A PRIMER FOR THE FIBONACCI NUMBERS XVII: GENERALIZED FIBONACCI NUMBERS SATISFYING $u_{n+1}u_{n-1} - u_n^2 = \pm 1$

V. E. HOGGATT, JR., and MARJORIE BICKNELL-JOHNSON
San Jose State University, San Jose, California 95192

There are many ways to generalize the Fibonacci sequence. Here, we examine some properties of integral sequences $\{u_n\}$ satisfying

(1) $$u_{n+1}u_{n-1} - u_n^2 = (-1)^n,$$

where necessarily $u_0 = 0$ and $u_1 = \pm 1$. The Fibonacci polynomials $f_n(x)$ given by

(2) $$f_{n+1}(x) = xf_n(x) + f_{n-1}(x), \quad f_0(x) = 0, \quad f_1(x) = 1,$$

evaluated at $x = b$ provide special sequences $\{u_n\}$. Of course, $f_n(1) = F_n$, the Fibonacci numbers 0, 1, 1, 2, 3, 5, ⋯, and $f_n(2) = P_n$, the Pell numbers 0, 1, 2, 5, 12, 29, ⋯. Divisibility properties of the Fibonacci polynomials [1] and properties of the Pell numbers and the general sequences $\{f_n(b)\}$ [2] have been examined in earlier Primer articles.

In the course of events, we will completely solve the Diophantine equations $y^2 - (a^2 \pm 4)x^2 = \pm 4$ and show that all of our generalized Fibonacci polynomials are special cases of Chebyshev polynomials of the first and second kinds.

1. SOLUTIONS TO $y^2 - (a^2 + 4)x^2 = \pm 4$

Theorem 1. Let $\{u_n\}$ be a sequence of integers such that $u_{n+1}u_{n-1} - u_n^2 = (-1)^n$ for all integers n. Then there exists an integer a such that

(3) $$u_{n+2} = au_{n+1} + u_n.$$

Proof. Set

$$u_2 = au_1 + bu_0, \qquad u_3 = au_2 + bu_1$$

for some real numbers a and b. By Cramer's rule,

$$b = \begin{vmatrix} u_1 & u_2 \\ u_2 & u_3 \end{vmatrix} \div \begin{vmatrix} u_1 & u_0 \\ u_2 & u_1 \end{vmatrix} = \frac{u_1u_3 - u_2^2}{u_1^2 - u_0u_2} = 1$$

since $u_1u_3 - u_2^2 = (-1)^2$ and $u_0u_2 - u_1^2 = (-1)^1$ by definition of $\{u_n\}$. Thus, a is an integer. In fact, $u_2 = au_1 + u_0$ and $u_3 = au_2 + u_1$ yield

$$a = \frac{u_3 - u_1}{u_2} = \frac{u_2 - u_0}{u_1}.$$

Assume that $u_{n+1} = au_n + u_{n-1}$. Then

$$a = \frac{u_{n+1} - u_{n-1}}{u_n}$$

and

$$au_{n+1} + u_n = \frac{u_{n+1} - u_{n-1}}{u_n} \cdot u_{n+1} + u_n = \frac{u_{n+1}^2 - u_{n-1}u_{n+1} + u_n^2}{u_n} = \frac{u_{n+1}^2 + (-1)^{n+1}}{u_n}.$$

But, $u_{n+2}u_n - u_{n+1}^2 = (-1)^{n+1}$ by definition of the sequence, so that

$$u_{n+2} = [u_{n+1}^2 + (-1)^{n+1}]/u_n, \quad \text{and} \quad u_{n+2} = au_{n+1} + u_n$$

for an integer a by the Axiom of Mathematical Induction.

Corollary 1.1. The sequence $\{u_n\}$ has starting values $u_0 = 0$, $u_1 = \pm 1$.

