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Continued Fractions and Generalized Patterns

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Babson and Steingrimsson (2000, Séminaire Lotharingien de Combinatoire, **B44b,** 18) introduced generalized permutation patterns that allow the requirement that two adjacent letters in a pattern must be adjacent in the permutation.

Let $f_{\tau;r}(n)$ be the number of 1-3-2-avoiding permutations on n letters that contain exactly r occurrences of τ, where τ is a generalized pattern on k letters. Let $F_{\tau;r}(x)$ and $F_\tau(x, y)$ be the generating functions defined by $F_{\tau;r}(x) = \sum_{n\geq 0} f_{\tau;r}(n)x^n$ and $F_\tau(x, y) = \sum_{r\geq 0} F_{\tau;r}(x)y^r$. We find an explicit expression for $F_\tau(x, y)$ in the form of a continued fraction for τ given as a generalized pattern: τ = 12-3-...-k, τ = 21-3-...-k, τ = 123...k, or τ = k...321. In particular, we find $F_\tau(x, y)$ for any τ generalized pattern of length 3. This allows us to express $F_{\tau;r}(x)$ via Chebyshev polynomials of the second kind and continued fractions.

1. INTRODUCTION

Let $[p] = \{1, \dots, p\}$ denote a totally ordered alphabet on p letters, and let $\pi = (\pi_1, \dots, \pi_m) \in [p_1]^m$, $\beta = (\beta_1, \dots, \beta_m) \in [p_2]^m$. We say that π is *order-isomorphic* to β if for all $1 \leq i < j \leq m$ one has $\pi_i < \pi_j$ if and only if $\beta_i < \beta_j$. For two permutations $\pi \in S_n$ and $\tau \in S_k$, an *occurrence* of τ in π is a subsequence $1 \leq i_1 < i_2 < \cdots < i_k \leq n$ such that $(\pi_{i_1}, \dots, \pi_{i_k})$ is order-isomorphic to τ; in such a context τ is usually called the *pattern* (classical pattern). We say that π *avoids* τ, or is τ*-avoiding*, if there is no occurrence of τ in π. More generally, we say π *containing* τ exactly r times, if there exists r different occurrences of τ in π.

The set of all τ-avoiding permutations of all possible sizes including the empty permutation is denoted $\mathcal{S}(\tau)$. Pattern avoidance proved to be a useful language in a variety of seemingly unrelated problems, from stack sorting [8] to singularities of Schubert varieties [10]. A complete study of pattern avoidance for the case $\tau \in S_3$ is carried out in [16].

On the other hand, [1] introduced generalized permutation patterns that allow the requirement that two adjacent letters in a pattern must be adjacent in the permutation. The idea of [1] introducing these patterns was the study of Mahonian statistics.

We write a classical pattern with dashes between any two adjacent letters of the pattern, say 1324, as 1-3-2-4, and if we write, say 24-1-3, then we mean that if this pattern occurs in permutation π, then the letters in the permutation π that correspond to 2 and 4 are adjacent. For example, the permutation π = 35421 has only two occurrences of the pattern 23-1, namely the subsequences 352 and 351, whereas π has four occurrences of the pattern 2-3-1, namely the subsequences 352, 351, 342 and 341.

Reference [3] presented a complete solution for the number of permutations avoiding any pattern of length three with exactly one adjacent pair of letters. Reference [4] presented a complete solution for the number of permutations avoiding any two patterns of length three with exactly one adjacent pair of letters. Reference [7] almost presented results avoiding two or more 3-patterns without internal dashes, that is, where the pattern corresponds to a contiguous subword in a permutation. Besides, [5] presented the following generating functions regarding the distribution of the number of occurrences of any generalized pattern of length 3:

$$\sum_{\pi\in\mathcal{S}} y^{(123)\pi}\frac{x^{|\pi|}}{|\pi|!} = \frac{2f(y)e^{\frac{1}{2}(f(y)-y+1)x}}{f(y)+y+1+(f(y)-y-1)e^{f(y)x}},$$

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$$\sum_{\pi\in\mathcal{S}} y^{(213)\pi}\frac{x^{|\pi|}}{|\pi|!} = \frac{1}{1-\int_0^x e^{(y-1)t^2/2}dt},$$

where $(\tau)\pi$ is the number of occurrences of τ in π, $f(y) = \sqrt{(y-1)(y+3)}$.

