

COMPUTATIONAL FORMULAS FOR CONVOLUTED
GENERALIZED FIBONACCI AND LUCAS NUMBERS

Hong Feng

Department of Applied Mathematics, Dalian University of Technology
Dalian 116024, P.R. China
E-mail: fenghongbi@263.net

Zhizheng Zhang

Department of Mathematics, Nanjing University
Nanjing, 210093, P.R. China

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1. INTRODUCTION

In the notation of Horadam [2], let $W_n = W_n(a, b; p, q)$, where

$$W_n = pW_{n-1} - qW_{n-2} \quad (n \geq 2) \tag{1}$$

$$W_0 = a, \quad W_1 = b.$$

If α and β , assumed distinct, are the roots of

$$\lambda^2 - p\lambda + q = 0,$$

we have the Binet form

$$W_n = \frac{A\alpha^n - B\beta^n}{\alpha - \beta}, \tag{2}$$

in which $A = b - a\beta$ and $B = b - a\alpha$.

The n^{th} terms of the well-known Fibonacci and Lucas numbers can be denoted by $F_n = W_n(0, 1; 1, -1)$ and $L_n = W_n(2, 1; 1, -1)$, respectively.

We also denote

$$U_n = W_n(0, 1; p, q) = \frac{\alpha^n - \beta^n}{\alpha - \beta}, \quad V_n = W_n(2, p; p, q) = \alpha^n + \beta^n. \tag{3}$$

By simple computing, we have

$$\sum_{n=0}^{\infty} W_{mn} x^n = \frac{a + (bU_m - aU_{m+1})x}{1 - V_m x + q^m x^2}. \tag{4}$$

Let $W = \{W_n\}$ be defined as above, with $W_0 = 0$. For any positive integer $k \geq 2$, W. Zhang [3] obtained the following summation:

$$\sum_{a_1 + a_2 + \dots + a_k = n} W_{a_1} W_{a_2} \dots W_{a_k} = \frac{b^{k-1}}{(p^2 - 4q)^{k-1} (k-1)!} [g_{k-1}(n)W_{n-k+1} + h_{k-1}(n)W_{n-k}],$$

where the summation is taken over all n -tuples with positive integer coordinates (a_1, a_2, \dots, a_k) such that $a_1 + a_2 + \dots + a_k = n$. Moreover, $g_{k-1}(x)$ and $h_{k-1}(x)$ are two effectively computable polynomials of degree $k - 1$ with coefficients only depending on p, q and k .

Recently, Z. Zhang and X. Wang [4] gave explicit expressions for $g_{k-1}(x)$ and $h_{k-1}(x)$. P. He and Z. Zhang [1] discussed generalized Lucas numbers. The purpose of this paper is to generalize the above results, i.e., to evaluate the following summation:

$$\sum_{a_1+a_2+\dots+a_k=n} W_{ma_1} W_{ma_2} \dots W_{ma_k}.$$

2. THE CONVOLUTED FORMULA OF GENERALIZED FIBONACCI NUMBERS

Throughout this section, with $W_0 = 0$, if we let

$$G_k(x) = \left(\frac{bU_m}{1 - V_m x + q^m x^2} \right)^k = \sum_{n=0}^{\infty} W_{mn}^{(k)} x^{n-1}, \tag{5}$$

then we have

$$\sum_{a_1+a_2+\dots+a_k=n} W_{ma_1} W_{ma_2} \dots W_{ma_k} = W_{m(n-k+1)}^{(k)}. \tag{6}$$

Theorem 2.1:

$$W_{mn}^{(k+1)} = \frac{bU_m}{k(V_m^2 - 4q^m)} \left\{ nV_m W_{m(n+1)}^{(k)} - 2q^m(n + 2k - 1)W_{mn}^{(k)} \right\}. \tag{7}$$

Proof: Noting that

$$\frac{d}{dx} (G_k(x)(V_m - 2q^m x)^k) = G_k'(x)(V_m - 2q^m x)^k + G_k(x)k(V_m - 2q^m x)^{k-1}(-2q^m)$$

and

$$\begin{aligned} \frac{d}{dx} (G_k(x)(V_m - 2q^m x)^k) &= \frac{d}{dx} \left(\frac{bU_m(V_m - 2q^m x)}{1 - V_m x + q^m x^2} \right)^k \\ &= k \left(\frac{bU_m(V_m - 2q^m x)}{1 - V_m x + q^m x^2} \right)^{k-1} bU_m \frac{2q^m(1 - V_m x + q^m x^2) + V_m^2 - 4q^m}{(1 - V_m x + q^m x^2)^2}, \end{aligned}$$

we have

$$G'_k(x)bU_m(V_m - 2q^m x) - 2bkU_m q^m G_k(x) = 2bkU_m q^m G_k(x) + k(V_m^2 - 4q^m)G_{k+1}(x).$$

Comparing the coefficients of both sides of the equation, our theorem holds.

