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CONVOLUTIONS OF BERNOULLI AND EULER POLYNOMIALS

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ABSTRACT. By means of the generating function technique, several convolution identities are derived for the polynomials of Bernoulli and Euler.

The numbers and polynomials of Bernoulli and Euler are very useful in classical analysis and numerical mathematics. Their basic properties can briefly be summarized as follows. More comprehensive coverage can be found in the monographs by Abramowitz–Stegun [1, §23], Comtet [7, §1.14], Graham–Knuth–Patashnik [11, §6.5] and Rosen [13, §3.1].

The Bernoulli and Euler numbers are defined respectively by the exponential generating functions

$$\frac{u}{e^u-1}=\sum_{n\geq 0}B_n\frac{u^n}{n!} \qquad \text{and} \qquad \frac{2e^u}{e^{2u}+1}=\sum_{n\geq 0}E_n\frac{u^n}{n!}.$$

Some related summation formulae and identities can be found in Agoh–Dilcher [2, 8] and Chu–Wang [5, 6]. For the corresponding polynomials, the generating functions read as

$$\frac{ue^{ux}}{e^u-1}=\sum_{n\geq 0}B_n(x)\frac{u^n}{n!} \qquad \text{and} \qquad \frac{2e^{ux}}{e^u+1}=\sum_{n\geq 0}E_n(x)\frac{u^n}{n!}.$$

Both polynomials are expressed through the respective numbers

$$B_n(x) \;=\; \sum_{k=0}^{n}\binom{n}{k}B_kx^{n-k} \quad \text{and} \quad E_n(x) \;=\; \sum_{k=0}^{n}\binom{n}{k}\frac{E_k}{2^k}\Big(x-\frac{1}{2}\Big)^{n-k}.$$

They satisfy the binomial relations

$$B_n(x+y) \;=\; \sum_{k=0}^{n}\binom{n}{k}B_k(x)y^{n-k} \quad \text{and} \quad E_n(x+y) \;=\; \sum_{k=0}^{n}\binom{n}{k}E_k(x)y^{n-k}$$

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differential equations

$$B_n'(x) \;=\; nB_{n-1}(x) \quad \text{and} \quad E_n'(x) \;=\; nE_{n-1}(x)$$

reciprocal relations

$$B_n(1-x) \;=\; (-1)^n B_n(x) \quad \text{and} \quad E_n(1-x) \;=\; (-1)^n E_n(x) \tag{1}$$

as well as difference equations

$$B_n(1+x) - B_n(x) = nx^{n-1} \quad \text{and} \quad E_n(1+x) + E_n(x) = 2x^n. \tag{2}$$

There is also an expression of Euler polynomials in terms of Bernoulli polynomials

$$E_n(x) = \frac{2}{n+1}\Big\{B_{n+1}(x) - 2^{n+1}B_{n+1}(x/2)\Big\}. \tag{3}$$

The generating function method (cf. Graham *et al* [11, Chapter 7] and Wilf [15]) is powerful in dealing with problems of combinatorial computations. Motivated by the recent work on bivariate Ω–polynomials due to Chu–Magli [4], the purpose of this paper is to utilize this tool to investigate the binomial convolutions defined by

$$\Omega_m\big(x,y\big|\,{}^{\alpha,\gamma}_{F,G}\big) = \sum_{k=0}^{m} \lambda^k \binom{m}{k} \frac{F_{k+\alpha}(x)}{(k+1)_\alpha} \frac{G_{m-k+\gamma}(y)}{(m-k+1)_\gamma} \tag{4}$$

where $\alpha,\ \gamma \in \mathbb{N}_0$, $\lambda \in \mathbb{C}$ and the shifted factorial is defined by

$$(z)_0 = 1 \quad \text{and} \quad (z)_n = z(z+1)\cdots(z+n-1) \quad \text{for} \quad n = 1, 2, \cdots.$$

Consider the exponential generating function

$$\sum_{m\geq 0} \frac{u^m}{m!} \Omega_m\big(x,y\big|\,{}^{\alpha,\gamma}_{F,G}\big) = \sum_{m\geq 0}\sum_{i=0}^{m} \lambda^i \binom{m}{i} \frac{F_{i+\alpha}(x)}{(i+1)_\alpha} \frac{G_{m-i+\gamma}(y)}{(m-i+1)_\gamma} \frac{u^m}{m!}.$$

Interchanging the summation order and then making the replacement $m \to i+j$, we can reformulate the generating function as follows

$$\sum_{m\geq 0} \frac{u^m}{m!} \Omega_m\big(x,y\big|\,{}^{\alpha,\gamma}_{F,G}\big) = \sum_{i\geq 0} \frac{(\lambda u)^i}{(i+\alpha)!} F_{i+\alpha}(x) \sum_{j\geq 0} \frac{u^j}{(j+\gamma)!} G_{j+\gamma}(y)$$

which leads us to the following product expression

$$\sum_{m\geq 0} \frac{u^m}{m!} \Omega_m\big(x,y\big|\,{}^{\alpha,\gamma}_{F,G}\big) = \frac{\lambda^{-\alpha}}{u^{\alpha+\gamma}} \Big\{ \sum_{i\geq \alpha} \frac{(\lambda u)^i}{i!} F_i(x) \Big\} \times \Big\{ \sum_{j\geq \gamma} \frac{u^j}{j!} G_j(y) \Big\}. \tag{5}$$

This generating function relation will be utilized to compute the convolutions defined by (4). Several identities will be established for $\Omega_m\big(x,y\big|\,{}^{\alpha,\gamma}_{F,G}\big)$ in the remaining three sections, respectively dealing with the cases of "$\alpha =$

$\gamma = 0$", "$\alpha = 1,\ \gamma = 0$" and "$\alpha = \gamma = 1$" where $\lambda = \pm 1,\ 2$ and $\{F, G\}$ will be specified by Bernoulli and Euler polynomials.

1. $\alpha = \gamma = 0$

In this section, we examine the convolution sums $\Omega_m\big(x, y\big|\,{}^{\alpha,\gamma}_{F,G}\big)$ defined in (4), when $\alpha = \gamma = 0$ and $\{F, G\}$ are specified by Bernoulli and Euler polynomials.

1.1. $F_k(x) = B_k(x)$ and $G_k(y) = B_k(y)$.