Proof. By Theorem 1, $u_2 = au_1 + u_0$. Thus,

$$u_2^2 = a^2u_1^2 + 2au_1u_0 + u_1^2 = au_1(au_1 + u_0) + u_0^2 = au_1u_2 + u_0^2.$$

Since also $u_0 = u_2 - au_1$, substituting above for u_0^2, we have

$$u_2^2 = au_1u_2 + (u_2^2 - 2au_1u_2 + a^2u_1^2), \qquad 0 = au_1(au_1 - u_2)$$

Now, either $a = 0$, or $u_1 = 0$, or $u_2 = au_1$. If $a = 0$, $u_2 = u_0$, and from $u_2u_0 - u_1^2 = -1$, $u_0 = 0$ and $u_1 = \pm 1$ give the only possible solutions. If $u_1 = 0$, then $u_2 = u_0$ leads to $u_2^2 = -1$, clearly impossible for integers. If $u_2 = au_1$, then $u_2 = au_1 = au_1 + u_0$ forces $u_0 = 0$, and again $u_1 = \pm 1$.

Theorem 2. Let $\{u_n\}$ be a sequence of integers such that $u_{n+1}u_{n+1} - u_n^2 = (-1)^n$ for all n. Then $x = u_n$ and $y = u_{n+1} + u_{n-1}$ are solutions for the Diophantine equation

$$y^2 - (a^2 + 4)x^2 = \pm 4, \tag{4}$$

where also $u_{n+1} = au_n + u_{n-1}$.

Proof. From Theorem 1, $u_{n+1} = au_n + u_{n-1}$. If $y = u_{n+1} + u_{n-1}$ and $x = u_n$, then

$$u_{n+1} = y - u_{n-1} = y - (u_{n+1} - au_n) = y - u_{n+1} - ax$$

yielding

$$u_{n+1} = (y - ax)/2.$$

Then

$$u_{n-1} = y - u_{n+1} = y - (y - ax)/2 = (y + ax)/2.$$

By definition of the sequence $\{u_n\}$,

$$u_{n+1}u_{n-1} - u_n^2 = (-1)^n,$$

$$\frac{y + ax}{2} \cdot \frac{y - ax}{2} - x^2 = \pm 1,$$

$$(y^2 - a^2x^2) - 4x^2 = \pm 4,$$

$$y^2 - (a^2 + 4)x^2 = \pm 4.$$

Now, let the generalized Lucas and Fibonacci numbers $\mathcal{L}_n$ and $\mathcal{F}_n$ be defined in terms of Fibonacci polynomials as in Eq. (2):

$$\begin{aligned} \mathcal{L}_n &= f_{n+1}(a) + f_{n-1}(a) \\ \mathcal{F}_n &= f_n(a). \end{aligned} \tag{5}$$

Since [2]

$$f_{n+1}(x)f_{n-1}(x) - f_n^2(x) = (-1)^n, \tag{6}$$

$$\mathcal{L}_n^2 - (a^2 + 4)\mathcal{F}_n^2 = \pm 4 \tag{7}$$

by Theorem 2. Thus, the generalized Lucas and Fibonacci numbers give solutions to the Diophantine equation (4).

Theorem 3. The generalized Lucas and Fibonacci numbers $\mathcal{L}_n$ and $\mathcal{F}_n$ are the only solutions to the Diophantine equation

$$y^2 - (a^2 + 4)x^2 = \pm 4. \tag{4}$$

Proof. Now, $y^2 - (a^2 + 4)x^2 = +4$ has solution $x = 0$, $y = 2$, as well as a solution $x = 1$, $y = 3$ if $a = 1$, but no solution for $x = 1$ when $a > 1$. The other equation $y^2 - (a^2 + 4)x^2 = -4$ has solution $x = 1$, $y = a$. The case $a = 1$ was solved by Ferguson [3]. We use a method of infinite descent which is an extension of the method of Ferguson [3], and take $a > 1$, $x > 1$. Thus, $y^2 - (a^2 + 4)x^2 = \pm 4$ implies that

$$ax < y < (a + 2)x$$

since

$$y^2 = (a^2+4)x^2 \pm 4 = a^2x^2 + 4x^2 \pm 4 < a^2x^2 + 4ax^2 + 4x^2$$

forces

$$(ax)^2 < y^2 < (a+2)^2x^2 .$$

Since y and ax must have the same parity, let

$$y = ax + 2t, \qquad 1 \leqslant t < x .$$

Assume that x is the smallest non-Fibonacci solution. Replace y with $ax + 2t$ in (4), yielding

