ON CONTINUED FRACTION EXPANSIONS WHOSE ELEMENTS ARE ALL ONES

GREGORY WULCZYN
Bucknell University, Lewisburg, Pennsylvania 17837

I. EVEN PERIOD EXPANSIONS

1. NUMBER THEORY REVIEW. Here is an example of an even continued fraction expansion of $\sqrt{D}$, D a non-square integer, with $D = 13$.

$$\sqrt{13} = 3+\sqrt{13}-3 = 3+\frac{\sqrt{13}+3}{4}$$

$$\frac{\sqrt{13}+3}{4} = 1+\frac{\sqrt{13}-1}{4} = 1+\frac{\sqrt{13}+1}{3}$$

$$\frac{\sqrt{13}+1}{3} = 1+\frac{\sqrt{13}-2}{3} = 1+\frac{\sqrt{13}+2}{3}$$

$$\frac{\sqrt{13}+2}{3} = 1+\frac{\sqrt{13}-1}{3} = 1+\frac{\sqrt{13}+1}{4}$$

$$\frac{\sqrt{13}+1}{4} = 1+\frac{\sqrt{13}-3}{4} = 1+\frac{\sqrt{13}+1}{1}$$

Hence $\sqrt{13} = \langle 3, 1, 1, 1, 1, 6 \rangle$ and the solution of the Pellian equations $x^2 - Dy^2 = d_i$ can be found from the table.

continued fraction elements c_i	3	1	1	1	1	6
signed denominators d_i	−4	3	−3	4	−1	
p convergents p_i	3	4	7	11	18	
q convergents q_i	1	1	2	3	5	

The q convergents are the Fibonacci numbers. The primitive solution of $x^2 - 13y^2 = -1$ is picked up from the half period. Thus

$$y = 1^2 + 2^2 = 5; \qquad x = 4 \times 1 + 7 \times 2 = 18.$$

In general for period $2r$,

$$y = q_r^2 + q_{r-1}^2 = q_{2r-1}; \qquad x = p_{r-1}q_{r-1} + p_r q_r = q_{2r-1}\,.$$

Also the representation of D as the sum of two squares can be found as

$$D = d_r^2 + (D - d_r^2) = d_r^2 + t^2\,,$$

where d_r is the middle denominator. Thus $13 = 3^2 + 2^2$. Finally for $D = 5$ (modulo 8), since a signed denominator is ± 4, the convergents under the -4 column are the coefficients of the cubic root of unity

$$\frac{3+\sqrt{13}}{2}$$

in the field $(1, \sqrt{13})$.

Since the period is even the x_0 of the quadratic congruence $x_0^2 \equiv -1 \pmod{13}$ is given by $x_0 \equiv x \equiv 18 \equiv 5$ (modulo 13).

2. FIBONACCI RELATIONS TO BE USED.

(a) $$(F_n, F_{n+1}) = 1.$$

(b) $$F_{2n}^2 + 1 = F_{2n-1}F_{2n+1}$$

(c) $$F_n^2 + F_{n+1}^2 = F_{2n+1}\,.$$

It may be noted that no odd Fibonacci number is ever divisible by a prime of the form $p = 4s + 3$ since from (b) $x^2 \equiv -1 \pmod p$ which is impossible.

3. EVEN VARIABLE DIFFERENCE TABLE: $D = m^2 + k$

		2n ones $n > 1$		
m	1	1 .... 1	1	$2m$
$-k$	———	———	-1	
m	$m+1$	$2m+1$	$mF_{2n+1} + F_{2n}$	
1	1	2	F_{2n+1}	

The supposition $(mF_{2n+1} + F_{2n})^2 - F_{2n+1}^2(m^2 + k) = -1$ leads to

$$2mF_{2n}F_{2n+1} + F_{2n}^2 - kF_{2n+1}^2 = -1$$

$$2mF_{2n}F_{2n+1} - kF_{2n+1}^2 = -(F_{2n}^2 + 1) = -F_{2n-1}F_{2n+1}$$

$$2mF_{2n} - kF_{2n+1} = F_{2n-1}$$

Recalling that $(F_n, F_{n+1}) = 1$ and that F_{3n} is always even this linear diophantine equation will have an infinite number of positive integer solutions for m and k unless $2n + 1 \equiv 0 \pmod 3$.

