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EXPLICIT DESCRIPTIONS OF SOME CONTINUED FRACTIONS

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In a previous paper [1], the author proved the following theorem.

THEOREM 1: Let

$$B(u, v) = \sum_{k=0}^{v} \left(\frac{1}{u}\right)^{2^k} \qquad (u \geq 3, \text{ an integer}).$$

Then we have

(a) $B(u, 0) = [0, u]$,
$B(u, 1) = [0, u - 1, u + 1]$.

(b) Suppose $B(u, v) = [a_0, a_1, \ldots, a_n]$. Then
$B(u, v + 1) = [a_0, a_1, \ldots, a_{n-1}, a_n + 1, a_n - 1, a_{n-1}, a_{n-2}, \ldots, a_2, a_1]$.

Repeated application of this theorem generates the continued fraction for

$$B(u, \infty) = \sum_{k=0}^{\infty} \left(\frac{1}{u}\right)^{2^k}.$$

For example, we find

(1) $B(3, \infty) = [0, 2, 5, 3, 3, 1, 3, 5, 3, 1, 5, 3, 1, \ldots]$,
(2) $B(u, \infty) = [0, u - 1, u + 2, u, u, u - 2, u, u + 2, u, u - 2, \ldots]$.

Recently, Bergman [2] provided an explicit, nonrecursive description of the partial quotients in (1), and by implication, in (2). (This description is our Theorem 3.) The purpose of this paper is to prove Bergman's result, and to provide similar results for the continued fractions given in [3] and [4].

We start off with some terminology about "strings." By a *string*, we mean a (finite or infinite) ordered sequence of symbols. Thus, for example, we may consider the partial quotients

$$[a_0, a_1, \ldots, a_n]$$

of a continued fraction to be a string. If w and x are strings, then by wx, the *concatenation* of w with x, we mean the juxtaposition of the elements of w with those of x. By $|w|$, we mean the *length* of w, i.e., the number of symbols in w. Note that $|w|$ may be either 0 or ∞. If w is a finite string, then by w^R, the *reversal* of w, we mean the symbols of w taken in reverse order. Finally, by the symbol w^n, we mean the string

$$\underbrace{ww \ldots w}_{n \text{ times}}$$

By w^0, we mean the *empty string*, denoted by $\emptyset$, with the property that $w\emptyset = \emptyset w = w$. Note that $(wx)^R = x^R w^R$, and so $(w^R)^n = (w^n)^R$.

THEOREM 2: Let A_0 and B be finite strings. Define $A_{n+1} = A_n B A_n^R$. Let the symbol A_∞ stand for the unique infinite string of which $A_0, A_1, \ldots$ are all prefixes.

Then $A_\infty = X_1 Y_1 X_2 Y_2 X_3 Y_3 \ldots$ where

(a) $X_k = \begin{cases} A_0 & \text{if } k \text{ is odd} \\ A_0^R & \text{if } k \text{ is even,} \end{cases}$

(b) $Y_k = \begin{cases} B^R & \text{if } k \in S \\ B & \text{if } k \notin S. \end{cases}$

and

$$S = \{n \geq 1\colon n = 2^i(1 + 2j),\ i,\ j \text{ integers} \geq 0 \text{ and } j \text{ is odd}\}$$
$$= \{3, 6, 7, 11, 12, 14, 15, \ldots\}.$$

To prove this result, we need a lemma.

LEMMA 1: Let A_0, A_n, and B be as in Theorem 2. Then

$$A_{n+1} = (A_0B^*A_0^RB^*)^{2^n-1}A_0B^*A_0^R$$

where by the symbol B^* we mean *either* B or B^R.

PROOF: We use induction on n. Clearly, the lemma is true for $n = 0$. Assume true for n. Then we find

$$A_{n+2} = A_{n+1}BA_{n+1}^R = (A_0B^*A_0^RB^*)^{2^n-1}A_0B^*A_0^RBA_0B^*A_0^R(B^*A_0B^*A_0^R)^{2^n-1}$$
$$= (A_0B^*A_0^RB^*)^{2^{n+1}-1}A_0B^*A_0^R$$

and the proof of the lemma is complete.