$$(ax+2t)^2 - (a^2+4)x^2 \pm 4 = 0$$

$$4x^2 - 4axt - 4t^2 \pm 4 = 0 .$$

Solve the quadratic for $2x$, yielding

$$2x = at \pm \sqrt{(a^2+4)t^2 \pm 4}$$

But, $2x$ is an integer, and therefore

$$(a^2+4)t^2 \pm 4 = s^2$$

for an integer s so that $t = u_n$ and $s = u_{n+1} + u_{n-1}$ are solutions by Theorem 2. Since $x > 0$,

$$\begin{aligned} 2x &= at + \sqrt{(a^2+4)t^2 \pm 4} \\ &= at + s \\ &= au_n + (u_{n+1} + u_{n-1}) \\ &= (au_n + u_{n-1}) + u_{n-1} \\ &= 2u_{n+1} \end{aligned}$$

so that $x = u_{n+1}$. But, if x is the smallest non-Fibonacci solution, then x cannot be the next larger Fibonacci solution after t. This is a contradiction, and there is no first non-Fibonacci solution. Thus, the Diophantine equation

$$y^2 - (a^2+4)x^2 = \pm 4$$

has solutions in integers if and only if

$$y = \pm\mathcal{L}_n = f_{n+1}(a) + f_{n-1}(a) \quad \text{and} \quad x = \pm\mathcal{F}_n = f_n(a) .$$

2. SPECIAL SEQUENCES $\{u_n\}$ AND THE EQUATION $y^2 - (a^2-4)x^2 = \pm 4$

Now, all of these sequences $\{u_n\}$ have starting values $u_0 = 0$ and $u_1 = \pm 1$. It is interesting to note some special cases. Notice that the sequence

$$\cdots, 1, 0, 1, 0, 1, 0, 1, 0, 1, 1, 2, 3, 5, \cdots$$

due to Bergum [4] satisfies $u_0 = 0$, $u_1 = 1$, and

$$u_{n+1}u_{n-1} - u_n^2 = (-1)^n ,$$

where the left-hand part of the sequence has

$$u_{n+2} = u_n = 0 \cdot u_{n+1} + u_n$$

while the right-hand part has

$$u_{n+2} = 1 \cdot u_{n+1} + u_n .$$

It is interesting to note that special cases of the sequences $\{u_n\}$ satisfying $u_{n+1}u_{n-1} - u_n^2 = (-1)^n$ occur from [2]

(8) $$\mathcal{F}_{n-k}\mathcal{F}_{n+k} - \mathcal{F}_n^2 = (-1)^{n+k+1}\mathcal{F}_k^2$$

for the generalized Fibonacci numbers given in Eq. (5). Let

$$\mathcal{F}_{nk-k}\mathcal{F}_{nk+k} - \mathcal{F}_{nk}^2 = (-1)^{nk+k+1}\mathcal{F}_k^2$$

be rewritten

$$\frac{\tau_{(n-1)k}}{\tau_k}\,\frac{\tau_{(n+1)k}}{\tau_k} - \frac{\tau_{nk}^2}{\tau_k^2} = (-1)^{(n+1)k+1}$$

Now, since τ_{nk}/τ_k is known to be an integer [1], let $u_n = \tau_{nk}/\tau_k$, and the equation above becomes

$$u_{n+1}u_{n-1} - u_n^2 = (-1)^{(n+1)k+1},$$

where $(-1)^{(n+1)k+1}$ is $(-1)^n$ if k is odd but (-1) if k is even. In particular, if $k = 2$, the sequence of Fibonacci numbers with even subscripts, $\{0, 1, 3, 8, 21, \cdots\}$, gives a solution to $u_{n+1}u_{n-1} - u_n^2 = -1$. Another solution is $u_n = n$, since $(n+1)(n-1) - n^2 = -1$ for all n.

Is there a sequence $\{u_n\}$ of positive terms for which $u_{n+1}u_{n-1} - u_n^2 = +1$? Considering Fibonacci numbers with odd subscripts, $\{1, 2, 5, 13, 34, \cdots\}$, we observe that $u_n = F_{2n+1}$ is a solution, and that $u_{n+1} = 3u_n - u_{n-1}$. Using $u_{n+1}u_{n-1} - u_n^2 = 1$ and solving $u_{n+1} = au_n + bu_{n-1}$ as in Theorem 1 yields $u_{n+1} = au_n - u_{n-1}$. If we let $y = u_{n+1} - u_{n-1}$ and $x = u_n$, proceeding as in Theorem 2, we are led to the Diophantine equation $y^2 - (a^2-4)x^2 = -4$. We summarize as