Example. $$D = m^2 + k, \qquad \sqrt{D} = \langle m, 1, 1, 1, 1, 1, 1, 2m \rangle$$

$$(13m + 8)^2 - 169(m^2 + k) = -1$$

$$16m - 13k = -5, \qquad k = m + \frac{3m+5}{13}$$

$$m = 7, \quad k = 7 + 2 = 9, \quad D = 58, \quad \sqrt{58} = \langle 7, 1, 1, 1, 1, 1, 1, 14 \rangle, \quad x^2 - 58y = -1$$

has primitive solution

$$x = 13m + 8 = 99, \quad y = 13.$$

$$m = 13 + 7 = 20, \quad k = 20 + 5 = 25, \quad D = 425, \quad \sqrt{425} = \langle 20, 1, 1, 1, 1, 1, 1, 40 \rangle, \quad x^2 - 425y^2 = -1$$

has primitive solution

$$x = 13m + 8 = 268, \quad y = 13.$$

In general if

$$D = 169m^2 - 140m + 29, \qquad \sqrt{D} = \langle 13m - 6, 1, 1, 1, 1, 1, 1, 26m - 12 \rangle$$

and the primitive solution of $x^2 - Dy^2 = -1$ is given by $x = 169m - 70$, $y = 13$.

II. ODD PERIOD EXPANSIONS

4. NUMBER THEORY REVIEW. Let $D = 135$

$$\sqrt{135} = 11 + \sqrt{135} - 11 = 11 + \frac{\sqrt{135} + 11}{14}$$

$$\frac{\sqrt{135} + 11}{14} = 1 + \frac{\sqrt{135} - 3}{14} = 1 + \frac{\sqrt{135} + 3}{9}$$

$$\frac{\sqrt{135} + 3}{9} = 1 + \frac{\sqrt{135} - 6}{9} = 1 + \frac{\sqrt{135} + 6}{11}$$

$$\frac{\sqrt{135} + 6}{11} = 1 + \frac{\sqrt{135} - 5}{11} = 1 + \frac{\sqrt{135} + 5}{10}$$

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$$\frac{\sqrt{135}+5}{10} = 1+\frac{\sqrt{135}-5}{10} = 1+\frac{\sqrt{135}+5}{11}$$

$$\frac{\sqrt{135}+5}{11} = 1+\frac{\sqrt{135}-6}{11} = 1+\frac{\sqrt{135}+6}{9}$$

$$\frac{\sqrt{135}+6}{9} = 1+\frac{\sqrt{135}-3}{9} = 1+\frac{\sqrt{135}+3}{14}$$

$$\frac{\sqrt{135}+3}{14} = 1+\frac{\sqrt{135}-11}{14} = 1+\sqrt{135}+11$$

$$\sqrt{135}+11 = 22$$

$$\sqrt{135} = \langle 11, 1, 1, 1, 1, 1, 1, 1, 22\rangle \quad .$$

The solutions of the Pellian equations $x^2 - Dy^2 = d_i$ can be found from the table.

c. f. elements	c_i	11	1	1	1	1	1	1	1	22
signed denominators	d_i	−14	9	−11	10	−11	9	−14	1	
p convergents	p_i	11	12	23	35	58	93	151	244	
q convergents	q_i	1	1	2	3	5	8	13	21	

The primitive solution of $x^2 - 135y^2 = 1$ is given by $x = p_8 = 244$, $y = q_8 = 21$. It can also be picked up from the half period. If the period is $2r+1$, $y = (q_r + q_{r-2})q_{r-1}$. Here

$$y = 3(2+5) = 21,$$

$$x = q_{r-1}p_{r-2} + q_r p_{r-1}.$$

Here $x = 3 \times 23 + 5 \times 35 = 244$.

5. FIBONACCI IDENTITIES TO BE USED.

(a) $$(F_{r-2}+F_r)F_{r-1} = F_{2r-2}$$

(b) $$F_{2n-1}^2 - 1 = F_{2n}F_{2n-2}$$

6. ODD VARIABLE DIFFERENCE TABLE : $D = m^2 + k$

m	1	......$2r-1$ ones......		1	$2m$
$-k$				1	
m	$m+1$	$2m+1$		$mF_{2r}+F_{2r-1}$	
1	1	2		F_{2r}	

The supposition $(mF_{2r}+F_{2r-1})^2 - F_{2r}^2(m+k) = 1$ leads to

$$2mF_{2r}F_{2r-1} + F_{2r-1}^2 - kF_{2r}^2 = 1$$

$$2mF_{2r}F_{2r-1} - F_{2r}^2 k = -(F_{2r-1}^2 - 1) = -F_{2r}F_{2r-2}$$

$$2mF_{2r-1} - kF_{2r} = -F_{2r-2}$$

Since $(F_{2r}, F_{2r-1}) = 1$, this linear diophantine equation will have an infinite number of positive integer solutions unless r is a multiple of 3. When $r = 3t$, F_{2r} is even, but F_{2r-2} is odd.