The exponential generating function corresponding to (5) becomes

$$\Big\{\sum_{i\geq 0}\frac{(\lambda u)^i}{i!}B_i(x)\Big\} \times \Big\{\sum_{j\geq 0}\frac{u^j}{j!}B_j(y)\Big\} = \frac{\lambda u^2 e^{u(\lambda x+y)}}{(e^{\lambda u}-1)(e^u-1)}. \tag{6}$$

$\boxed{\lambda = 1}$ The generating function displayed in (6) can be expressed as

$$\frac{u^2 e^{u(x+y)}}{(e^u-1)^2} = (x+y-1)\frac{u^2 e^{u(x+y-1)}}{e^u-1} - \frac{u^2 d}{du}\frac{e^{u(x+y-1)}}{e^u-1}.$$

Extracting the coefficient of $\frac{u^m}{m!}$ across the last equation leads to the convolution

$$\sum_{k=0}^{m}\binom{m}{k}B_k(x)B_{m-k}(y) = (1-m)B_m(x+y-1)+(x+y-1)mB_{m-1}(x+y-1).$$

Applying the relation of difference in (2) for Bernoulli polynomials, we can simplify the last equation to the following formula.

Theorem 1. (Hansen [12, Eq 50.11.2])

$$\sum_{k=0}^{m}\binom{m}{k}B_k(x)B_{m-k}(y) = (1-m)B_m(x+y) + (x+y-1)mB_{m-1}(x+y).$$

We remark that the special case $x = y = 0$ of the last identity

$$\sum_{k=2}^{\ell-2}\binom{\ell}{k}B_k B_{\ell-k} = -(\ell+1)B_\ell$$

is originally due to Euler and Ramanujan (cf. [8, Eq 1.2], [9, Eq 1.2] and [10]).

$\boxed{\lambda = -1}$ Replacing x by $1-x$ in Theorem 1, then invoking both (1) and (2) on Bernoulli polynomials, we deduce another convolution formula.

Corollary 2. (Dilcher [8, Eq 3.2] and Hansen [12, Eq 50.11.1])

$$\sum_{k=0}^{m}(-1)^k\binom{m}{k}B_k(x)B_{m-k}(y) = (1-m)B_m(y-x) + (y-x)mB_{m-1}(y-x).$$

$\boxed{\lambda = 2}$ The generating function displayed in (6) can be reformulated as

$$\frac{2u^2e^{u(2x+y)}}{(e^u-1)^2(e^u+1)} = (2x+y-1)\frac{u^2e^{u(2x+y-1)}}{e^u-1} - \frac{u^2e^{u(2x+y)}}{e^{2u}-1} - \frac{u^2d}{du}\frac{e^{u(2x+y-1)}}{e^u-1}.$$

Extracting the coefficient of $\frac{u^m}{m!}$ across this equation and then appealing to (2) and (3), we derive the following convolution identity.

Theorem 3. (Hansen [12, Eq 50.11.4])

$$\sum_{k=0}^{m} 2^k\binom{m}{k}B_k(x)B_{m-k}(y) = (1-m)B_m(2x+y) + \tfrac{m(m-1)}{4}E_{m-2}(2x+y) + m\big(2x+y-\tfrac{3}{2}\big)B_{m-1}(2x+y).$$

1.2. $F_k(x) = E_k(x)$ **and** $G_k(y) = E_k(y)$**.** The exponential generating function corresponding to (5) reads as

$$\Big\{\sum_{i\geq 0}\frac{(\lambda u)^i}{i!}E_i(x)\Big\} \times \Big\{\sum_{j\geq 0}\frac{u^j}{j!}E_j(y)\Big\} = \frac{4e^{u(\lambda x+y)}}{(e^{\lambda u}+1)(e^u+1)}. \tag{7}$$

$\boxed{\lambda = 1}$ The generating function displayed in (7) can be stated as

$$\frac{4e^{u(x+y)}}{(e^u+1)^2} = 4(x+y-1)\frac{e^{u(x+y-1)}}{e^u+1} - \frac{4d}{du}\frac{e^{u(x+y-1)}}{e^u+1}.$$

Extracting the coefficient of $\frac{u^m}{m!}$ across this equation and then appealing to (2), we derive the following convolution formula.

Theorem 4. (Dilcher [8, Eq 4.2] and Hansen [12, Eq 51.6.2])

$$\sum_{k=0}^{m}\binom{m}{k}E_k(x)E_{m-k}(y) = 2(1-x-y)E_m(x+y) + 2E_{m+1}(x+y).$$

$\boxed{\lambda = -1}$ Replacing x by $1-x$ in Theorem 4, then applying the relations of the difference and reciprocity in (1) and (2) for Euler polynomials, we can express the resulting equation as another identity.

Corollary 5. (Hansen [12, Eq 51.6.1])

$$\sum_{k=0}^{m}(-1)^k\binom{m}{k}E_k(x)E_{m-k}(y) = 2(y-x)E_m(y-x) - 2E_{m+1}(y-x).$$

$\boxed{\lambda = 2}$ The generating function displayed in (7) can be expressed as

$$\frac{4e^{u(2x+y)}}{(e^{2u}+1)(e^u+1)} = \frac{2e^{u(2x+y)}}{e^u+1} - \frac{2e^{u(2x+y+1)}}{e^{2u}+1} + \frac{2e^{u(2x+y)}}{e^{2u}+1}.$$

Extracting the coefficient of $\frac{u^m}{m!}$ across this equation recovers directly the following convolution formula.

Theorem 6. (Hansen [12, Eq 51.6.5])

$$\sum_{k=0}^{m} 2^k \binom{m}{k} E_k(x) E_{m-k}(y) = E_m(2x+y) - 2^m E_m(\tfrac{2x+y+1}{2}) + 2^m E_m(\tfrac{2x+y}{2}).$$

1.3. $F_k(x) = B_k(x)$ **and** $G_k(y) = E_k(y)$**.** The exponential generating function corresponding to (5) is given by

$$\Big\{\sum_{i\geq 0} \frac{(\lambda u)^i}{i!} B_i(x)\Big\} \times \Big\{\sum_{j\geq 0} \frac{u^j}{j!} E_j(y)\Big\} = \frac{2\lambda u e^{u(\lambda x+y)}}{(e^{\lambda u}-1)(e^u+1)}. \tag{8}$$

$\boxed{\lambda = 1}$ The generating function displayed in (8) can be expressed as

$$\frac{2ue^{u(x+y)}}{e^{2u}-1} = \frac{ue^{u(x+y)}}{e^u-1} - \frac{ue^{u(x+y)}}{e^u+1}.$$

Extracting the coefficient of $\frac{u^m}{m!}$ across the last equation and then taking into account of (3), we establish the following identity.