Theorem 4. If $\{u_n\}$ is a sequence of integers such that

$$u_{n+1}u_{n-1} - u_n^2 = +1$$

for all n, then there exists an integer a such that

$$u_{n+2} = au_{n+1} - u_n$$

and $y = u_{n+1} - u_{n-1}$ and $x = u_n$ are solutions of the Diophantine equation

(9) $$y^2 - (a^2-4)x^2 = -4.$$

Theorem 5. The odd-subscripted Fibonacci and Lucas numbers give the only solutions to the Diophantine equation

(9) $$y^2 - (a^2-4)x^2 = -4.$$

Proof. We show that (9) has no integral solutions if $|a| \neq 3$, proceeding in the manner of the proof of Theorem 3. Here,

$$(a-2)x < y < ax.$$

Since y and ax must have the same parity, let

$$y = ax - 2t, \qquad 1 \leq t < x.$$

Notice that, if $x = 1$, $y^2 - (a^2-4) = -4$ becomes $a^2 - y^2 = 8$, which is solved only by $a = 3$, $y = 1$.

Let x be the first solution greater than one. Replace y with $ax - 2t$ in (9), yielding

$$(ax-2t)^2 - (a^2-4)x^2 + 4 = 0$$
$$4x^2 - 4axt + 4t^2 + 4 = 0.$$

Solving the quadratic for $2x$ gives

$$2x = at \pm \sqrt{(a^2-4)t^2 - 4}.$$

Since $2x$ is integral, we must have $(a^2-4)t^2 - 4 = s^2$ for some integer s. By Theorem 4, $t = u_n$ is a solution where $t > 1$. But, since x is the first solution greater than 1, and $x > t$, we have a contradiction, and

$$y^2 - (a^2-4)x^2 = -4$$

is not solvable in positive integers unless $a = 3$. When $a = 3$, the equation becomes $y^2 - 5x^2 = -4$, which is solved only by

$$y = L_{2n+1}, \qquad x = F_{2n+1},$$

odd-subscripted Lucas and Fibonacci numbers [5].

Theorem 6. If $\{u_n\}$ is a sequence of integers such that

$$u_{n+1}u_{n-1} - u_n^2 = -1$$

for all n, then there exists an integer a such that

$$u_{n+2} = au_{n+1} - u_n \quad \text{and} \quad y = u_{n+1} - u_{n-1} \quad \text{and} \quad x = u_n$$

are solutions of the Diophantine equation

$$y^2 - (a^2-4)x^2 = +4. \tag{10}$$

Proof. Proceed as in Theorem 4.

Theorem 7. The Fibonacci and Lucas numbers with even subscripts give solutions to the Diophantine equation

$$y^2 - (a^2-4)x^2 = +4.$$

Proof. Set $a = 3$ and refer to Lind [5].

3. GENERALIZED FIBONACCI POLYNOMIALS

Next, in order to write solutions for the Diophantine equation (10), we consider a type of generalized Fibonacci polynomial. Let

$$h_0(x) = 0, \quad h_1(x) = 1, \quad \text{and} \quad h_{n+2}(x) = xh_{n+1}(x) - h_n(x) \tag{11}$$

and

$$g_0(x) = 2, \qquad g_1(x) = x,$$

where

$$g_{n+2}(x) = xg_{n+1}(x) + g_{n-1}(x).$$

We note that $\{h_n(a)\}$ is a special sequence $\{u_n\}$ since

$$h_{n+1}(a)h_{n-1}(a) - h_n^2(a) = -1.$$

Then

$$h_n(x) = \frac{a_1^n(x) - a_2^n(x)}{a_1(x) - a_2(x)}, \quad x \neq 2; \qquad h_n(2) = n,$$

$$g_n(x) = a_1^n(x) + a_2^n(x) = h_{n+1}(x) - h_{n-1}(x),$$

where $a_1(x)$ and $a_2(x)$ are roots of

$$\lambda^2 - \lambda x + 1 = 0.$$

(By way of comparison, the Fibonacci polynomials $f_n(x)$ have the analogous relationship to the roots of

$$\lambda^2 - \lambda x - 1 = 0.$$

Also note that $h_n(3) = F_{2n}$.)

