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The q-Stirling numbers, continued fractions and the q-Charlier and q-Laguerre polynomials

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Abstract

We give a simple proof of the continued fraction expansions of the ordinary generating functions of the q-Stirling numbers of both kinds. By generalizing the method of Touchard (1956) and Milne (1978), we obtain the explicit formulas and measure of one set of the polynomials whose moments are the q-Stirling numbers.

Keywords: q-Stirling numbers; Continued fractions

1. Introduction

For $q \in \mathbb{R}$ such that $|q| < 1$ define

$$[x] = \frac{q^x - 1}{q - 1}.$$

For $n \geqslant 0$ let $[x]_0 = 1$, $[x]_n = [x]\ [x-1] \cdots [x-n+1]$ and $[n]! = [n]_n$. Also for $n \geqslant k \geqslant 0$ define

$$\begin{bmatrix} n \\ k \end{bmatrix} = \frac{[n]!}{[k]![n-k]!}.$$

Recall that the two related q-Stirling numbers of the second kind [8, 17] may be defined by recurrence:

$$\begin{aligned} &S_q(n+1, k+1) = S_q(n,k) + [k+1] S_q(n, k+1), \\ &S_q(0,k) = \delta_{0k}, \qquad S_q(n,0) = \delta_{n0} \end{aligned} \tag{1.1}$$

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and

$$\begin{aligned} &\tilde{S}_q(n+1,k+1) = q^{k-1}\tilde{S}_q(n,k) + [k+1]\tilde{S}_q(n,k+1), \\ &\tilde{S}_q(0,k) = \delta_{0k}, \qquad \tilde{S}_q(n,0) = \delta_{n0}. \end{aligned} \tag{1.2}$$

Note that $\tilde{S}_q(n,k) = q^{\binom{k}{2}}S_q(n,k)$.

Similarly, the q-Stirling numbers of the first kind $c_q(n,k)$ [8] can be defined by

$$\begin{aligned} &c_q(n+1,k+1) = c_q(n,k) + [n]c_q(n,k+1), \\ &c_q(0,k) = \delta_{0k}, \qquad c_q(n,0) = \delta_{n0}. \end{aligned} \tag{1.3}$$

For convenience, we shall take $s_q(n,k) = q^{n-k}c_q(n,k)$ as our q-Stirling numbers of the first kind.

Our first purpose is to give a short proof of the continued fraction expansions of the ordinary generating functions of the forementioned three q-Stirling numbers.

Theorem 1. *In the ring of formal power series of x, the following identities hold*:

$$1 + \sum_{n\geqslant 1}\sum_{k=1}^{n} S_q[n,k]a^k x^n = \cfrac{1}{1 - ax - \cfrac{aqx^2}{\cfrac{\cdots}{1 - b_k x - \cfrac{\lambda_{k+1}x^2}{\cdots}}}}, \tag{1.4}$$

where $b_k = aq^k + [k]$, $\lambda_{k+1} = aq^{k+1}[k+1]$ *for* $k \geqslant 0$;

$$1 + \sum_{n\geqslant 1}\sum_{k=1}^{n} \tilde{S}_q[n,k]a^k x^n = \cfrac{1}{1 - ax - \cfrac{a(1+a(q-1))x^2}{\cfrac{\cdots}{1 - b_k x - \cfrac{\lambda_{k+1}x^2}{\cdots}}}}, \tag{1.5}$$

where $b_k = aq^{2k} + [k](1 + aq^{k-1}(q-1))$, $\lambda_{k+1} = aq^{2k}[k+1](1 + aq^k(q-1))$ *for* $k \geqslant 0$;

$$1 + \sum_{n\geqslant 1}\sum_{k=1}^{n} S_q[n,k]a^k x^n = \cfrac{1}{1 - ax - \cfrac{aqx^2}{\cfrac{\cdots}{1 - b_k x - \cfrac{\lambda_{k+1}x^2}{\cdots}}}}, \tag{1.6}$$

where $b_k = (a + q[k])q^k + q^k[k]$, $\lambda_{k+1} = (a + q[k])[k+1]q^{2k+1}$ *for* $k \geqslant 0$.