Theorem 7.

$$\sum_{k=0}^{m} \binom{m}{k} B_k(x) E_{m-k}(y) = B_m(x+y) - \tfrac{m}{2} E_{m-1}(x+y) = 2^m B_m(\tfrac{x+y}{2}).$$

We point out that the special case $x = 0$ of the last theorem recovers the identity due to Cheon [3, Eq 13] and Srivastava–Pinter [14, Eq 11]:

$$\sum_{k=0}^{m} \binom{m}{k} B_k E_{m-k}(y) = 2^m B_m(\tfrac{y}{2}).$$

$\boxed{\lambda = -1}$ Replacing x by $1-x$ in Theorem 7 and then applying (1) and (2) to the resulting equation, we get another identity.

Corollary 8.

$$\sum_{k=0}^{m} (-1)^k \binom{m}{k} B_k(x) E_{m-k}(y) = B_m(y-x) + \tfrac{m}{2} E_{m-1}(y-x).$$

$\boxed{\lambda = 2}$ The generating function displayed in (8) can be expressed as

$$\frac{4ue^{u(2x+y)}}{(e^{2u}-1)(e^u+1)} = \frac{2ue^{u(2x+y)}}{e^{2u}-1} + 2\frac{ud}{du}\frac{e^{u(2x+y-1)}}{e^u+1} - 2(2x+y-1)\frac{ue^{u(2x+y-1)}}{e^u+1}.$$

Extracting the coefficient of $\frac{u^m}{m!}$ across the last equation gives the convolution

$$\sum_{k=0}^{m} 2^k \binom{m}{k} B_k(x) E_{m-k}(y) = 2^m B_m(\tfrac{2x+y}{2}) + mE_m(2x+y-1) \\ - m(2x+y-1)E_{m-1}(2x+y-1).$$

By means of (1), (2) and (3), this equation simplifies to the following identity.

Theorem 9.

$$\sum_{k=0}^{m} 2^k \binom{m}{k} B_k(x) E_{m-k}(y) = B_m(2x+y) \\ - m\big\{E_m(2x+y) - (2x+y-\tfrac{3}{2})E_{m-1}(2x+y)\big\}.$$

$\boxed{\lambda = 1/2}$ The generating function displayed in (8) can be expressed as the partial fractions

$$\frac{e^{uy}}{e^u+1} \times \frac{ue^{\frac{ux}{2}}}{e^{u/2}-1} = \frac{ue^{\frac{u}{2}(x+2y)}}{2(e^{u/2}-1)} - \frac{ue^{\frac{u}{2}(1+x+2y)}}{2(e^u+1)} - \frac{ue^{\frac{u}{2}(x+2y)}}{2(e^u+1)}.$$

Extracting the coefficient of $\frac{u^m}{m!}$ across this equation, then interchanging x and y and finally multiplying the resulting equation by 2^m, we derive the following convolution identity.

Theorem 10.

$$\sum_{k=0}^{m} 2^k \binom{m}{k} E_k(x) B_{m-k}(y) = B_m(2x+y) \\ - 2^{m-2}m\Big\{E_{m-1}\big(\tfrac{2x+y+1}{2}\big) + E_{m-1}\big(\tfrac{2x+y}{2}\big)\Big\}.$$

2. $\alpha = 1$ AND $\gamma = 0$

In this section, we examine the convolution sums $\Omega_m\big(x, y\big|{\alpha,\gamma \atop F,G}\big)$ defined in (4), when $\alpha = 1$, $\gamma = 0$ and $\{F, G\}$ are specified by Bernoulli and Euler polynomials.

2.1. $F_k(x) = B_k(x)$ and $G_k(y) = B_k(y)$.

The exponential generating function corresponding to (5) becomes

$$\frac{\lambda^{-1}}{u}\Big\{\sum_{i\geq 1} \frac{(\lambda u)^i}{i!} B_i(x)\Big\} \times \Big\{\sum_{j\geq 0} \frac{u^j}{j!} B_j(y)\Big\} = \frac{1}{\lambda u}\Big\{\frac{\lambda u e^{\lambda ux}}{e^{\lambda u}-1} - 1\Big\}\frac{ue^{uy}}{e^u-1}. \quad (9)$$

$\boxed{\lambda = 1}$ The generating function displayed in (9) can be expressed as

$$\frac{1}{u}\left\{\frac{ue^{ux}}{e^u-1}-1\right\}\frac{ue^{uy}}{e^u-1}=(x+y-1)\frac{ue^{u(x+y-1)}}{e^u-1}-\frac{ud}{du}\frac{e^{u(x+y-1)}}{e^u-1}-\frac{e^{uy}}{e^u-1}.$$

Extracting the coefficient of $\frac{u^m}{m!}$ across this equation and then appealing to (2), we derive the following convolution identity.

Theorem 11.

$$\sum_{k=0}^{m}\binom{m}{k}\frac{B_{k+1}(x)}{k+1}B_{m-k}(y)=(x+y-1)B_m(x+y)-\tfrac{m}{m+1}B_{m+1}(x+y)-\tfrac{B_{m+1}(y)}{m+1}.$$

$\boxed{\lambda = -1}$ Replacing x by $1-x$ in Theorem 11 results in another formula.

Corollary 12.

$$\sum_{k=0}^{m}(-1)^k\binom{m}{k}\frac{B_{k+1}(x)}{k+1}B_{m-k}(y)$$
$$=(x-y)B_m(y-x)+\tfrac{m}{m+1}B_{m+1}(y-x)+\tfrac{B_{m+1}(y)}{m+1}.$$

$\boxed{\lambda = 2}$ The generating function displayed in (9) can be expressed as

$$\frac{1}{2u}\left\{\frac{2ue^{2ux}}{e^{2u}-1}-1\right\}\frac{ue^{uy}}{e^u-1}=(2x+y-1)\frac{ue^{u(2x+y-1)}}{2(e^u-1)}-\frac{ue^{u(2x+y)}}{2(e^{2u}-1)}$$
$$-\frac{e^{uy}}{2(e^u-1)}-\frac{ud}{2du}\frac{e^{u(2x+y-1)}}{e^u-1}.$$

Extracting the coefficient of $\frac{u^m}{m!}$ across this equation and then applying (2) and (3), we obtain the following convolution formula.