It is easy to establish from $a_1(x)a_2(x) = 1$ that

$$2a_1^n = g_n(x) + [a_1(x) - a_2(x)]h_n(x)$$

$$2a_2^n = g_n(x) - [a_1(x) - a_2(x)]h_n(x)$$

with $a_1(x) - a_2(x) = \sqrt{x^2-4}$. From this it readily follows that

$$1 = a_1^n(x)a_2^n(x) = [g_n^2(x) - (x^2-4)h_n^2(x)]/4$$

or

$$g_n^2(x) - (x^2-4)h_n^2(x) = +4.$$

Now, we are interested in the sequences of integers formed by evaluating $h_n(x)$ and $g_n(x)$ at $x = a$. Thus

$$g_n^2(a) - (a^2-4)h_n^2(a) = +4. \tag{12}$$

and we do have solutions to

$$y^2 - (a^2-4)x^2 = +4.$$

Theorem 8. The generalized Fibonacci numbers $\{h_n(a)\}$ and generalized Lucas numbers $\{g_n(a)\}$ provide the only solutions to the Diophantine equation

$$y^2 - (a^2-4)x^2 = +4. \tag{10}$$

Proof. Note that if $x = 1$, then $y = a$, and if $x = 0$, then $y = 2$. Now one can proceed as follows. We can write, as before,

$$(a-2)x < y \leqslant ax.$$

Clearly, y and ax must have the same parity, so that we can let

$$y = ax - 2t, \qquad 1 \leqslant t < x,$$

where x is the first positive integer which is greater than 1, not equal to $h_m(a)$, and a solution. Then, as before, replace y with $ax - 2t$ in (10), yielding

$$(ax-2t)^2 - (a^2-4)x^2 - 4 = 0$$

$$4x^2 - 4axt + 4t^2 - 4 = 0.$$

Solving the quadratic for $2x$,

$$2x = at \pm \sqrt{(a^2-4)t^2+4}\,. \tag{13}$$

Since $2x$ is an integer, there exists an integer s such that

$$(a^2-4)t^2 + 4 = s^2,$$

with a solution given by

$$t = h_n(a) \quad \text{and} \quad s = g_n(a) = h_{n+1}(a) - h_{n-1}(a)$$

by Eq. (12). Then, (13) taken with the plus sign gives

$$2x = ah_n(a) + h_{n+1}(a) - h_{n-1}(a) = 2h_{n+1}(a)$$

and $x = h_{n+1}(a)$, a contradiction, since x was defined as not having the form $h_m(a)$.

Next, we consider the case of Eq. (13) taken with the minus sign. The cases $a = 1$ or $a = 0$ are not very interesting. We need a lemma:

Lemma. For $a > 1$, the sequence $\{h_n(a)\}$ is a strictly increasing sequence.

Proof of the Lemma.

$$h_0(a) = 0, \quad h_1(a) = 1, \quad h_2(a) = a, \quad h_{n+2}(a) = ah_{n+1}(a) - h_n(a).$$

Since

$$h_{n+1}(a) = ah_n(a) - h_{n-1}(a) > (a-1)h_n(a)$$

if

$$h_{n-1}(a) < h_n(a),$$

then

$$h_{n+1}(a) > h_n(a).$$

Thus, if we choose the minus sign in Eq. (13), then we have

$$2x = ah_n(a) - (h_{n+1}(a) - h_{n-1}(a))$$

$$= ah_n(a) - h_{n+1}(a) + h_{n-1}(a) = 2h_{n-1}(a)$$

or $x = h_{n-1}(a)$ which contradicts the restriction that $t < x$. Thus, we must choose the plus sign in (13), which yielded $x = h_{n+1}(a)$. So, even if x is the first integer greater than one for which we have a solution for

$$y^2 - (a^2-4)x^2 = +4$$

and where $x \neq h_m(a)$, we find $x = h_{n+1}(a)$. This shows that there is no first positive integer which solves Eq. (10) which is not of the form $x = h_m(a)$. This concludes the proof of Theroem 8.

We note that the case $a = 2$ yields $y = \pm 2$ and x any integer. The recurrence

$$u_{n+2} = 2u_{n+1} - u_n$$

is satisfied by any arithmetic progression $b, b+d, b+2d, \cdots, B+nd, \cdots$. However, the restriction

$$u_{n+1}u_{n-1} - u_n^2 = -1$$

limits these to the integers $n = u_n$.