We should point out that all the above formulas are valid only in the *formal* sense (cf. [5]). Actually the Hamburger moment problems associated with the orthogonal polynomials corresponding to the above continued fractions are not always determinate, see for example [2, pp. 197–198] or [10]. We refer the reader to [2, pp. 6–10] for the formal definition of orthogonality.

It is well known [2, p. 85] that Theorem 1 can be restated in terms of orthogonal polynomials as follows.

Theorem 2. (a) *The polynomials $U_n^{(a)}(x;q)$ orthogonal to the moments $\mu_n^{(1)}(q)$,*

$$\mu_n^{(1)}(q) = \sum_{k=1}^{n} S_q[n,k]a^k, \tag{1.7}$$

are defined by $U_0^{(a)}(x;q) = 1$, $U_1^{(a)}(x;q) = x - a$ and for $n \geqslant 1$ by

$$U_{n+1}^{(a)}(x;q) = (x - aq^n - [n])U_n^{(a)}(x;q) - aq^{n-1}[n]U_{n-1}^{(a)}(x;q).$$

(b) *The polynomials $V_n^{(a)}(x;q)$ orthogonal to the moments $\mu_n^{(2)}(q)$,*

$$\mu_n^{(2)}(q) = \sum_{k=1}^{n} \tilde{S}_q[n,k]a^k, \tag{1.8}$$

are defined by $V_0^{(a)}(x;q) = 1$, $V_1^{(a)}(x;q) = x - a$ and for $n \geqslant 1$ by

$$V_{n+1}^{(a)}(x;q) = (x - b_n)V_n^{(a)}(x;q) - \lambda_n V_{n-1}^{(a)}(x;q),$$

where $b_n = aq^{2n} + [n](1 + aq^{n-1}(q-1))$, and $\lambda_{n+1} = aq^{2n}[n+1](1 + aq^n(q-1))$.

(c) *The polynomials $W_n^{(a)}(x;q)$ orthogonal to the moments $v_n(q)$,*

$$v_n(q) = \sum_{k=1}^{n} s_q(n,k)a^k, \tag{1.9}$$

are defined by $W_0^{(a)}(x;q) = 1$, $W_1^{(a)}(x;q) = x - a$ and for $n \geqslant 1$ by

$$W_{n+1}^{(a)}(x;q) = (x - b_n)W_n^{(a)}(x;q) - \lambda_n W_{n-1}^{(a)}(x;q),$$

where $b_n = (a + q + \cdots + q^n)q^n + q^n[n]$, and $\lambda_n = (a + q + \cdots + q^{n-1})[n]q^{2n-1}$.

We note that if $q = 1$, the polynomials in (a) and (b) reduce to the Charlier polynomials, and the polynomials in (c) reduce to the Laguerre polynomials [2]. The polynomials in (a) and (b) can then be regarded as two q-analogs of the Charlier polynomials, while the polynomials in (c) as q-analogs of the Laguerre polynomials.

Conversely, if we first establish Theorem 2, we automatically get Theorem 1. Actually, Ismail and Stanton [14] have earlier noticed that the polynomials in part (a) are a rescaled version of the Al-Salam–Carlitz polynomials [2, pp. 197–198]. So we can also prove part (a) by using the moments of the latter polynomials.

Parts (b) and (c) have been presented by the author at the 27th session of the Séminaire Lotharingien in 1991. Part (b) was proved by using the methods developed in this paper, while part (c) was derived from a more general result in [18].

Recently, De Médicis and Viennot [3] have given a bijective proof of part (c) and noticed that the polynomials in part (c) are a special case of the "little" q-Jacobi polynomials introduced by Hahn (see [6, p. 166]). Hence part (c) can also be derived from the known measure of the "little q-Jacobi" polynomials as the q-Charlier polynomials. Finally, after seeing an earlier version of this paper, Stanton [14] informed us that the polynomials in (b) are a rescaled version of the classical

q-Charlier polynomials (see [6, p. 187]). So part (b) can also be derived from the explicit measure of the classical q-Charlier polynomials.

In contrast with the other proofs, the three expansions of Theorem 1 are proved from scratch by means of the same method, inspired by the work of Rogers [13]. The continued fraction method used to proving Theorem 2 has the merit to be short and elementary.