Theorem 13.

$$\sum_{k=0}^{m}2^k\binom{m}{k}\frac{B_{k+1}(x)}{k+1}B_{m-k}(y)=\tfrac{1}{2}\big\{(2x+y-\tfrac{3}{2})B_m(2x+y)$$
$$-\tfrac{m}{m+1}B_{m+1}(2x+y)-\tfrac{B_{m+1}(y)}{m+1}+\tfrac{m}{4}E_{m-1}(2x+y)\big\}.$$

$\boxed{\lambda = 1/2}$ The generating function displayed in (9) can be reformulated as

$$\frac{ue^{uy}}{e^u-1}\times\frac{2}{u}\left\{\frac{ue^{\frac{ux}{2}}}{2(e^{u/2}-1)}-1\right\}=(x+2y-1)\frac{ue^{\frac{u}{2}(x+2y-1)}}{2(e^{u/2}-1)}-\frac{ue^{u(\frac{x}{2}+y)}}{2(e^u-1)}$$
$$-\frac{2e^{uy}}{e^u-1}-\frac{ud}{du}\frac{e^{\frac{u}{2}(x+2y-1)}}{e^{u/2}-1}.$$

Extracting the coefficient of $\frac{u^m}{m!}$ across this equation, then interchanging x and y and finally simplifying the resulting equation by means of (2) and (3), we derive the following convolution identity.

Theorem 14.

$$\sum_{k=0}^{m} 2^k \binom{m}{k} B_k(x) \frac{B_{m-k+1}(y)}{m-k+1} = (2x+y-\tfrac{3}{2})B_m(2x+y)$$
$$-\tfrac{m}{m+1}B_{m+1}(2x+y) - \tfrac{B_{m+1}(2x)}{m+1} + \tfrac{E_m(2x)}{2} + \tfrac{m}{4}E_{m-1}(2x+y).$$

2.2. $F_k(x) = E_k(x)$ **and** $G_k(y) = E_k(y)$**.** The exponential generating function corresponding to (5) is given by

$$\frac{\lambda^{-1}}{u}\left\{\sum_{i\geq 1}\frac{(\lambda u)^i}{i!}E_i(x)\right\} \times \left\{\sum_{j\geq 0}\frac{u^j}{j!}E_j(y)\right\} = \frac{1}{\lambda u}\left\{\frac{2e^{\lambda ux}}{e^{\lambda u}+1}-1\right\}\frac{2e^{uy}}{e^u+1}. \quad (10)$$

$\boxed{\lambda = 1}$ The generating function displayed in (10) can be expressed as

$$\frac{1}{u}\left\{\frac{2e^{ux}}{e^u+1}-1\right\}\frac{2e^{uy}}{e^u+1} = 4(x+y-1)\frac{e^{u(x+y-1)}}{u(e^u+1)} - \frac{4d}{udu}\frac{e^{u(x+y-1)}}{e^u+1} - \frac{2e^{uy}}{u(e^u+1)}.$$

Extracting the coefficient of $\frac{u^m}{m!}$ across this equation and then appealing to (2), we derive the following convolution formula.

Theorem 15.

$$\sum_{k=0}^{m}\binom{m}{k}\frac{E_{k+1}(x)}{k+1}E_{m-k}(y)$$
$$= \frac{2(1-x-y)E_{m+1}(x+y) + 2E_{m+2}(x+y) - E_{m+1}(y)}{m+1}.$$

$\boxed{\lambda = -1}$ Replacing x by $1-x$ in Theorem 15 and then invoking (1) and (2), we can express the resulting equation as the following identity.

Corollary 16.

$$\sum_{k=0}^{m}(-1)^k\binom{m}{k}\frac{E_{k+1}(x)}{k+1}E_{m-k}(y)$$
$$= \frac{2(x-y)E_{m+1}(y-x) + 2E_{m+2}(y-x) + E_{m+1}(y)}{m+1}.$$

$\boxed{\lambda = 2}$ The generating function displayed in (10) can be reformulated as

$$\frac{1}{2u}\left\{\frac{2e^{2ux}}{e^{2u}+1}-1\right\}\frac{2e^{uy}}{e^u+1} = \frac{e^{u(2x+y)}}{u(e^u+1)} - \frac{e^{u(2x+y+1)}}{u(e^{2u}+1)} + \frac{e^{u(2x+y)}}{u(e^{2u}+1)} - \frac{e^{uy}}{u(e^u+1)}.$$

Extracting the coefficient of $\frac{u^m}{m!}$ across this equation yields to the following convolution formula.

Theorem 17.

$$\sum_{k=0}^{m} 2^k \binom{m}{k} \frac{E_{k+1}(x)}{k+1} E_{m-k}(y) = \tfrac{1}{2(m+1)} \big\{ E_{m+1}(2x+y) - 2^{m+1} E_{m+1}(\tfrac{2x+y+1}{2}) + 2^{m+1} E_{m+1}(\tfrac{2x+y}{2}) - E_{m+1}(y) \big\}.$$

$\boxed{\lambda = 1/2}$ The generating function displayed in (10) can be decomposed into partial fractions

$$\frac{2e^{uy}}{e^u+1} \times \frac{2}{u} \Big\{ \frac{2e^{\frac{ux}{2}}}{e^{u/2}+1} - 1 \Big\} = \frac{4e^{\frac{u}{2}(x+2y)}}{u(e^{u/2}+1)} - \frac{4e^{u\frac{x+2y+1}{2}}}{u(e^u+1)} + \frac{4e^{u\frac{x+2y}{2}}}{u(e^u+1)} - \frac{4e^{uy}}{u(e^u+1)}.$$

Extracting the coefficient of $\frac{u^m}{m!}$ across this equation, then interchanging x and y and finally simplifying the resulting equation times 2^m, we obtain the following convolution identity.