Example: $D = m^2 + k$, $\sqrt{D} = \langle m, 1, 1, 1, 2m\rangle$ $(3m+2)^2 - 9(m^2+k) = 1$

$$4m - 3k = -1, \quad k = m + \frac{m+1}{3}$$

$$m = 2, \quad k = 3, \quad D = 7, \quad \sqrt{7} = \langle 2, 1, 1, 1, 4\rangle.$$

$x^2 - 7y^2 = 1$ has solution $x = 3 \times 2 + 2 = 8 \quad y = 3.$

Since $m = 2+3 = 5, \quad k = 5+2 = 7, \quad D = 32$ follows from $k = m + \frac{m+1}{3}$.

$x^2 - 32y^2 = 1$ has primitive solution $x = 3 \times 5 + 2 = 17, \quad y = 3.$ In general,

$$D = 9m^2 - 2m, \quad \sqrt{D} = \langle 3m-1, 1, 1, 1, 6m-2 \rangle .$$

The primitive solution of $x^2 - Dy^2 = 1$ is tiven by $x = 9m-1, \quad y = 3.$

7. $D = m^2 + k, \quad 2mF_r - kF_{r+1} = -F_{r-1}$

$$\sqrt{D} = m + \sqrt{D} - m = m + \frac{\sqrt{D}+m}{k}$$

$$\frac{\sqrt{D}+m}{k} = 1 + \frac{\sqrt{D}-(k-m)}{k} = 1 + \frac{\sqrt{D}+k-m}{2m+1-k}$$

$$\frac{\sqrt{D}+k-m}{2m+1-k} = 1 + \frac{\sqrt{D}-(3m+1-2k)}{2m+1-k} = 1 + \frac{\sqrt{D}+3m+1-2k}{4k-4m-1}$$

$$\frac{\sqrt{D}+3m+1-2k}{4k-4m-1} = 1 + \frac{\sqrt{D}-(6k-7m-2)}{4k-4m+1} = 1 + \frac{\sqrt{D}+6k-7m-2}{12m-9k+4}$$

$$\frac{\sqrt{D} + F_sF_{s-1}k - (1+2F_1F_2+\cdots+2F_{s-2}F_{s-1})m - (F_1^2+F_2^2+\cdots+F_{s-2}^2)}{2mF_sF_{s-1} - kF_s^2 + F_{s-1}^2}$$

$$\text{(A)}$$

$$= 1 + \frac{\sqrt{D} - [(1+2F_1F_2+\cdots+2F_{s-1}F_s\ m) - F_sF_{s+1}k + (F_1^2F_2^2+\cdots+F_{s-1})]}{2mF_sF_{s-1} - kF_s^2 + F_{s-1}^2}$$

$$= 1 + \frac{D + \text{(A)}}{kF_{s+1}^2 - 2mF_sF_{s+1} - F_s^2} \quad .$$

For this last assumption to be valid,

$$(2mF_sF_{s-1} - kF_s^2 + F_{s-1}^2)(kF_{s+1}^2 - 2mF_{s+1}F_s - F_s^2) \equiv m^2 + k - \text{(A)}^2 .$$

This identity will be proved by equating coefficients:

1. Coefficient of $-m^2$

$$4F_s^2F_{s-1}F_{s+1} = 4F_s^2[F_s^2 + (-1)^s] = 4F_s^4 + 4(-1)^sF_s^2 = \frac{4}{25}(L_{4s} + L_{2s} - 4) = [F_{s+2}F_s - F_{s+1}F_{s-2}]^2 - 1 .$$

2. Coefficient of $-k^2$

$$F_s^2\,F_{s+1}^2 = F_s^2F_{s+1}^2 .$$

3. Constant term:

$$-F_s^2F_{s-1}^2 = -(F_1^2 + F_2^2 + \cdots + F_{s-1}^2)^2 .$$

4. Coefficient of $2mk$

$$F_{s-1}F_sF_{s+1}^2 + F_s^3F_{s+1} = F_sF_{s+1}(F_{s-1}F_{s+1} + F_s^2) = [2L_{2s} + (-1)^s]F_sF_{s+1}$$

$$F_sF_{s+1}(1 + 2F_1F_2 + \cdots + 2F_{s-1}F_s = F_sF_{s+1}(F_{s+2}F_s - F_{s+1}F_{s-2}) = [2L_{2s} + (-1)^s].F_sF_{s+1}$$

