distributions, such as the binomial distribution.

Let $\alpha$ be an umbra with g.f. $f(t)$.

**Definition 5.1.** An umbra $\varphi_\alpha$ is said to be an $\alpha$-factorial umbra if

$$\varphi_\alpha \equiv \alpha.\chi$$

where $\chi$ is the singleton umbra.

**Example 5.2** (*$\varepsilon$-Factorial Umbra*). Since $\varepsilon \equiv \varepsilon.\chi$, the $\varepsilon$-factorial umbra is similar to the umbra $\varepsilon$, i.e. $\varphi_\varepsilon \equiv \varepsilon$.

**Example 5.3** (*$u$-Factorial Umbra*). Since $\chi \equiv u.\chi$, the $u$-factorial umbra is similar to the umbra $\chi$, i.e. $\varphi_u \equiv \chi$.

**Example 5.4** (*$\beta$-Factorial Umbra*). Since $u \equiv \beta.\chi$ from (18), the $\beta$-factorial umbra is similar to the unity umbra $u$, i.e. $\varphi_\beta \equiv u$. From Example 2.1, a Poisson r.v. of parameter 1 has factorial moments equal to 1.

**Example 5.5** (*$\chi$-Factorial Umbra*). From Example 4.6, it is $\chi.\chi \equiv u^{\langle -1 \rangle}$. The $\chi$-factorial umbra turns out to be $u^{\langle -1 \rangle}$, i.e. $\varphi_\chi \equiv u^{\langle -1 \rangle}$.

**Proposition 5.6.** *The $\alpha$-factorial umbra has g.f.*

$$g(t) = f[\log(1+t)]. \tag{32}$$

**Proof.** See Table 1. □

The $\alpha$-factorial umbra has moments equal to the factorial moments of the umbra $\alpha$, as the following proposition shows.

**Proposition 5.7.** *Let $\varphi_\alpha$ be an $\alpha$-factorial umbra. Then*

$$\varphi_\alpha^n \simeq (\alpha)_n, \qquad n = 0, 1, 2, \ldots.$$

**Proof.** By Eq. (6) and Definition 5.1 it is

$$E[(\varphi_\alpha)^n] = E[(\alpha.\chi)^n] = \sum_{k=0}^{n} (a)_k B_{n,k}(\delta_{1,1}, \delta_{1,2}, \ldots, \delta_{1,n-k+1}) \tag{33}$$

where $(a)_k$ are the factorial moments of the umbra $\alpha$ and $\delta_{1,i}$ are the moments of the umbra $\chi$. By (15) we have

$$B_{n,k}(\delta_{1,1}, \delta_{1,2}, \ldots, \delta_{1,n-k+1}) \simeq \binom{n}{k} \chi^{.k}(k.\overline{\chi})^{n-k}.$$

Since the umbra $\overline{\chi}$ has moments equal to 0 and $\chi^{.k} \simeq 1$, then

$$B_{n,k}(\delta_{1,1}, \delta_{1,2}, \ldots, \delta_{1,n-k+1}) = \begin{cases} 0, & \text{if } n > k \\ 1, & \text{if } n = k. \end{cases} \tag{34}$$

Hence the Eq. (33) becomes $E[(\varphi_\alpha)^n] = (a)_n$. □

**Example 5.8** (*Factorial Umbra of the Central Umbra*). From property (c) of Proposition 2.7,

$$\alpha^{a_1}.\chi \equiv (\alpha - a_1.u).\chi \equiv \alpha.\chi - a_1.\chi' \equiv \varphi_\alpha - \varphi_{a_1.u},$$

with $\chi' \equiv \chi$. Then the factorial umbra of the central umbra $\alpha^{a_1}$ is the difference between the factorial umbra of $\alpha$ and the factorial umbra of the umbra having moments equal to $a_1$. By (32) its g.f. becomes $f[\log(1+t)](1-t)^{a_1}$.