Theorem 18.

$$\sum_{k=0}^{m} 2^k \binom{m}{k} E_k(x) \frac{E_{m-k+1}(y)}{m-k+1} = \tfrac{1}{m+1} \Big\{ E_{m+1}(2x+y) - 2^{m+1} E_{m+1}(\tfrac{2x+y+1}{2}) + 2^{m+1} E_{m+1}(\tfrac{2x+y}{2}) - 2^{m+1} E_{m+1}(x) \Big\}.$$

2.3. $F_k(x) = B_k(x)$ **and** $G_k(y) = E_k(y)$**.** The exponential generating function corresponding to (5) reads as

$$\frac{\lambda^{-1}}{u} \Big\{ \sum_{i \ge 1} \frac{(\lambda u)^i}{i!} B_i(x) \Big\} \times \Big\{ \sum_{j \ge 0} \frac{u^j}{j!} E_j(y) \Big\} = \frac{1}{\lambda u} \Big\{ \frac{\lambda u e^{\lambda u x}}{e^{\lambda u} - 1} - 1 \Big\} \frac{2e^{uy}}{e^u+1}. \quad (11)$$

$\boxed{\lambda = 1}$ The generating function displayed in (11) can be stated as

$$\frac{1}{u} \Big\{ \frac{u e^{ux}}{e^u - 1} - 1 \Big\} \frac{2e^{uy}}{e^u+1} = \frac{2e^{u(x+y)}}{e^{2u}-1} - \frac{2e^{uy}}{u(e^u+1)}.$$

Extracting the coefficient of $\frac{u^m}{m!}$ across this equation and then appealing to (3), we derive the following convolution identity.

Theorem 19.

$$\sum_{k=0}^{m} \binom{m}{k} \frac{B_{k+1}(x)}{k+1} E_{m-k}(y) = \tfrac{B_{m+1}(x+y)}{m+1} - \tfrac{1}{2} E_m(x+y) - \tfrac{E_{m+1}(y)}{m+1}.$$

$\boxed{\lambda = -1}$ Replacing x by $1-x$ in Theorem 19 and then applying (1) and (2), we can reformulate the corresponding identity as follows.

Corollary 20.

$$\sum_{k=0}^{m}(-1)^k\binom{m}{k}\frac{B_{k+1}(x)}{k+1}E_{m-k}(y)=\tfrac{E_{m+1}(y)}{m+1}-\tfrac{B_{m+1}(y-x)}{m+1}-\tfrac{1}{2}E_m(y-x).$$

$\boxed{\lambda=2}$ The generating function displayed in (11) can be expressed as

$$\frac{1}{2u}\left\{\frac{2ue^{2ux}}{e^{2u}-1}-1\right\}\frac{2e^{uy}}{e^u+1}=\frac{e^{u(2x+y)}}{e^{2u}-1}-(2x+y-1)\frac{e^{u(2x+y-1)}}{e^u+1}$$
$$+\frac{d}{du}\frac{e^{u(2x+y-1)}}{e^u+1}-\frac{e^{uy}}{u(e^u+1)}.$$

Extracting the coefficient of $\frac{u^m}{m!}$ across this equation and then appealing to (2), we derive the following convolution formula.

Theorem 21.

$$\sum_{k=0}^{m}2^k\binom{m}{k}\frac{B_{k+1}(x)}{k+1}E_{m-k}(y)=\tfrac{1}{2}\Big\{\tfrac{B_{m+1}(2x+y)}{m+1}-E_{m+1}(2x+y)$$
$$+(2x+y-\tfrac{3}{2})E_m(2x+y)-\tfrac{E_{m+1}(y)}{m+1}\Big\}.$$

$\boxed{\lambda=1/2}$ The generating function displayed in (11) can be expressed as the partial fractions

$$\frac{2e^{uy}}{e^u+1}\times\frac{2}{u}\left\{\frac{ue^{\frac{ux}{2}}}{2(e^{u/2}-1)}-1\right\}=\frac{e^{\frac{u}{2}(x+2y)}}{e^{u/2}-1}-\frac{e^{u\frac{x+2y+1}{2}}}{e^u+1}-\frac{e^{u\frac{x+2y}{2}}}{e^u+1}-\frac{4e^{uy}}{u(e^u+1)}.$$

Extracting the coefficient of $\frac{u^m}{m!}$ across this equation, then interchanging x and y and finally simplifying the resulting equation times 2^m, we derive the following convolution identity.

Theorem 22.

$$\sum_{k=0}^{m}2^k\binom{m}{k}E_k(x)\frac{B_{m-k+1}(y)}{m-k+1}=\tfrac{B_{m+1}(2x+y)}{m+1}-2^{m-1}E_m(\tfrac{2x+y+1}{2})$$
$$-2^{m-1}E_m(\tfrac{2x+y}{2})-2^{m+1}\tfrac{E_{m+1}(x)}{m+1}.$$

2.4. $F_k(x)=E_k(x)$ **and** $G_k(y)=B_k(y)$**.** The exponential generating function corresponding to (5) is given below

$$\frac{\lambda^{-1}}{u}\left\{\sum_{i\geq1}\frac{(\lambda u)^i}{i!}E_i(x)\right\}\times\left\{\sum_{j\geq0}\frac{u^j}{j!}B_j(y)\right\}=\frac{1}{\lambda u}\left\{\frac{2e^{\lambda ux}}{e^{\lambda u}+1}-1\right\}\frac{ue^{uy}}{e^u-1}. \quad (12)$$

$\boxed{\lambda = 1}$ The generating function displayed in (12) can be expressed as

$$\frac{1}{u}\left\{\frac{2e^{ux}}{e^u+1}-1\right\}\frac{ue^{uy}}{e^u-1}=\frac{2e^{u(x+y)}}{e^{2u}-1}-\frac{e^{uy}}{e^u-1}.$$

Extracting the coefficient of $\frac{u^m}{m!}$ across this equation and then appealing to (3), we derive the following convolution identity.

Theorem 23.

$$\sum_{k=0}^{m}\binom{m}{k}\frac{E_{k+1}(x)}{k+1}B_{m-k}(y)=\tfrac{B_{m+1}(x+y)}{m+1}-\tfrac{1}{2}E_m(x+y)-\tfrac{B_{m+1}(y)}{m+1}.$$

$\boxed{\lambda = -1}$ Replacing x by $1-x$ in Theorem 23 and then utilizing (1) and (2), we can express the resulting equation as follows.