We can now prove Theorem 2. Part (a) follows immediately from the lemma. To prove part (b) we will prove, by induction on n, that the theorem is true for all $k \leq 2^n$.

Clearly, part (b) is true for $n = 0$. Assume true for all $k \leq 2^n$. Then we wish to show part (b) is true for all k such that $2^n < k \leq 2^{n+1}$.

Assume $2^n < k < 2^{n+1}$. Since $A_{n+1} = A_nBA_n^R$, we see that if $Y_k = B^R$ then $Y_{2^{n+1}-k} = B$; similarly, if $Y_k = B$ then $Y_{2^{n+1}-k} = B^R$.

We note that every positive integer can be written uniquely in the form

$$2^i(1 + 2j),$$

where i and j are nonnegative integers. Thus, it suffices to show that (for $2^n < k < 2^{n+1}$) if $k = 2^i(1 + 2j)$, then $2^{n+1} - k = 2^{i'}(1 + 2j')$, where j and j' are of opposite parity.

If $2^n < k < 2^{n+1}$, then the largest power of 2 dividing k is 2^{n-1}; hence, $0 \leq i \leq n - 1$. Therefore,

$$2^{n+1} - k = 2^{n+1} - 2^i(1 + 2j) = 2^i(2^{n+1-i} - 1 - 2j)$$
$$= 2^i(1 + 2(2^{n-i} - j - 1)) = 2^{i'}(1 + 2j').$$

But $n - i \geq 1$; hence, j and j' are indeed of opposite parity.

Finally, we must examine the case $k = 2^{n+1}$. But it is easy to see from Lemma 1 that $Y_k = B$ if k is a power of 2.

Now that we have built up some machinery, we can state and prove the explicit description of the continued fraction for $B(u, \infty)$.

THEOREM 3 (Bergman):

$$B(u, \infty) = [0, u - 1, U_1, V_1, U_2, V_2, U_3, V_3, \ldots]$$

where

$$U_k = \begin{cases} (u + 2, u) & \text{if } k \text{ is odd} \\ (u, u + 2) & \text{if } k \text{ is even,} \end{cases}$$

$$V_k = \begin{cases} (u, u - 2) & \text{if } k \notin S \\ (u - 2, u) & \text{if } k \in S. \end{cases}$$

S is as in Theorem 2.

Bergman's result follows immediately from Theorem 2 and the following lemma.

LEMMA 2: Let $A_0 = (u + 2, u)$; $B = (u, u - 2)$; let A_n and A_∞ be as in Theorem 2. Then

(a) $B(u, v + 2) = [0, u - 1, A_v, u - 1]$,

(b) $B(u, \infty) = [0, u - 1, A_\infty]$.

PROOF: To prove part (a), we use induction on v. From Theorem 1, we have

$$B(u, 2) = [0, u - 1, u + 2, u, u - 1] = [0, u - 1, A_0, u - 1].$$

Hence, the lemma is true for $v = 0$.

Now assume true for v. We have

$$B(u, v + 2) = [0, u - 1, A_v, u - 1].$$

But by Theorem 1,

$$B(u, v + 3) = [0, u - 1, A_v, u, u - 2, A_v^R, u - 1] \tag{3}$$
$$B(u, v + 3) = [0, u - 1, A_{v+1}, u - 1].$$

This proves part (a) of the lemma. To prove part (b), we simply let v approach ∞ in both sides of (3).

Note: Lemma 2 was independently discovered by M. Kmošek [5].

These results provide a different proof of the fact, proved in [1], that $B(u, \infty)$ is not a quadratic irrational. This is an implication of the following more general result.

THEOREM 4: Let A_0, B, A_n, and A_∞ be as in Theorem 2, A_0 and B not both empty. Then A_∞ is eventually periodic if and only if B is a palindrome.