**Example 5.9** (*Factorial Moments of the Binomial r.v.*). Since the factorial moments characterize a binomial r.v., we show how to evaluate them by umbral methods. As showed in Example 3.6, the umbral counterpart of a binomial r.v. is $n.\chi.p.\beta$. Due to (18) and (7) the corresponding factorial umbra is $n.(\chi.p.\beta).\chi \equiv n.\chi.p \equiv p(n.\chi)$. Its g.f. is

$$g(t) = (1+tp)^n = \sum_{j=0}^{n} (n)_j \, p^j \frac{t^j}{j!}$$

so the factorial moments are $(n)_j \, p^j$. If $n = 1$, the factorial umbra is $p\chi$ and, from Example 3.5, the first factorial moment of a Bernoulli r.v. is equal to $p$ while the others are equal to 0.

**Example 5.10** (*Factorial Umbra of the Cumulant Umbra*). If $\kappa_\alpha$ is the cumulant umbra of $\alpha$, then $\kappa_\alpha.\chi$ is the factorial cumulant umbra of $\alpha$, with g.f. $1 + \log[f(\log(1+t))]$ by (32).

The following theorem produces the umbra $\alpha$ from its factorial umbra $\varphi_\alpha$.

**Theorem 5.11** (*Inversion Theorem*). *Let $\varphi_\alpha$ be the factorial umbra of $\alpha$. It is*

$$\alpha \equiv \varphi_\alpha.\beta$$

*with $\beta$ the Bell umbra.*

**Proof.** By Proposition 3.3,

$$\varphi_\alpha.\chi \equiv \alpha.\chi.\beta \equiv \alpha. \quad \square$$

**Corollary 5.12.** *Let $\varphi_\alpha$ be the factorial umbra of $\alpha$ and $\kappa_\alpha$ its cumulant umbra. It is*

$$\varphi_\alpha.\beta \equiv \beta.\kappa_\alpha.$$

### 5.1. Factorial moments of a Poisson r.v.'s

Since $(\gamma.\beta.\alpha).\chi \equiv \gamma.\beta.(\varphi_\alpha)$ the factorial umbra of the umbral composition $\gamma.\beta.\alpha$ is the umbral composition of $\gamma$ and the factorial umbra of $\alpha$. From (13) a compound randomized Poisson r.v. $S_N = X_1 + \cdots + X_N$ with $N$ a Poisson r.v. with parameter a r.v. $Y$ has factorial moments

$$\sum_{k=1}^{n} g_k B_{n,k}[(\mu)_1, (\mu)_2, \ldots, (\mu)_{n-k+1}] \tag{35}$$

where $(\mu)_i$ are the factorial moments of the r.v. $X$ and $g_k$ are the moments of the r.v. $Y$. Now setting $\gamma \equiv x.u$ in $\gamma.\beta.\alpha$, we have $g_k = x^k$. Then from (35)

$$\sum_{k=1}^{n} x^k B_{n,k}[(\mu)_1, (\mu)_2, \ldots, (\mu)_{n-k+1}] \tag{36}$$

are the factorial moments of a compound Poisson r.v. with parameter $x$. Set $\alpha \equiv u$ in $x.\beta.\alpha$. We have $(x.\beta.u).\chi \equiv x.\beta.\chi \equiv x.u$ so that the factorial moments of $x.\beta.\alpha$ are equals to $x^n$ as well as for its probabilistic counterpart, a Poisson r.v. with parameter $x$.

Finally, set $\alpha \equiv u$ in $\gamma.\beta.\alpha$. We have $(\gamma.\beta.u).\chi \equiv \gamma.\beta.\chi \equiv \gamma$ so that the factorial moments of $\gamma.\beta$ are equal to the moments of $\gamma$. Then a randomized Poisson r.v. with parameter a r.v. $Y$ has factorial moments equal to the moments of the r.v. $Y$.

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