Corollary 24.

$$\sum_{k=0}^{m}(-1)^k\binom{m}{k}\frac{E_{k+1}(x)}{k+1}B_{m-k}(y)=-\tfrac{B_{m+1}(y-x)}{m+1}-\tfrac{1}{2}E_m(y-x)+\tfrac{B_{m+1}(y)}{m+1}.$$

$\boxed{\lambda = 2}$ The generating function displayed in (12) can be expressed as

$$\frac{1}{2u}\left\{\frac{2e^{2ux}}{e^{2u}+1}-1\right\}\frac{ue^{uy}}{e^u-1}=\frac{e^{u(2x+y)}}{2(e^u-1)}-\frac{e^{u(2x+y+1)}}{2(e^{2u}+1)}-\frac{e^{u(2x+y)}}{2(e^{2u}+1)}-\frac{e^{uy}}{2(e^u-1)}.$$

Extracting the coefficient of $\frac{u^m}{m!}$ across this equation and then appealing to (2), we derive the following convolution formula.

Theorem 25.

$$\sum_{k=0}^{m}2^k\binom{m}{k}\frac{E_{k+1}(x)}{k+1}B_{m-k}(y)=\tfrac{B_{m+1}(2x+y)}{2(m+1)}-2^{m-2}E_m(\tfrac{2x+y+1}{2})$$
$$-2^{m-2}E_m(\tfrac{2x+y}{2})-\tfrac{B_{m+1}(y)}{2(m+1)}.$$

$\boxed{\lambda = 1/2}$ The generating function displayed in (12) can be expressed as the partial fractions

$$\frac{ue^{uy}}{e^u-1}\times\frac{2}{u}\left\{\frac{2e^{\frac{ux}{2}}}{e^{u/2}+1}-1\right\}=\frac{2e^{\frac{u}{2}(x+2y)}}{e^u-1}-2(x+2y-1)\frac{e^{\frac{u}{2}(x+2y-1)}}{e^{u/2}+1}$$
$$-\frac{2e^{uy}}{e^u-1}+\frac{4d}{du}\frac{e^{\frac{u}{2}(x+2y-1)}}{e^{u/2}+1}.$$

Extracting the coefficient of $\frac{u^m}{m!}$ across this equation, then interchanging x and y and finally simplifying the resulting equation through (2) and (3), we derive the following convolution identity.

Theorem 26.

$$\sum_{k=0}^{m} 2^k \binom{m}{k} B_k(x) \frac{E_{m-k+1}(y)}{m-k+1} = \tfrac{B_{m+1}(2x+y)}{m+1} - \tfrac{B_{m+1}(2x)}{m+1} - E_{m+1}(2x+y) + (2x+y-\tfrac{3}{2})E_m(2x+y) + \tfrac{E_m(2x)}{2}.$$

3. $\alpha = \gamma = 1$

In this section, we examine the convolution sums $\Omega_m\big(x,y\big|\,{}^{\alpha,\gamma}_{F,G}\big)$ defined in (4), when $\alpha=\gamma=1$ and $\{F,G\}$ are specified by Bernoulli and Euler polynomials.

3.1. $F_k(x) = B_k(x)$ and $G_k(y) = B_k(y)$.

The exponential generating function corresponding to (5) becomes

$$\frac{\lambda^{-1}}{u^2}\Big\{\sum_{i\ge1}\frac{(\lambda u)^i}{i!}B_i(x)\Big\}\times\Big\{\sum_{j\ge1}\frac{u^j}{j!}B_j(y)\Big\} = \frac{1}{\lambda u^2}\Big\{\frac{\lambda u e^{\lambda u x}}{e^{\lambda u}-1}-1\Big\}\times\Big\{\frac{ue^{uy}}{e^u-1}-1\Big\}. \tag{13}$$

$\boxed{\lambda=1}$ The generating function displayed in (13) can be expressed as

$$\frac{1}{u^2}\Big\{\frac{ue^{ux}}{e^u-1}-1\Big\}\times\Big\{\frac{ue^{uy}}{e^u-1}-1\Big\} = (x+y-1)\frac{e^{u(x+y-1)}}{e^u-1} - \frac{d}{du}\frac{e^{u(x+y-1)}}{e^u-1} - \frac{e^{ux}}{u(e^u-1)} - \frac{e^{uy}}{u(e^u-1)} + \frac{1}{u^2}.$$

Extracting the coefficient of $\frac{u^m}{m!}$ across this equation and then appealing to (2), we derive the following convolution identity.

Theorem 27.

$$\sum_{k=0}^{m}\binom{m}{k}\frac{B_{k+1}(x)}{k+1}\frac{B_{m-k+1}(y)}{m-k+1} = (x+y-1)\tfrac{B_{m+1}(x+y)}{m+1} - \tfrac{B_{m+2}(x+y)}{m+2} - \tfrac{B_{m+2}(x)}{(m+1)(m+2)} - \tfrac{B_{m+2}(y)}{(m+1)(m+2)}.$$

$\boxed{\lambda=-1}$ Replacing x by $1-x$ in Theorem 27 and then applying (1) and (2), we can restate the resulting equation as the following identity.

Corollary 28.

$$\sum_{k=0}^{m}(-1)^k\binom{m}{k}\frac{B_{k+1}(x)}{k+1}\frac{B_{m-k+1}(y)}{m-k+1} = (x-y)\tfrac{B_{m+1}(y-x)}{m+1} + \tfrac{B_{m+2}(y-x)}{m+2} + \tfrac{B_{m+2}(1-x)}{(m+1)(m+2)} + \tfrac{B_{m+2}(y)}{(m+1)(m+2)}.$$

$\boxed{\lambda = 2}$ The generating function displayed in (13) can be expressed as

$$\frac{1}{2u^2}\left\{\frac{2ue^{2ux}}{e^{2u}-1}-1\right\}\times\left\{\frac{ue^{uy}}{e^u-1}-1\right\}=\frac{1}{2u^2}-\frac{e^{u(2x+y)}}{2(e^{2u}-1)}$$
$$+(2x+y-1)\frac{e^{u(2x+y-1)}}{2(e^u-1)}-\frac{e^{2ux}}{u(e^{2u}-1)}-\frac{e^{uy}}{2u(e^u-1)}-\frac{d}{2du}\frac{e^{u(2x+y-1)}}{e^u-1}.$$

Extracting the coefficient of $\frac{u^m}{m!}$ across this equation and then appealing to (2) and (3), we derive the following convolution formula.