In summary, we have set down the complete solutions to the Diophantine equations

$$y^2 - (a^2 \pm 4)x^2 = \pm 4.$$

$y^2 - (a^2+4)x^2$ has solution $x = 0$, $y = 2$, for all a. For

$$y^2 - (a^2+4)x^2 = -4,$$

we get $x = 1$, $y = a$. Both solutions are starting pairs for the recurrence

$$u_{n+2} = au_{n+1} + u_n,$$

and $y = 2, a, \cdots$ leads to $f_{n+1}(a) + f_{n-1}(a)$, and $x = 0, 1, \cdots$, leads to $f_n(a)$, where $f_n(x)$ are the Fibonacci polynomials. Here, $u_{n+1}u_{n-1} - u_n^2 = (-1)^n$ lead to $y^2 - (a^2+4)x^2 = \pm 4$ via $u_{n+2} = au_{n+1} + u_n$. But either

$$u_{n+1}u_{n-1} - u_n^2 = -1 \quad \text{or} \quad u_{n+1}u_{n-1} - u_n^2 = +1$$

lead to the recurrence $u_{n+2} = au_{n+1} - u_n$, and lead to $y^2 - (a^2-4)x^2 = \pm 4$. Now $y^2 - (a^2-4)x^2 = +4$ allows $x = 0$, $y = 2$ and $x = 1$, $y = a$ as starting solutions, where $x = 0, 1, \cdots$, leads to $h_n(a)$, and $y = 2, a, \cdots$, leads to $h_{n+1}(a) - h_{n-1}(a)$ for the generalized Fibonacci polynomials $h_n(x)$. Finally, $y^2 - (a^2-4)x^2 = -4$ has solution $x = 1$, $y = 1$ when $|a| = 3$, but no solution if $|a| \neq 3$. This then becomes $y^2 - 5x^2 = -4$ which is satisfied only by the oddly subscripted Fibonacci and Lucas numbers, which satisfy the recurrence $u_{n+1} = 3u_n - u_{n-1}$, so that

$$F_{2n+1} = h_{n+1}(3) - h_n(3),$$

and, of course, $F_{2n+1} = f_{2n+1}(1)$. In all cases, the only solutions arise from sequences of Fibonacci polynomials $f_n(x)$ evaluated at $x = a$, or generalized Fibonacci polynomials $h_n(x)$ evaluated at $x = a$. We can then state

Theorem 9. The Diophantine equations

$$y^2 - (a^2-4)x^2 = \pm 4$$
$$y^2 - (a^2+4)x^2 = \pm 4$$

have solutions in positive integers if and only if

$$y^2 - (a^2-4)x^2 = -4$$

has a solution $x = 1$ or

$$y^2 - (a^2+4)x^2 = -4$$

has a solution $x = 1$. Every solution is given by terms of a sequence of Fibonacci polynomials evaluated at a, $\{f_n(a)\}$, or generalized Fibonacci polynomials evaluated at $x = a$, $\{h_n(a)\}$.

4. CHEBYSHEV POLYNOMIALS

There are Chebyshev polynomials of two kinds:

$$U_{n+2}(x) = 2xU_{n+1}(x) - U_n(x)$$
$$T_{n+2}(x) = 2xT_{n+1}(x) - T_n(x)$$

with $T_0(x) = 1$ and $T_1(x) = x$, and $U_0(x) = 1$ and $U_1(x) = 2x$. The $T_n(x)$ are the Chebyshev polynomials of the first kind, and the $U_n(x)$ are the Chebyshev polynomials of the second kind [8]. There are also related polynomials

$$S_n(x) = U_n(x/2) \quad \text{and} \quad C_n(x) = 2T_n(x/2)$$

which are tabulated in [8]. Our $h_n(x)$ and $g_n(x)$ are related to $S_n(x)$ and $C_n(x)$ as follows:

$$h_n(x) = S_{n+1}(x) \qquad \text{and} \qquad g_n(x) = C_n(x).$$

An early article by Paul F. Byrd [10] explains the close connection between Fibonacci and Lucas polynomials and the $U_n(x)$ and $T_n(x)$. See also Hoggatt [9], and Buschman [11].