5. Coefficient of k.

$$2F_sF_{s+1}(F_1^2 + F_2^2 + \cdots + F_{s-1}^2) + 1 = 2F_s^2F_{s-1}F_{s+1} + 1 = 1 + 2F_s^2[F_s^2 + (-1)^s] = 2F_s^4 + 2F_s^2(-1)^s + 1$$

$$F_{s-1}^2F_{s+1}^2 + F_s^4 = F_s^4 + [F_s^2 + (-1)^s]^2 = 2F_s^4 + 2(-1)^sF_s^2 + 1 .$$

6. Coefficient of $-2m$

$$F_s^3F_{s-1} + F_{s-1}^2F_sF_{s+1} = F_{s-1}F_s[F_s^2 + F_{s-1}F_{s+1}] = F_{s-1}F_s[F_s(F_{s+2} - F_{s+1}) + F_{s-1}F_{s+1}]$$

$$= F_{s-1}F_s[F_sF_{s+2} - F_{s+1}(F_s - F_{s-1})] = F_{s-1}F_s(F_sF_{s+2} - F_{s+1}F_{s-2}).$$

$$(F_1^2 + F_2^2 + \cdots + F_{s-1}^2)(1 + 2F_1F_2 + 2F_sF_3 + \cdots + 2F_{s-1}F_s = F_{s-1}F_s[F_sF_{s+2} - F_{s+1}F_{s-2}]$$

In proving this identity the following Fibonacci identities were used:

(a) $$1 + 2F_1F_2 + \cdots + 2F_{s-1}F_s = F_sF_{s+2} - F_{s+1}F_{s-2}$$

(b) $$F_1^2 + F_2^2 + \cdots + F_s^2 = F_{s-1}F_s$$

(c) $$F_{s-1}F_{s+1} = F_s^2 + (-1)^s\ .$$

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A MORE GENERAL FIBONACCI MULTIGRADE

DONALD CROSS
St. Luke's College, Exeter, England

In a recent article I gave examples of multigrades based on Fibonacci series in which

$$F_{n+2} = F_{n+1} + F_n\ .$$

Here I first give a more general multigrade for series in which

$$F_{n+2} = yF_{n+1} + xF_n\ .$$

Consider

$$1 \quad 3 \quad 7 \quad 17 \quad 47 \quad (\text{where } x = 1, y = 2).$$

By inspection we notice that

$$1^m + 3^m + 3^m + 7^m = 0^m + 4^m + 4^m + 6^m$$

$$3^m + 7^m + 7^m + 17^m = 0^m + 10^m + 10^m + 14^m\ , \text{etc.}$$

(where $m = 1, 2$).

We can look at other series of a like kind:

$$1 \quad 3 \quad 10 \quad 33 \quad 109 \quad (\text{where } x = 1,\ y = 3).$$

Here

$$1^m + 3^m + 3^m + 3^m + 10^m + 10^m = 0^m + 0^m + 7^m + 7^m + 7^m + 9^m$$

$$3^m + 10^m + 10^m + 10^m + 33^m + 33^m = 0^m + 0^m + 23^m + 23^m + 23^m + 30^m,\ \text{etc.}$$

(where $m = 1, 2$)

$$1 \quad 3 \quad 11 \quad 39 \quad 139 \quad (\text{where } x = 2,\ y = 3).$$

Here

$$1^m + 1^m + 3^m + 3^m + 3^m + 11^m + 11^m + 11^m = 0^m + 0^m + 0^m + 8^m + 8^m + 8^m + 10^m + 10^m$$

$$3^m + 3^m + 11^m + 11^m + 11^m + 39^m + 39^m + 39^m = 0^m + 0^m + 0^m + 28^m + 28^m + 28^m + 36^m + 36^m,\ \text{etc.}$$

(where $m = 1, 2$)

The general series

$$a \quad b \quad ax + by \quad bx + axy + by^2$$

gives

$$x(a)^m + y(b)^m + (x + y - 2)(ax + by)^m = (x + y - 2)0^m + y(ax + by - b)^m + x(ax + by - a)^m$$

(where $m = 1, 2$).

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