Theorem 29.

$$\sum_{k=0}^{m}2^k\binom{m}{k}\frac{B_{k+1}(x)}{k+1}\frac{B_{m-k+1}(y)}{m-k+1}=\tfrac{E_m(2x+y)}{8}+(2x+y-\tfrac{3}{2})\tfrac{B_{m+1}(2x+y)}{2(m+1)}$$
$$-\tfrac{B_{m+2}(2x+y)}{2(m+2)}-\tfrac{B_{m+2}(2x)}{2(m+1)(m+2)}+\tfrac{E_{m+1}(2x)}{4(m+1)}-\tfrac{B_{m+2}(y)}{2(m+1)(m+2)}.$$

3.2. $F_k(x) = E_k(x)$ **and** $G_k(y) = E_k(y)$**.** The exponential generating function corresponding to (5) reads as

$$\frac{\lambda^{-1}}{u^2}\left\{\sum_{i\geq 1}\frac{(\lambda u)^i}{i!}E_i(x)\right\}\times\left\{\sum_{j\geq 1}\frac{u^j}{j!}E_j(y)\right\}=\frac{1}{\lambda u^2}\left\{\frac{2e^{\lambda ux}}{e^{\lambda u}+1}-1\right\}\times\left\{\frac{2e^{uy}}{e^u+1}-1\right\}. \tag{14}$$

$\boxed{\lambda = 1}$ The generating function displayed in (14) can be expressed as

$$\frac{1}{u^2}\left\{\frac{2e^{ux}}{e^u+1}-1\right\}\times\left\{\frac{2e^{uy}}{e^u+1}-1\right\}=4(x+y-1)\frac{e^{u(x+y-1)}}{u^2(e^u+1)}$$
$$-\frac{4d}{u^2du}\frac{e^{u(x+y-1)}}{e^u+1}-\frac{2e^{ux}}{u^2(e^u+1)}-\frac{2e^{uy}}{u^2(e^u+1)}+\frac{1}{u^2}.$$

Extracting the coefficient of $\frac{u^m}{m!}$ across this equation and then appealing to (2), we derive the following convolution identity.

Theorem 30.

$$\sum_{k=0}^{m}\binom{m}{k}\frac{E_{k+1}(x)}{k+1}\frac{E_{m-k+1}(y)}{m-k+1}=2(1-x-y)\tfrac{E_{m+2}(x+y)}{(m+1)(m+2)}+2\tfrac{E_{m+3}(x+y)}{(m+1)(m+2)}$$
$$-\tfrac{E_{m+2}(x)}{(m+1)(m+2)}-\tfrac{E_{m+2}(y)}{(m+1)(m+2)}.$$

$\boxed{\lambda = -1}$ Replacing x by $1-x$ in Theorem 30 and then using (1) and (2), we obtain the following formula.

Corollary 31.

$$\sum_{k=0}^{m}(-1)^k\binom{m}{k}\frac{E_{k+1}(x)}{k+1}\frac{E_{m-k+1}(y)}{m-k+1}=2(x-y)\tfrac{E_{m+2}(y-x)}{(m+1)(m+2)}+2\tfrac{E_{m+3}(y-x)}{(m+1)(m+2)}+\tfrac{E_{m+2}(1-x)}{(m+1)(m+2)}+\tfrac{E_{m+2}(y)}{(m+1)(m+2)}.$$

$\boxed{\lambda=2}$ The generating function displayed in (14) can be expressed as

$$\frac{1}{2u^2}\Big\{\frac{2e^{2ux}}{e^{2u}+1}-1\Big\}\times\Big\{\frac{2e^{uy}}{e^u+1}-1\Big\}=\frac{e^{u(2x+y)}}{u^2(e^u+1)}-\frac{e^{u(2x+y+1)}}{u^2(e^{2u}+1)}+\frac{e^{u(2x+y)}}{u^2(e^{2u}+1)}-\frac{e^{2ux}}{u^2(e^{2u}+1)}-\frac{e^{uy}}{u^2(e^u+1)}+\frac{1}{2u^2}.$$

Extracting the coefficient of $\frac{u^m}{m!}$ across this equation and then appealing to (2), we derive the following convolution formula.

Theorem 32.

$$\sum_{k=0}^{m}2^k\binom{m}{k}\frac{E_{k+1}(x)}{k+1}\frac{E_{m-k+1}(y)}{m-k+1}=\tfrac{1}{2(m+1)(m+2)}E_{m+2}(2x+y)-\tfrac{2^{m+1}}{(m+1)(m+2)}E_{m+2}(\tfrac{2x+y+1}{2})+\tfrac{2^{m+1}}{(m+1)(m+2)}E_{m+2}(\tfrac{2x+y}{2})-\tfrac{2^{m+1}}{(m+1)(m+2)}E_{m+2}(x)-\tfrac{1}{2(m+1)(m+2)}E_{m+2}(y).$$

3.3. $F_k(x)=B_k(x)$ **and** $G_k(y)=E_k(y)$**.** The exponential generating function corresponding to (5) is given by

$$\frac{\lambda^{-1}}{u^2}\Big\{\sum_{i\ge1}\frac{(\lambda u)^i}{i!}B_i(x)\Big\}\times\Big\{\sum_{j\ge1}\frac{u^j}{j!}E_j(y)\Big\}=\frac{1}{\lambda u^2}\Big\{\frac{\lambda ue^{\lambda ux}}{e^{\lambda u}-1}-1\Big\}\times\Big\{\frac{2e^{uy}}{e^u+1}-1\Big\}. \tag{15}$$

$\boxed{\lambda=1}$ The generating function displayed in (15) can be expressed as

$$\frac{1}{u^2}\Big\{\frac{ue^{ux}}{e^u-1}-1\Big\}\times\Big\{\frac{2e^{uy}}{e^u+1}-1\Big\}=\frac{2e^{u(x+y)}}{u(e^{2u}-1)}-\frac{e^{ux}}{u(e^u-1)}-\frac{2e^{uy}}{u^2(e^u+1)}+\frac{1}{u^2}.$$

Extracting the coefficient of $\frac{u^m}{m!}$ across this equation and then appealing to (3), we derive the following convolution identity.