The purpose of this paper is to point out an analogue of [15], and some interesting consequences of this analogue. Generalizations of this theorem have already been given in [6, 9, 12]. In the present paper we study the generating function for the number 1-3-2-avoiding permutations in S_n that contain a prescribed number of generalized pattern τ. The study of the obtained continued fraction allows us to recover and to present an analogue of the results of [2, 6, 9, 12] that relates the number of 1-3-2-avoiding permutations that contain no 12-3-...-k (or 21-3-...-k) patterns to Chebyshev polynomials of the second kind.

Let $f_{\tau;r}(n)$ stand for the number of 1-3-2-avoiding permutations in S_n that contain exactly r occurrences of τ. We denote by $F_{\tau;r}(x)$ and $F_\tau(x, y)$ the generating function of the sequence $\{f_{\tau;r}(n)\}_{n\geq 0}$ and $\{f_{\tau;r}(n)\}_{n,r\geq 0}$, respectively, that is,

$$F_{\tau;r}(x) = \sum_{n\geq 0} f_{\tau;r}(n)x^n, \qquad F_\tau(x, y) = \sum_{r\geq 0} F_{\tau;r}(x)y^r.$$

The paper is organized as follows. The cases $\tau = 12\text{-}3\text{-}\dots\text{-}k$, $\tau = 21\text{-}3\text{-}\dots\text{-}k$, $\tau = 123\dots k$, and $\tau = k\dots 321$ are treated in Section 2. In Section 3, we present the cases $\tau =$ 123, 213, 231, 312, and 321, that is, τ is a 3-letters generalized pattern without dashes. In Section 4, we treat the cases when τ is a 3-letters generalized pattern with one dash. Finally, in Section 5, we present examples of restricted more than one generalized pattern of 3-letters.

2. Four General Cases

In this section, we study the following four cases: $\tau = 12\text{-}3\text{-}\dots\text{-}k$, $\tau = 21\text{-}3\text{-}\dots\text{-}k$, $\tau = 12\dots k$, and $\tau = k\dots 21$, by the following three subsections.

2.1. *Pattern* 12-3-...-k.

Our first result is a natural analogue of the main theorems of [9, 12, 15].

THEOREM 2.1. *The generating function $F_{12\text{-}3\text{-}\dots\text{-}k}(x, y)$ for $k \geq 2$ is given by the continued fraction*

$$\cfrac{1}{1-x+xy^{d_1}-\cfrac{xy^{d_1}}{1-x+xy^{d_2}-\cfrac{xy^{d_2}}{1-x+xy^{d_3}-\cfrac{xy^{d_3}}{\cdots}}}},$$

where $d_i = \binom{i-1}{k-2}$, and $\binom{a}{b}$ is assumed 0 whenever $a < b$ or $b < 0$.

PROOF. Following [12] we define $\eta_j(\pi)$, $j \geq 3$, as the number of occurrences of 12-3-...-j in π. Define $\eta_2(\pi)$ for any π, as the number of occurrences of 12 in π, $\eta_1(\pi)$ as the number of letters of π, and $\eta_0(\pi) = 1$ for any π, which means that the empty pattern occurs exactly once in each permutation. The *weight* of a permutation π is a monomial in k independent variables $q_1, \dots, q_k$ defined by

$$w_k(\pi) = \prod_{j=1}^{k} q_j^{\eta_j(\pi)}.$$

The *total weight* is a polynomial

$$W_k(q_1, \dots, q_k) = \sum_{\pi\in\mathcal{S}(1\text{-}3\text{-}2)} w_k(\pi).$$

The following proposition is implied immediately by the definitions.