In Section 2 we shall first prove Theorems 1 and 2 from scratch. In Section 3 we comment on the combinatorial interpretations of the q-Stirling numbers. In Section 4 we give an explicit formula and measure of the polynomials $V_n^{(a)}(x;q)$ by generalizing the method of Touchard [15] and Milne [11]. These polynomials turn out to be a rescaled version of the classical q-Charlier polynomials [6, p. 187].

2. Continued fractions expansions

We first expand the ordinary generating functions of the q-Stirling numbers as Stieltjes continued fractions.

Lemma 3. *The following identities hold:*

$$1+\sum_{n,k\geqslant 1} S_q(n,k)a^k x^n = \cfrac{1}{1-\cfrac{a\cdot x}{1-\cfrac{1\cdot x}{\cfrac{\cdots}{1-\cfrac{\lambda_n\cdot x}{\cdots}}}}}, \tag{2.1}$$

where $\lambda_{2n-1}=aq^{n-1}$, $\lambda_{2n}=[n]$ *for* $n\geqslant 1$;

$$1+\sum_{n,k\geqslant 1} \tilde{S}_q(n,k)a^k x^n = \cfrac{1}{1-\cfrac{a\cdot x}{1-\cfrac{(1+a(q-1))\cdot x}{\cfrac{\cdots}{1-\cfrac{\lambda_n\cdot x}{\cdots}}}}}, \tag{2.2}$$

where $\lambda_{2n-1}=aq^{2n}$, $\lambda_{2n}=[n](1+aq^{n-1}(q-1))$ *for* $n\geqslant 1$;

$$1+\sum_{n,k\geqslant 1} s_q(n,k)a^k x^n = \cfrac{1}{1-\cfrac{a\cdot x}{1-\cfrac{q\cdot x}{\cfrac{\cdots}{1-\cfrac{\lambda_n\cdot x}{\cdots}}}}}, \tag{2.3}$$

where $\lambda_{2n-1}=(a+q+\cdots+q^{n-1})q^{n-1}$, $\lambda_{2n}=[n]\,q^n$ *for* $n\geqslant 1$.

Proof. Let $f(a,x)$ be the left-hand side of (2.1). It then follows from (1.1) that

$$f(a,x) = \sum_{k \geqslant 0} \frac{a^k x^k}{(1-[1]x)(1-[2]x)\cdots(1-[k]x)} = 1 + \frac{ax}{1-x} f\left(\frac{a}{q}, \frac{qx}{1-x}\right). \tag{2.4}$$

Assume that

$$f(a,x) = \cfrac{1}{1 - \cfrac{c_1(a)\,x}{1 - \cfrac{c_2(a)\,x}{1 - \cfrac{c_3(a)\,x}{\cdots}}}}. \tag{2.5}$$

Contracting the continued fraction (2.5) starting from the first row and the second row yields, respectively,

$$f(a,x) = 1 + \cfrac{c_1(a)x}{1 - (c_1(a) + c_2(a))x - \cfrac{c_2(a)c_3(a)\,x^2}{1 - (c_3(a) + c_4(a))\,x - \cfrac{c_4(a)c_5(a)\;x^2}{\cdots}}} \tag{2.6}$$

and

$$f(a,x) = \cfrac{1}{1 - c_1(a)x - \cfrac{c_1(a)c_2(a)\,x^2}{1 - (c_2(a) + c_3(a))\,x - \cfrac{c_3(a)c_4(a)\;x^2}{\cdots}}}. \tag{2.7}$$

Therefore

$$\frac{ax}{1-x} f\left(a, \frac{qx}{1-x}\right) = \cfrac{ax}{1 - (1 + qc_1(a))x - \cfrac{c_1(a)c_2(a)\,q^2 x^2}{1 - (1 + qc_2(a) + qc_3(a))\,x \cfrac{c_3(a)c_4(a)\,q^2\,x^2}{\cdots}}}. \tag{2.8}$$

Now substitute (2.6) and (2.8) in the functional equation (2.4) and identify the corresponding terms. We successively obtain that

$$\text{(I)}\qquad \begin{aligned} &c_1(a) = a, \\ &c_1(a) + c_2(a) = 1 + qc_1(a/q) \;\Rightarrow\; c_2(a) = 1, \\ &c_2(a)c_3(a) = c_1(a/q)c_2(a/q)q^2 \;\Rightarrow\; c_3(a) = aq, \\ &c_3(a) + c_4(a) = 1 + qc_2(a/q) + qc_3(a/q) \;\Rightarrow\; c_4(a) = 1 + q, \\ &\cdots \qquad\qquad\qquad\qquad\qquad\qquad \cdots \end{aligned}$$