PROOF: We say the infinite string w is *eventually periodic* if and only if $w = xy^\infty$ where, by the symbol y^∞, we mean the infinite string $yyyy\ldots$. The string y is called the *repeating portion*, or the *period*.

Suppose B is a palindrome. Then by Lemma 1,

$$A_{n+1} = (A_0B^*A_0^RB^*)^{2^n-1}A_0B^*A_0^R.$$

But $B = B^R$; so B^* always equals B. Hence,

$$A_\infty = (A_0BA_0^RB)^\infty.$$

Now assume A_∞ is eventually periodic, i.e., $A_\infty = xy^\infty$. Since

$$|A_n| = 2^n(|A_0| + |B|) - |B|,$$

we may choose n such that $|x| \leq |A_n|$. Then since $A_nBA_n^R$ is a prefix of A_∞, we may assume (by renaming x and y, if necessary) that $x = A_n$.

Now let $z = y^{|A_n| + |B|}$. Clearly, $A_\infty = xy^\infty = xz^\infty$. If y is a repeating portion, then so is z. The string z consists of groups of $B^*A_n^*$'s; hence, if we can show that groups of $B^*A_n^*$'s repeat only if $B = B^R$, we will be done.

By renaming "A_n" to be "A_0," we may use the result given in Theorem 2 to describe the positions of the B's in A_∞. We will show that for all integers $i \geq 1$, there exist $c_i \in S$ and $d_i \notin S$ such that $c_i - d_i = i$. This shows that in A_∞ there exists a B and a B^R exactly $|z|$ symbols apart; hence, if z really is a repeating portion, we must have $B = B^R$.

Let i be written in base 2 as a string of ones and zeros. Then, clearly, for some $m \geq 0$, $n \geq 1$, this expression has the form

$$z\ 0\ 1^n\ 0^m,$$

where z is an arbitrary string of ones and zeros.

Let c_i be the number represented by the binary string $z\ 1\ 1^n\ 0^m$ and let d_i be the number represented by $1\ 0^n\ 0^m$. Then, clearly, $c_i - d_i = i$, and it is easily verified that $c_i \in S$ and $d_i \notin S$.

Thus, $B = B^R$ and Theorem 4 is proved.

Note: Theorem 4 was stated without proof in [6].

COROLLARY 1: $B(u, \infty)$ is not a quadratic irrational.

PROOF: From Lemma 2, we have $B(u, \infty) = [0, u - 1, A_\infty]$, where $A_0 = (u + 2, u)$ and $B = (u, u - 2)$. Since $B \neq B^R$, A_∞ cannot be eventually periodic. Hence, by a well-known theorem (see Hardy & Wright [7]), $B(u, \infty)$ is not a quadratic irrational.

COROLLARY 2: Suppose each element x of the strings A_0 and B satisfies $0 \leq x < b$, where b is an integer ≥ 2. Then we may consider A_∞ to be the base b representation of a number between 0 and 1. Then Theorem 4 implies that this number is irrational if and only if $B = B^R$.

As the last result of this paper, we state a theorem giving a description similar to that in Theorem 3 for another type of continued fraction.

In [3] and [4], the following result is proved.

THEOREM 5: Let $\{c(k)\}_{k=0}^{\infty}$ be a sequence of positive integers such that $c(v + 1) \geq 2c(v)$ for all $v \geq v'$. Let $d(v) = c(v + 1) - 2c(v)$. Define $S(u, v)$ as follows:

$$S(u, v) = \sum_{k=0}^{v} u^{-c(k)} \qquad (u \geq 2, \text{ an integer}).$$

Then, if $v \geq v'$ and $S(u, v) = [a_0, a_1, \ldots, a_n]$ and n is even,

$$S(u, v + 1) = [a_0, a_1, \ldots, a_n, u^{d(v)} - 1, 1, a_n - 1, a_{n-1}, a_{n-2}, \ldots, a_2, a_1].$$

It is possible to use the techniques above to get an explicit description of the continued fraction for $S(u, \infty)$ similar to that for $B(u, \infty)$. This description is somewhat more complicated due to the extra terms given in Theorem 5. If we assume that $v' = 0$, $c(v + 1) > 2c(v)$ and $u \geq 3$, then the description becomes somewhat more manageable.