PROPOSITION 2.2. *$F_{12\text{-}3\text{-}\dots\text{-}k}(x, y) = W_k(x, 1, \dots, 1, y)$ for $k \geq 2$.*

We now find a recurrence relation for the numbers $\eta_j(\pi)$. Let $\pi \in S_n$, so that $\pi = (\pi', n, \pi'')$.

PROPOSITION 2.3. *For any nonempty $\pi \in \mathcal{S}(1\text{-}3\text{-}2)$*

$$\eta_j(\pi) = \eta_j(\pi') + \eta_j(\pi'') + \eta_{j-1}(\pi'),$$

where $j \neq 2$. Besides, if π' is nonempty then

$$\eta_2(\pi) = \eta_2(\pi') + \eta_2(\pi'') + 1,$$

otherwise

$$\eta_2(\pi) = \eta_2(\pi'').$$

PROOF. Let $l = \pi^{-1}(n)$. Since π avoids 1-3-2, each number in π' is greater than any of the numbers in π''. Therefore, π' is a 1-3-2-avoiding permutation of the numbers $\{n - l + 1, n - l + 2, \dots, n - 1\}$, while π'' is a 1-3-2-avoiding permutation of the numbers $\{1, 2, \dots, n - l\}$. On the other hand, if π' is an arbitrary 1-3-2-avoiding permutation of the numbers $\{n-l+1, n-l+2, \dots, n-1\}$ and π'' is an arbitrary 1-3-2-avoiding permutation of the numbers $\{1, 2, \dots, n - l\}$, then $\pi = (\pi', n, \pi'')$ is 1-3-2-avoiding. Finally, if $(i_1, \dots, i_j)$ is an occurrence of 12-3-$\dots$-j in π then either $i_j < l$, and so it is also an occurrence of 12-3-$\dots$-j in π', or $i_1 > l$, and so it is also an occurrence of 12-$\dots$-j in π'', or $i_j = l$, and so $(i_1, \dots, i_{j-1})$ is an occurrence of 12-3-$\dots$-$(j - 1)$ in π', where $j \neq 2$. For $j = 2$ the proposition is trivial. The result follows. □

Now we are able to find the recurrence relation for the total weight W. Indeed, by Proposition 2.3,

$$\begin{aligned} W_k(q_1, \dots, q_k) &= 1 + \sum_{\varnothing \neq \pi \in \mathcal{S}(1\text{-}3\text{-}2)} \prod_{j=1}^{k} q_j^{\eta_j(\pi)} \\ &= 1 + \sum_{\varnothing \neq \pi' \in \mathcal{S}(1\text{-}3\text{-}2)} \sum_{\pi'' \in \mathcal{S}(1\text{-}3\text{-}2)} \prod_{j=1}^{k} q_j^{\eta_j(\pi'')} \cdot q_1^{\eta_1(\pi')+1} q_2 \cdot \\ &\quad \prod_{j=2}^{k-1} (q_j q_{j+1})^{\eta_j(\pi')} \cdot q_k^{\eta_k(\pi')} + \sum_{\pi'' \in \mathcal{S}(1\text{-}3\text{-}2)} q_1 \prod_{j=1}^{k} q_j^{\eta_j(\pi'')}. \end{aligned}$$

Hence

$$\begin{aligned} W_k(q_1, \dots, q_k) &= 1 + q_1 W_k(q_1, \dots, q_k) \\ &\quad + q_1 q_2 W_k(q_1, \dots, q_k)(W_k(q_1, q_2 q_3, \dots, q_{k-1} q_k, q_k) - 1). \end{aligned} \tag{1}$$

Following [12], for any $d \geq 0$ and $2 \leq m \leq k$ define

$$\mathbf{q}^{d,m} = \prod_{j=2}^{k} q_j^{\binom{d}{j-m}};$$

recall that $\binom{a}{b} = 0$ if $a < b$ or $b < 0$. The following proposition is implied immediately by the well-known properties of binomial coefficients.