One can show by induction

$$c_{2n-1}(a) = aq^n, \quad c_{2n}(a) = [n], \qquad n \geqslant 1. \tag{2.9}$$

Putting the above values in (2.5) yields formula (2.1). Now let $g(a,x)$ be the left-hand side of (2.2). It then follows from (1.2) that

$$\begin{aligned} g(a,x) &= \sum_{k\geqslant 0} \frac{a^k q^{\binom{k}{2}} x^k}{(1-[1]x)(1-[2]x)\cdots(1-[k]x)} \\ &= 1 + \frac{ax}{1-x} g\left(a, \frac{qx}{1-x}\right). \end{aligned} \tag{2.10}$$

If we replace $f(a,x)$ by $g(a,x)$ in (2.5)–(2.8), we get from (2.10) that

$$\text{(II)} \quad \begin{aligned} &c_1(a) = a, \\ &c_1(a) + c_2(a) = 1 + qc_1(a) \;\Rightarrow\; c_2(a) = 1 + a(q-1), \\ &c_2(a)c_3(a) = c_1(a)c_2(a/q)q^2 \;\Rightarrow\; c_3(a) = aq^2, \\ &c_3(a) + c_4(a) = 1 + qc_2(a) + qc_3(a) \;\Rightarrow\; c_4(a) = (1+q)(1+aq(q-1)), \\ &\cdots \qquad\qquad\qquad\qquad \cdots \end{aligned}$$

and more generally

$$c_{2n-1}(a) = aq^{2(n-1)}, \quad c_{2n}(a) = [n](1 + aq^{n-1}(q-1)), \qquad n \geqslant 1. \tag{2.11}$$

Finally, it follows from (1.3) that

$$\sum_{k=0}^{n} s_q(n,k)a^k = a(a+q)\cdots(a+q+\cdots+q^{n-1}).$$

Therefore, if we let $h(a,x)$ be the left-hand side of (2.3), we have

$$\begin{aligned} h(a,x) &= 1 + ax + a(a+q)x^2 + a(a+q)(a+q+q^2)x^3 + \cdots \\ &= 1 + ax(1 + (1+a/q)(qx) + (1+a/q)(1+a/q+q)(qx)^2) \\ &= 1 + axh(1+a/q, qx). \end{aligned} \tag{2.12}$$

Similarly, if we replace $f(a,x)$ by $h(a,x)$ in (2.5)–(2.7), we get from (2.12) that

$$\text{(III)} \quad \begin{aligned} &c_1(a) = a, \\ &c_1(a) + c_2(a) = c_1(1+a/q)q \;\Rightarrow\; c_2(a) = q, \\ &c_2(a)c_3(a) = c_1(1+a/q)c_2(1+a/q)q^2 \;\Rightarrow\; c_3(a) = (a+q)q, \\ &c_3(a) + c_4(a) = qc_2(1+a/q) + qc_3(1+a/q) \;\Rightarrow\; c_4(a) = (1+q)q^2, \\ &\cdots \qquad\qquad\qquad\qquad \cdots \end{aligned}$$

and more generally

$$c_{2n-1}(a) = (a + q + \cdots + q^{n-1})q^{n-1}, \quad c_{2n}(a) = [n]q^n, \qquad n \geqslant 1. \tag{2.13}$$

Thus we have completed the proof of the lemma. □

Remark. Rogers [13] seems to be the first to have used the "contracting" and "functional equation" techniques to derive continued fraction expansions of power series. In the case $q = 1$, Dumont [4] has proved Lemma 3 in a similar manner.

By contraction of the continued fractions in Lemma 3 (cf. (2.7)) we get immediately Theorem 1.