THEOREM 6: Let $S(u, \infty) = \lim_{v \to \infty} S(u, v)$. Let us write $n = 2^{i_n}(1 + 2j_n)$ where i_n and j_n are nonnegative integers; however, put $j_n = -1$ for $n = 0$. Define $p(n)$, the *parity* of an integer n as 0 if n is even, and 1 if n is odd. Then under the simplifying assumptions of the previous paragraph,

$$S(u, \infty) = [0, A_0, B_1, C_1, B_2, A_1, B_3, C_2, B_4, A_2, B_5, \ldots],$$

where

$$A_n = (u^{c(0)} + p(j_n) - 2, 1, u^{d(0)} - 1, u^{c(0)} - p(n))$$
$$C_n = (u^{c(0)} - p(n), u^{d(0)} - 1, 1, u^{c(0)} - 1 - p(j_n))$$
$$B_n = \begin{cases} (u^{d(1+i_n)} - 1, 1) \text{ if } j_n \text{ is even,} \\ (1, u^{d(1+i_n)} - 1) \text{ if } j_n \text{ is odd.} \end{cases}$$

PROOF: The proof is a straightforward (though tedious) application of previous techniques, and is omitted here.

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THE NONEXISTENCE OF QUASIPERFECT NUMBERS OF CERTAIN FORMS

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1. INTRODUCTION

A natural number n is called perfect, multiperfect, or quasiperfect according as $\sigma(n) = 2n$, $\sigma(n) = kn$ ($k \geq 2$, an integer), or $\sigma(n) = 2n+1$, respectively, where $\sigma(n)$ is the sum of the positive divisors of n.

No odd multiperfect numbers are known. In many papers concerned with odd perfect numbers (summarized in McDaniel & Hagis [5]), values have been obtained which cannot be taken by the even exponents on the prime factors of such numbers, if all those exponents are equal. McDaniel [4] has given results of a similar nature for odd multiperfect numbers.

No quasiperfect numbers have been found. It is known [Cattaneo [1]) that if there are any they must be odd perfect squares, and it has recently been shown by Hagis & Cohen [3] that such a number must have at least seven distinct prime factors and must exceed 10^{35}. In this paper we shall give results analogous to those described for odd multiperfect numbers, but with extra generality. In particular, we shall show that no perfect fourth power is quasiperfect, and no perfect sixth power, prime to 3, is quasiperfect. We are unable to prove the nonexistence of quasiperfect numbers of the form m^2, where m is squarefree, but will show that any such numbers must have more than 230,000 distinct prime factors, so the chance of finding any is slight!

All italicized letters here denote nonnegative integers, with p and q primes, $p > 2$.

2. SOME LEMMAS

The following result is due to Cattaneo [1].

LEMMA 1: If n is quasiperfect and $r \mid \sigma(n)$, then $r \equiv 1$ or $3 \pmod 8$.

We shall need

LEMMA 2: Suppose n is quasiperfect and $p^{2a} \| n$. If $q \mid 2a + 1$, then

$$(q - 1)(p + 1) \equiv 0 \text{ or } 4 \pmod{16}.$$

PROOF: Notice first that if b is odd, then, modulo 8,

$$(1) \qquad \sigma(p^{b-1}) = 1 + p + p^2 + \cdots + p^{b-1} \equiv 1 + (p + 1) + \cdots + (p + 1) = 1 + \tfrac{1}{2}(b - 1)(p + 1).$$

Let $F_d(\xi)$ denote the cyclotomic polynomial of order d. It is well known that

$$\xi^m - 1 = \prod_{d \mid m} F_d(\xi) \qquad (m > 0),$$

so

$$(2) \qquad \sigma(p^{2}) = \prod_{\substack{d \mid 2a+1 \\ d > 1}} F_d(p).$$