We denote by $\sigma_i(n, k)$ the summation of all products of choosing i elements from $n + k - i + 1, n + k - i + 2, \dots, n + 2k - 1$ but not containing any two consecutive elements, i.e.,

$$\sigma_i(n, k) = \sum \prod_{t=1}^i (n + k - i + j_t) \tag{8}$$

where the summation is taken over all i -tuples with positive integer coordinates (j_1, j_2, \dots, j_i) such that $1 \leq j_1 < j_2 < \dots < j_i \leq k + i - 1$ and $|j_r - j_s| \geq 2$ for $1 \leq r \neq s \leq i$.

We note that $\sigma_0(n, k) = 1$. It is easy to prove that

$$(n + 2k - 1)\sigma_{k-1}(n, k - 1) = \sigma_k(n, k), \tag{9}$$

and

$$(n + 2k - 1)\sigma_{i-1}(n, k - 1) + \sigma_i(n + 1, k - 1) = \sigma_i(n, k), \tag{10}$$

which yields

Theorem 2.2:

$$W_{mn}^{(k+1)} = \frac{(bU_m)^k}{k!(V_m^2 - 4q^m)^k} \sum_{i=0}^k (-2q^m)^i V_m^{k-i} \langle n \rangle_{k-i} \sigma_i(n, k) W_{m(n+k-i)}, \tag{11}$$

where $\langle n \rangle_k = n(n + 1)(n + 2) \dots (n + k - 1)$.

Proof: We prove the theorem by induction on k . When $k = 0, 1$, the theorem is true by applying Theorem 2.1. Assume the theorem is true for a positive integer $k - 1$. Then

$$\begin{aligned}
 W_{mn}^{(k+1)} &= \frac{bU_m}{k(V_m^2 - 4q^m)} \left\{ nV_m W_{m(n+1)}^{(k)} - 2q^m(n + 2k - 1)W_{mn}^{(k)} \right\} \\
 &= \frac{bU_m}{k(V_m^2 - 4q^m)} \left\{ nV_m \frac{(bU_m)^{k-1}}{(k-1)!(V_m^2 - 4q^m)^{k-1}} \sum_{i=0}^{k-1} (-2q^m)^i V_m^{k-i-1} \langle n+1 \rangle_{k-i-1} \times \right. \\
 &\quad \sigma_i(n+1, k-1)W_{m(n+k-i)} + (-2q^m)(n-1+2k) \frac{(bU_m)^{k-1}}{(k-1)!(V_m^2 - 4q^m)^{k-1}} \times \\
 &\quad \left. \sum_{i=0}^{k-1} (-2q^m)^i V_m^{k-i-1} \langle n \rangle_{k-i-1} \sigma_i(n, k-1)W_{m(n+k-i-1)} \right\} \\
 &\quad \frac{(bU_m)^k}{k!(V_m^2 - 4q^m)^k} \{ V_m^k n \langle n+1 \rangle_{k-1} \sigma_0(n+1, k-1)W_{m(n+k)} \\
 &\quad + \sum_{i=1}^{k-1} (-2q^m)^i V_m^{k-i} n \langle n+1 \rangle_{k-i} \sigma_i(n+1, k-1)W_{m(n+k-i)} \\
 &\quad + \sum_{i=1}^k (-2q^m)^i V_m^{k-i} (n+2k-1) \langle n \rangle_{k-i} \sigma_{i-1}(n, k-1)W_{m(n+k-i)} \} \\
 &= \frac{(bU_m)^k}{k!(V_m^2 - 4q^m)^k} \{ V_m^k \langle n \rangle_k \sigma_0(n, k)W_{m(n+k)} \\
 &\quad + \sum_{i=1}^{k-1} (-2q^m)^i V_m^{k-i} \langle n \rangle_{k-i} W_{m(n+k-i)} [\sigma_i(n+1, k-1) + (n+2k-1)\sigma_{i-1}(n, k-1)] \\
 &\quad + (-2q^m)^k (n+2k-1)\sigma_{k-1}(n, k-1)W_{mn} \}
 \end{aligned}$$