3. Remarks on the combinatorial interpretations of the moments

Let $\mathfrak{S}_n$ be the set of permutations of $\{1,2, \ldots ,n\}$. For any permutation $\sigma = \sigma(1)\sigma(2) \cdots \sigma(n)$, a *left–right maximum* is a $\sigma(i)$ such that $\sigma(i) > \sigma(j)$ for all $j < i$ and an *inversion* is a pair $(\sigma(i), \sigma(j))$ where $\sigma(i) > \sigma(j)$ for all pairs (i, j) such that $1 \leqslant i < j \leqslant n$. Denote by $\operatorname{lrm}\sigma$ and $\operatorname{inv}\sigma$ the numbers of the left–right maxima and inversions of σ. From the *inversion table* we immediately obtain

$$\sum_{\sigma \in \mathfrak{S}_n} a^{\operatorname{lrm}\sigma} q^{\operatorname{inv}\sigma} = a(a + q) \cdots (a + q + \cdots + q^{n-1}). \tag{3.1}$$

Thus from (1.9)

$$s_q(n, k) = \sum_{\sigma \in \mathfrak{S}_n(k)} q^{\operatorname{inv}\sigma}, \tag{3.2}$$

where $\mathfrak{S}_n(k)$ denotes the set of permutations of $\{1,2, \ldots ,n\}$ with k left–right maxima. The interpretation (3.1) seems to appear first in [7].

Let $\Pi_n(k)$ be the set of ordered partitions into k blocks of $\{1,2, \ldots ,n\}$, i.e., the blocks of each partition are arranged in increasing order of their minima. Let $\pi = (B_1, B_2, \ldots , B_k)$ be such a partition. An *inversion* of π is a pair (b_i, B_j) such that $b_i \in B_i$ where $i < j$ and $b_i > \min B_j$. A *dual inversion of* π is a pair (B_i, b_j) such that $b_j \in B_j$ where $i < j$ and $\min B_i < b_j$. Let $\operatorname{inv}\sigma$ and $\widetilde{\operatorname{inv}}\,\sigma$ be the number of inversions and dual inversions of σ. It is easy to see, by verifying the recursions (1.7) and (1.8), that

$$S_q(n, k) = \sum_{\pi \in \Pi_n(k)} q^{\operatorname{inv}\pi}, \tag{3.3}$$

$$\tilde{S}_q(n, k) = \sum_{\pi \in \Pi_n(k)} q^{\widetilde{\operatorname{inv}}\,\pi}. \tag{3.4}$$

These inv interpretations are due to Milne [12]. As pointed out by Wachs and White [17], but calling $\operatorname{inv}(\pi) = lb(\pi)$ and $\widetilde{\operatorname{inv}}\,\pi = ls(\pi)$, the above combinatorial interpretations of the q-Stirling numbers of the second kind are "easy" and there are also some "hard" statistics on the set of partitions, which also have the q-Stirling numbers of the second kind as their generating functions. However it is not easy to verify this fact. Stanton raised the question how this "hard" statistics could be the same as the easy ones, and Wachs and White [17] proved it by constructing an explicit bijection between these "hard" statistics and the easy ones. Once established Theorem 1, we can also derive this result as follows. According to a theorem due to Flajolet [5] we can rewrite the left-hand side of (1.4) as the generating functions of certain *Motzkin paths* with respect to some weights. This leads to the *rs* statistics of partitions of [17] via a classical bijection due to Flajolet [5], Françon and Viennot (see [16, II-14] or [5]). Similarly, a combinatorial interpretation of (1.6) in terms of Motzkin paths also leads to hard interpretations of the q-Stirling numbers of the first kind on permutations, see [3]. Note that one of the motivations of this paper is due to these "hard" statistics.

4. The classical q-analog of Charlier polynomials

The classical q-Charlier polynomials [6, p. 187] are defined by

$$C_n(x; a, q) = {}_2\varphi_1\left(\begin{matrix} q^{-n}, x \\ 0 \end{matrix}; q, -\frac{q^{n+1}}{a}\right) \tag{4.1}$$

and satisfy the orthogonality

$$\sum_{x=0}^{\infty} C_m(q^{-x}; a, q) C_n(q^{-x}; a, q) \frac{a^x}{(q;q)_x} q^{\binom{x}{2}}$$
$$= (-a; q)_\infty (-qa^{-1}; q)_n (q;q)_n q^{-n} \delta_{mn}. \tag{4.2}$$

Although it is possible to verify directly that the polynomials $V_n^{(a)}(x; q)$ are actually a rescaled version of $C_n(x; a, q)$ by checking the three terms recurrence satisfied by these polynomials, we prefer to give an alternative argument to derive naturally the explicit expression and measure from the q-Stirling numbers as Touchard [15] and Milne [11] did in some special cases.