Theorem 33.

$$\sum_{k=0}^{m}\binom{m}{k}\frac{B_{k+1}(x)}{k+1}\frac{E_{m-k+1}(y)}{m-k+1}=\tfrac{B_{m+2}(x+y)}{(m+1)(m+2)}-\tfrac{E_{m+1}(x+y)}{2(m+1)}-\tfrac{B_{m+2}(x)}{(m+1)(m+2)}-\tfrac{E_{m+2}(y)}{(m+1)(m+2)}.$$

$\boxed{\lambda = -1}$ Replacing x by $1-x$ in Theorem 33 and then invoking (1) and (2), we can reformulate the resulting equation as the following formula.

Corollary 34.

$$\sum_{k=0}^{m}(-1)^k\binom{m}{k}\frac{B_{k+1}(x)}{k+1}\frac{E_{m-k+1}(y)}{m-k+1} = -\tfrac{B_{m+2}(y-x)}{(m+1)(m+2)} - \tfrac{E_{m+1}(y-x)}{2(m+1)} + \tfrac{B_{m+2}(1-x)}{(m+1)(m+2)} + \tfrac{E_{m+2}(y)}{(m+1)(m+2)}.$$

$\boxed{\lambda = 2}$ The generating function displayed in (15) can be expressed as

$$\frac{1}{2u^2}\left\{\frac{2ue^{2ux}}{e^{2u}-1}-1\right\}\times\left\{\frac{2e^{uy}}{e^u+1}-1\right\} = \frac{e^{u(2x+y)}}{u(e^{2u}-1)} - (2x+y-1)\frac{e^{u(2x+y-1)}}{u(e^u+1)} + \frac{d}{udu}\frac{e^{u(2x+y-1)}}{e^u+1} - \frac{e^{2ux}}{u(e^{2u}-1)} - \frac{e^{uy}}{u^2(e^u+1)} + \frac{1}{2u^2}.$$

Extracting the coefficient of $\frac{u^m}{m!}$ across this equation and then appealing to (2) and (3), we derive the following convolution formula.

Theorem 35.

$$\sum_{k=0}^{m}2^k\binom{m}{k}\frac{B_{k+1}(x)}{k+1}\frac{E_{m-k+1}(y)}{m-k+1} = \tfrac{B_{m+2}(2x+y)}{2(m+1)(m+2)} + (2x+y-\tfrac{3}{2})\tfrac{E_{m+1}(2x+y)}{2(m+1)} - \tfrac{E_{m+2}(2x+y)}{2(m+1)} - \tfrac{B_{m+2}(2x)}{2(m+1)(m+2)} + \tfrac{E_{m+1}(2x)}{4(m+1)} - \tfrac{E_{m+2}(y)}{2(m+1)(m+2)}.$$

$\boxed{\lambda = 1/2}$ The generating function displayed in (15) can be expressed as the partial fractions

$$\left\{\frac{2e^{uy}}{e^u+1}-1\right\}\times\frac{2}{u^2}\left\{\frac{ue^{\frac{ux}{2}}}{2(e^{u/2}-1)}-1\right\} = \frac{2}{u^2} + \frac{e^{\frac{u}{2}(x+2y)}}{u(e^{u/2}-1)} - \frac{e^{\frac{u}{2}x}}{u(e^{u/2}-1)} - \frac{e^{u\frac{x+2y}{2}}}{u(e^u+1)} - \frac{e^{u\frac{x+2y+1}{2}}}{u(e^u+1)} - \frac{4e^{uy}}{u^2(e^u+1)}.$$

Extracting the coefficient of $\frac{u^m}{m!}$ across this equation, then interchanging x and y, we derive the following convolution identity.

Theorem 36.

$$\sum_{k=0}^{m} 2^k \binom{m}{k} \frac{E_{k+1}(x)}{k+1} \frac{B_{m-k+1}(y)}{m-k+1} = \tfrac{B_{m+2}(2x+y)}{2(m+1)(m+2)} - \tfrac{B_{m+2}(y)}{2(m+1)(m+2)}$$
$$- \tfrac{2^{m+1}E_{m+2}(x)}{(m+1)(m+2)} - \tfrac{2^{m-1}}{m+1}\Big\{E_{m+1}(\tfrac{2x+y}{2}) + E_{m+1}(\tfrac{2x+y+1}{2})\Big\}.$$

By manipulating the generating function equation (5), one can derive other convolution formulae for $\Omega_m\big(x,y\big|\,{}^{\alpha,\gamma}_{F,G}\big)$ with different α,γ and λ values. Because the resulting convolution identities are generally quite complicated, we confine ourselves to presenting only two further summation formulae which may serve to show the existence of similar convolution sums.

$\boxed{\alpha = 1,\ \gamma = 2,\ \lambda = 1 \text{ and } F_k(x) = G_k(x) = B_k(x)}$

$$\sum_{k=0}^{m} \binom{m}{k} \frac{B_{k+1}(x)}{k+1} \frac{B_{m-k+2}(y)}{(m-k+1)_2} = (x+y-1)\tfrac{B_{m+2}(x+y)}{(m+1)(m+2)} - \tfrac{B_{m+3}(x+y)}{(m+1)(m+3)}$$
$$-(y-\tfrac{1}{2})\tfrac{B_{m+2}(x)}{(m+1)(m+2)} - \tfrac{B_{m+3}(x)}{(m+1)(m+2)(m+3)} - \tfrac{B_{m+3}(y)}{(m+1)(m+2)(m+3)}.$$

$\boxed{\alpha = 1,\ \gamma = 0,\ \lambda = 3 \text{ and } F_k(y) = G_k(y) = E_k(y)}$

$$\sum_{k=0}^{m} 3^k \binom{m}{k} \frac{E_{k+1}(x)}{k+1} E_{m-k}(y) = \tfrac{2\cdot 3^{m-1}}{m+1}(3x+y-1)E_{m+1}(\tfrac{3x+y-1}{3})$$
$$- \tfrac{2\cdot 3^m}{m+1}E_{m+2}(\tfrac{3x+y-1}{3}) - \tfrac{2\cdot 3^{m-1}}{m+1}(3x+y-2)E_{m+1}(\tfrac{3x+y-2}{3}) + \tfrac{2\cdot 3^m}{m+1}E_{m+2}(\tfrac{3x+y-2}{3})$$
$$- \tfrac{2\cdot 3^{m-1}}{m+1}(3x+y-3)E_{m+1}(\tfrac{3x+y}{3}) + \tfrac{2\cdot 3^m}{m+1}E_{m+2}(\tfrac{3x+y}{3}) - \tfrac{E_{m+1}(y)}{3(m+1)}.$$

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