(Apply (9), (10))

$$\begin{aligned}
 &= \frac{(bU_m)^k}{k!(V_m^2 - 4q^m)^k} \{V_m^k \langle n \rangle_k \sigma_0(n, k) W_{m(n+k)} \\
 &+ \sum_{i=1}^{k-1} (-2q^m)^i V_m^{k-i} \langle n \rangle_{k-i} \sigma_i(n, k) W_{m(n+k-i)} + (-2q^m)^k \sigma_k(n, k) W_{mn}\} \\
 &= \frac{(bU_m)^k}{k!(V_m^2 - 4q^m)^k} \sum_{i=0}^k (-2q^m)^i V_m^{k-i} \langle n \rangle_{k-i} \sigma_i(n, k) W_{m(n+k-i)}.
 \end{aligned}$$

Hence the theorem is also true for k . This completes the proof.

Theorem 2.3:

$$\begin{aligned}
 &\sum_{a_1+a_2+\dots+a_k=n} W_{ma_1} W_{ma_2} \dots W_{ma_k} \\
 &= \frac{(bU_m)^{k-1}}{(k-1)!(V_m^2 - 4q^m)^{k-1}} \sum_{i=0}^{k-1} (-2q^m)^i V_m^{k-1-i} \langle n-k+1 \rangle_{k-1-i} \sigma_i(n-k+1, k-1) W_{m(n-i)}. \quad (12)
 \end{aligned}$$

Proof: Use (6) and Theorem 2.2.

Lemma 2.4:

$$U_m W_{m(k+n)} = U_{mn} W_{m(k+1)} - q^m U_{m(n-1)} W_{mk}. \quad (13)$$

Proof: Use (2), (3).

Let

$$g_{k-1}^{(m)}(n) = \sum_{i=0}^{k-1} (-2q^m)^i V_m^{k-1-i} \langle n-k+1 \rangle_{k-1-i} \sigma_i(n-k+1, k-1) U_{m(k-i)} \quad (14)$$

and

$$h_{k-1}^{(m)}(n) = -qM \sum_{i=0}^{k-1} (-2q^m)^i V_m^{k-1-i} \langle n-k+1 \rangle_{k-1-i} \sigma_i(n-k+1, k) U_{m(k-1-i)}. \quad (15)$$

Then we have the following theorem.

Theorem 2.5:

$$\sum_{a_1+a_2+\dots+a_k=n} W_{ma_1} W_{ma_2} \dots W_{ma_k}$$

$$= \frac{(b^{k-1}U_m^{k-2}}{(k-1)!(V_m^2 - 4q^m)^{k-1}} \left\{ g_{k-1}^{(m)}(n)W_{m(n-k+1)} + h_{k-1}^{(m)}(n)W_{m(n-k)} \right\}. \tag{16}$$

Proof: Use Theorem 2.3 and Lemma 2.4.
Corollary 2.6:

$$\begin{aligned} & \sum_{a+b=n} W_{ma}W_{mb} \\ &= \frac{b}{V_m^2 - 4q^m} \{ [(n-1)V_mU_{2m} - 2q_m nU_m]W_{m(n-1)} - q^m(n-1)V_mU_mW_{m(n-2)} \}, \\ & \sum_{a+b+c=n} W_{ma}W_{mb}W_{mc} \\ & \frac{b^2U_m}{2(V_m^2 - 4q^m)^2} \{ (n-2)(n-1)V_m^2U_{3m} - 2q_m(n-2)(2n+1)V_mU_{2m} + 4q^{2m}(n-1) \\ & (n+1)U_m]W_{m(n-2)} - q^m[(n-2)(n-1)V_m^2U_{2m} - 2q^m(n-2)(2n+1)V_mU_m]W_{m(n-3)} \}. \end{aligned}$$

Proof: Take $k = 2, 3$ in Theorem 2.5.
 From (16), we have
Corollary 2.7:

$$b^{k-1}U_m^{k-2} \left\{ g_{k-1}^{(m)}(n)W_{m(n-k+1)} + h_{k-1}^{(m)}(n)W_{m(n-k)} \right\} \equiv 0 \pmod{(k-1)!(V_m^2 - 4q^m)^{k-1}}. \tag{17}$$