We define the linear function φ on the vector space $\mathbb{C}[q^x]$ by

$$\varphi([x]^n) = \sum_{k=0}^{n} \tilde{S}_q(n,k) a^k. \tag{4.3}$$

It is easy to see that the q-Stirling numbers $\tilde{S}_q(n,k)$ satisfy

$$[x]^n = \sum_{k=0}^{n} \tilde{S}_q(n,k) [x]_k$$

and $\tilde{S}_q(n,n) = 1$. Since $\{[x]_n\}_{n \geqslant 0}$ and $\{[x]^n\}_{n \geqslant 0}$ are two bases of the vector space $\mathbb{C}[q^x]$, if we define

$$[x]_n = \sum_{k=0}^{n} s_q^*(n,k) [x]^k,$$

we should have

$$\sum_{k=0}^{n} \tilde{S}_q(n,k) s_q^*(k,m) = \delta_{mn} \quad \text{for } m, n \in \mathbb{N}.$$

It follows that

$$\varphi([x]_n) = \sum_{k=0}^{n} s_q^*(n,k)\,\varphi([x]^k) = \sum_{k=0}^{n} s_q^*(n,k) \sum_{l=0}^{k} S_q(k,l)a^l = a^n. \tag{4.4}$$

Recall that the two classical q-analogs of the exponential e^x [1,6] are defined by

$$e_q(x) = \sum_{k=0}^{\infty} \frac{x^k}{[k]!} \quad \text{and} \quad E_q(x) = \sum_{k=0}^{\infty} \frac{q^{\binom{k}{2}} x^k}{[k]!}.$$

Note that $(e_q(x))^{-1} = E_q(-x)$. Hence

$$\varphi([x]_n) = a^n = \frac{1}{e_q(a)} \sum_{k=0}^{\infty} \frac{a^{n+k}}{[k]!} = \frac{1}{e_q(a)} \sum_{k=0}^{\infty} \frac{[k]_n}{[k]!} a^k. \tag{4.5}$$

Since $\{[x]_n\}$ is a basis of $\mathbb{C}[q^x]$ and φ linear, we obtain the following result.

Proposition 4. *For any polynomial $P(x)$ of q^x, we have*

$$\varphi(P(x)) = \frac{1}{e_q(a)} \sum_{k=0}^{\infty} \frac{P(k)}{[k]!} a^k.$$

Setting $P(x) = [x]^n$ in Proposition 4 we get then a q-analog of Touchard's formula [15].

Corollary 5. *We have*

$$B_{q,n}(a) = \sum_{k=0}^{n} \tilde{S}_q(n,k)a^k = \frac{1}{e_q(a)} \sum_{k=0}^{\infty} \frac{[k]^n}{[k]!} a^k.$$

Note that the above formula generalizes Milne's q-analog of Dobinski's identity [11], which corresponds to the $a = 1$ case.

Lemma 6. *Let $P(x)$ be a polynomial of q^x and $k \geqslant 0$, then*

$$\varphi([x]_k P(x)) = a^k \varphi(P(x+k)).$$

Proof. We first remark that $\varphi([x][x-1]_n) = a^{n+1} = a\varphi([x]_n)$. So $\varphi([x]P(x-1)) = a\varphi(P(x))$ for any polynomial $P(x)$ of q^x. Therefore $\varphi([x]_k P(x)) = \varphi([x][x-1]_{k-1}P(x)) = a\varphi([x]_{k-1}P(x))$, and the proof is complete by induction. □

We need the following version of the q-binomial theorem (see [1, p. 225] for a combinatorial proof using vector spaces over a finite field).

$$\prod_{i=0}^{n-1} (X - Zq^i) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} \prod_{j=0}^{k-1} (Y - Zq^j) \prod_{l=0}^{n-k-1} (X - Yq^l). \tag{4.6}$$

Putting $X = q^x$, $Y = 1$ and $Z = 0$ in (4.6) yields

$$(q^x)^n = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} (q-1)^k q^{\binom{k}{2}} [x]_k. \tag{4.7}$$