5. ANOTHER CONSEQUENCE OF $u_{n+1}u_{n-1} - u_n^2 = (-1)^n$

Finally, we examine another consequence of

$$u_{n+1}u_{n-1} - u_n^2 = (-1)^n.$$

We note that

$$(u_n, u_{n+1}) = 1, \qquad (u_n, u_{n-1}) = 1.$$

Note that $1, -1, -u_{n-1}, u_{n-1}$ are incongruent modulo u_n, $u \geq 5$, and form a multiplicative subgroup of the multiplicative group of integers modulo u_n. Since the order of the multiplicative group of integers mod u_n is $\varphi(u_n)$, where $\varphi(n)$ denotes the number of integers less than n and prime to n, and since the order of subgroup divides the order of a group, $4|\varphi(u_n)$. This method of proof was given by Montgomery [6] as solution to the problem of showing that $\varphi(F_n)$ is divisible by 4 if $n \geq 5$. The same problem also appeared in a slightly different form in the *Fibonacci Quarterly* [7]. We can generalize to

$$2^{m+2}|\varphi(\tau_{2^m n}), \qquad n \geq 5,$$

for the generalized Fibonacci numbers $\tau_n = f_n(a)$ by virtue of $\varphi(s) = 2k \geq 2$ for positive integers $s > 2$, and $\tau_{2t} = \tau_t \mathcal{L}_t$. Since $(\tau_t, \mathcal{L}_t) = 1$ or 2, then

$$\varphi(\tau_{2t}) = \varphi(\tau_t)\varphi(a),$$

where $a = \mathcal{L}_t$ or $\mathcal{L}_t/2$ so that $\varphi(a) = 2k \geq 2$. Thus,

$$\tau_{2^m n} = \tau_n \mathcal{L}_n \mathcal{L}_{2n} \mathcal{L}_{4n}, \cdots,$$

where

$$\varphi(\tau_n)\varphi(\mathcal{L}_n \mathcal{L}_{2n} \mathcal{L}_{4n} \cdots) = 4 \cdot 2^m r$$

for some integer $r \geq 1$.

REFERENCES

1. Marjorie Bicknell, "A Primer for the Fibonacci Numbers: Part VII, An Introduction to Fibonacci Polynomials and Their Divisibility Properties," *The Fibonacci Quarterly,* Vol. 8, No. 4 (Oct. 1970), pp. 407–420.
2. Marjorie Bicknell, "A Primer on the Pell Sequence and Related Sequences," *The Fibonacci Quarterly,* Vol. 13, No. 4 (Dec. 1975), pp. 345–349.
3. David E. Ferguson, "Letter to the Editor," *The Fibonacci Quarterly,* Vol. 8, No. 1 (Feb. 1970), p. 88.
4. Gerald E. Bergum, private communication.
5. D. A. Lind, "The Quadratic Field $Q(\sqrt{5})$ and a Certain Diophantine Equation," *The Fibonacci Quarterly,* Vol. 6, No. 3 (June, 1968), p. 91.
6. Peter L. Montgomery, solution to E 2581, *Amer. Math. Monthly,* Vol. 84, No. 6 (June-July, 1977), p. 488.
7. Problem H-54, proposed by Douglas A. Lind, Solution by John L. Brown, Jr., *The Fibonacci Quarterly,* Vol. 4, No. 4 (Dec. 1966), pp. 334–335.
8. *Tables of Chebyshev Polynomials $S_n(x)$ and $C_n(x)$,* U. S. Dept. of Commerce, Nat. Bureau of Standards, Applied Math. Series 9. Introduction by Cornellius Lanczos, p. v.
9. V. E. Hoggatt, Jr., "Fibonacci Numbers and Generalized Binomial Coefficients," *The Fibonacci Quarterly,* Vol. 5, No. 4 (Nov. 1967), pp. 383–400.
10. Paul F. Byrd, "Expansion of Analytic Functions in Terms Involving Lucas Numbers of Similar Number Sequences," *The Fibonacci Quarterly,* Vol. 3, No. 2 (April, 1965), pp. 101–114.
11. R. G. Buschman, "Fibonacci Numbers, Chebyshev Polynomials, Generalizations, and Difference Equations," *The Fibonacci Quarterly,* Vol. 1, No. 4 (Dec. 1963), pp. 1–7.

****