3. THE CONVOLUTED FORMULA OF GENERALIZED LUCAS NUMBERS

Let $\Delta = 2U_{m+1} - pU_m$. Taking $a = 2, b = p$ and using (4) we have

$$\sum_{n=0}^{\infty} V_{mn}x^n = \frac{2 - \Delta x}{1 - V_mx + q^m x^2}.$$

Let

$$H_k(x) = \sum_{n=0}^{\infty} V_{mn}^{(k)}x^n = \left(\frac{2 - \Delta x}{1 - V_mx + q^m x^2} \right)^k.$$

Obviously, $V_{mn}^{(1)} = V_{mn}$. From these, we have

$$\sum_{a_1+a_2+\dots+a_k=n} V_{ma_1}V_{ma_2}\dots V_{ma_k} = V_{mn}^{(k)}. \quad (18)$$

Theorem 3.1:

$$k(\Delta V_m - 4q^m)V_{mn}^{(k+1)} = 4(n+2)V_{m(n+2)}^{(k)} - 2(2n+k+2)\Delta V_{m(n+1)}^{(k)} + (n+k)\Delta^2 V_{mn}^{(k)}. \quad (19)$$

Proof: Noting that

$$\begin{aligned} \frac{d}{dx}(H_k(x)) &= \frac{d}{dx} \left(\frac{2 - \Delta x}{1 - V_m x + q^m x^2} \right)^k \\ &= k \left(\frac{2 - \Delta x}{1 - V_m x + q^m x^2} \right)^{k-1} \frac{-\Delta + 2V_m - 4q^m x + q^m \Delta x^2}{(1 - V_m x + q^m x^2)^2} \\ &= k \left(\frac{2 - \Delta x}{1 - V_m x + q^m x^2} \right)^{k-1} \frac{\Delta[1 - V_m x + q^m x^2] + [\Delta V_m - 4q^m]}{(1 - V_m x + q^m x^2)^2} \\ &= k \frac{\Delta}{2 - \Delta x} \left(\frac{2 - \Delta x}{1 - V_m x + q^m x^2} \right)^k + k \frac{\Delta V_m - 4q^m}{(2 - \Delta x)^2} x \left(\frac{2 - \Delta x}{1 - V_m x + q^m x^2} \right)^{k+1}, \end{aligned}$$

we have

$$(2 - \Delta x)^2 \frac{d}{dx} H_k(x) = k[\Delta V_m - 4q^m]xH_{k+1}(x) + k\Delta(2 - \Delta x)H_k(x).$$

This implies

$$k[\Delta V_m - 4q^m]xH_{k+1}(x) = (4 - 4\Delta x + \Delta^2 x^2) \frac{d}{dx} H_k(x) - k\Delta(2 - \Delta x)H_k(x).$$

Comparing the coefficients of both sides in the above equation, the theorem holds.

Theorem 3.2:

$$\begin{aligned} &\sum_{a+b=n} V_{ma}V_{mb} \\ &= \frac{1}{\Delta V_m - 4q^m} \{4(n+2)V_{m(n+2)} + 2(2n+3)\Delta V_{m(n+1)} + (n+1)\Delta^2 V_{mn}\}. \quad (20) \end{aligned}$$

$$\sum_{a+b+c=n} V_{ma}V_{mb}V_{mc} = \frac{1}{2(\Delta V_m - 4q^m)^2} \{16(n+4)(n+2)V_{m(n+4)} + 8((n+2)(2n+7) + (n+3)(2n+4))\Delta V_{m(n+3)} + 4((n+2)(n+3) + (2n+4)(2n+5) + (n+2)^2)\Delta^2 V_{m(n+2)} + 2(n+2)(4n+7)\Delta^3 V_{m(n+1)} + (n+1)(n+2)\Delta^4 V_{mn}\}. \quad (21)$$

Proof: Use (18) and Theorem 3.1.

In Theorem 3.1, taking $m = 1, 2$ gives the main results of paper [1].

Corollary 3.3:

$$k(p^2 - 4q)V_n^{(k+1)} = 4(n+2)V_{n+2}^{(k)} - 2p(2n+k+2)V_{n+1}^{(k)} + p^2(n+k)V_n^{(k)}, \quad (22)$$

$$kp^2(p^2 - 4q)V_{2n}^{(k+1)} = 4(n+2)V_{2(n+2)}^{(k)} - 2(2n+k+2)(p^2 - 2q)V_{2(n+1)}^{(k)} + (n+k)(p^2 - 2q)^2V_{2n}^{(k)}. \quad (23)$$

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