Applying φ to (4.7) and then applying (4.6) with $X = 1$, $Y = 0$ and $Z = (q-1)a$ leads to

$$\varphi((q^x)^n) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} (q-1)^k q^{\binom{k}{2}} a^k = (a(q-1); q)_n. \tag{4.8}$$

As usual we define, for any function $f(x)$ of x, the shift operator E: $Ef(x) = f(x+1)$, the identity operator I: $If(x) = f(x)$ and the q-difference operator by means of

$$\Delta_q^0 f(x) = f(x), \qquad \Delta_q^{n+1} f(x) = (E - q^n I)\Delta_q^n f(x) = \Delta^n f(x+1) - q^n \Delta_q^n f(x).$$

Note that

$$\Delta_q^n f(x) = (E - q^{n-1} I)(E - q^{n-2} I) \cdots (E - I) f(x). \tag{4.9}$$

It follows from (4.6) that

$$\Delta_q^n f(x) = \sum_{k=0}^{n} (-1)^k \begin{bmatrix} n \\ k \end{bmatrix} q^{\binom{k}{2}} f(x+n-k). \tag{4.10}$$

We require the following easily verified formula:

$$\Delta_q^n [x]_m = \begin{cases} [m]_n [x]_{m-n} q^{n(x+n-m)} & \text{if } n \leqslant m, \\ 0 & \text{otherwise}. \end{cases} \tag{4.11}$$

Theorem 7. *Let*

$$H_n(q^x; a, q) = \sum_{k=0}^{n} (-a)^k \begin{bmatrix} n \\ k \end{bmatrix} q^{k(k-1)/2} [x]_{n-k}. \tag{4.12}$$

We have the orthogonality

$$\varphi(H_m(q^x; a, q) H_n(q^x; a, q)) = a^n [n]! (a(1-q); q)_n \delta_{mn}. \tag{4.13}$$

Proof. Assume that $m \leqslant n$, then (4.11) reduces to $\Delta_q^n [x]_m = [n]! q^{nx} \delta_{mn}$. By Lemma 6 and (4.10) we have

$$\begin{aligned} \varphi([x]_m H_n(q^x; a, q)) &= \sum_{k=0}^{n} (-a)^k \begin{bmatrix} n \\ k \end{bmatrix} q^{k(k-1)/2} a^{n-k} \varphi([x+n-k]_m) \\ &= a^n \varphi(\Delta_q^n [x]_m) = a^n [n]! \varphi(q^{nx}) \delta_{mn}. \end{aligned}$$

Therefore

$$\varphi(H_m(q^x;a,q)H_n(q^x;a,q)) = \varphi([x]_m H_n(q^x;a,q)) = a^n[n]!\,\varphi(q^{nx})\delta_{mn}.$$

The proof is complete in virtue of (4.8). □

Remark. The special cases of Theorem 7 have been proved, respectively, by Touchard [15] for $q = 1$ and Milne [11] for $a = 1$ by similar methods. Note that our right-hand side of (4.13) is simpler than that of Milne [11] even in the case $a = 1$.

In (4.12) if we set $z = [x]$ then we get the explicit expression for $V_n^{(a)}(x;q)$:

$$V_n^{(a)}(z;q) = q^{\binom{n}{2}} H_n((q-1)z+1;a,q) = \sum_{k=0}^{n} (-aq^{n-1})^{n-k} \begin{bmatrix} n \\ k \end{bmatrix} \prod_{i=0}^{k-1} (z - [i]). \tag{4.14}$$

The polynomials $H_n(q^x;a,q)$ may also be written in terms of hypergeometric functions as

$$H_n(q^x;a,q) = \frac{(q^{-x};q)_n}{(q-1)^n} q^{nx-\binom{n}{2}}\, {}_1\varphi_1\left(\begin{matrix} q^{-n} \\ q^{1-n+x} \end{matrix}; q, aq^n(1-q)\right). \tag{4.15}$$

If we replace q by $1/q$ and then q^{-x} by x in the above formula, we obtain the classical q-Charlier polynomials [6, p. 187], that is

$$H_n\left(x;\frac{a}{1-q},\frac{1}{q}\right) = \left(\frac{-a}{1-q}\right)^n q^{-\binom{n}{2}}\, {}_2\varphi_1\left(\begin{matrix} q^{-n}, x \\ 0 \end{matrix}; q, -\frac{q^{n+1}}{a}\right). \tag{4.16}$$

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