PROPOSITION 2.4. *For any $d \geq 0$ and $2 \leq m \leq k$*

$$\mathbf{q}^{d,m}\mathbf{q}^{d,m+1} = \mathbf{q}^{d+1,m}.$$

Observe now that $W_k(q_1, \ldots, q_k) = W_k(q_1, \mathbf{q}^{0,2}, \ldots, \mathbf{q}^{0,k})$ and that by (1) and Proposition 2.4

$$W_k(q_1, \mathbf{q}^{d,2}, \ldots, \mathbf{q}^{d,k}) = 1 + q_1 W_k(q_1, \mathbf{q}^{d,2}, \ldots, \mathbf{q}^{d,k}) + q_1 q^{d,2} W_k(q_1, q^{d,2}, \ldots, \mathbf{q}^{d,k})(W_k(q_1, \mathbf{q}^{d+1,2}, \ldots, \mathbf{q}^{d+1,k}) - 1).$$

Therefore

$$W_k(q_1, \ldots, q_k) = \cfrac{1}{1 - q_1 + q_1\mathbf{q}^{0,2} - \cfrac{q_1\mathbf{q}^{0,2}}{1-q_1+q_1\mathbf{q}^{1,2} - \cfrac{q_1\mathbf{q}^{1,2}}{1-q_1+q_1\mathbf{q}^{2,2} - \cfrac{q_1\mathbf{q}^{2,2}}{\cdots}}}}.$$

To obtain the continued fraction representation for $F(x, y; k)$ it is enough to use Proposition 2.2 and to observe that

$$q_1\mathbf{q}^{d,2}\Big|_{q_1=x, q_2=\ldots=q_{k-1}=1, q_k=y} = xy^{\binom{d}{k-2}}.$$

□

COROLLARY 2.5.

$$F(x, y; 2) = \frac{1 - x + xy - \sqrt{(1-x)^2 - 2x(1+x)y + x^2y^2}}{2xy},$$

in other words, for any $r \geq 1$

$$f_{12}(n) = \frac{r+1}{n(n-r)}\binom{n}{r+1}^2.$$

PROOF. For $k = 2$, Theorem 2.1 yields

$$F_{12}(x, y) = \cfrac{1}{1 - x + xy - \cfrac{xy}{1-x+xy - \cfrac{xy}{1-x+xy - \cfrac{xy}{\cdots}}}}.$$

which means that

$$F_{12}(x, y) = \frac{1}{1 - x + xy - xyF_{12}(x, y)}.$$

So the rest is easy to see. □

Now, we find an explicit expression for $F_{12\text{-}3\text{-}\ldots\text{-}k;r}(x)$ where $0 \leq r \leq k-2$. Following [12], consider a recurrence relation

$$T_j = \frac{1}{1 - xT_{j-1}}, \qquad j \geq 1. \tag{2}$$

The solution of (2) with the initial condition $T_0 = 0$ is denoted by $R_j(x)$, and the solution of (2) with the initial condition

$$T_0 = G_{12\text{-}3\text{-}\ldots\text{-}k}(x, y) = \cfrac{1}{1 - x + xy^{\binom{k-2}{0}} - \cfrac{xy^{\binom{k-2}{0}}}{1-x+xy^{\binom{k-1}{1}} - \cfrac{xy^{\binom{k-1}{1}}}{1-x+xy^{\binom{k}{2}} - \cfrac{xy^{\binom{k}{2}}}{\cdots}}}}$$

is denoted by $S_j(x, y; k)$, or just S_j when the value of k is clear from the context. Our interest in (2) is stipulated by the following relation, which is an easy consequence of Theorem 2.1:

$$F_{12\text{-}3\text{-}\ldots\text{-}k}(x, y) = S_{k-2}(x, y; k). \tag{3}$$

Following [12, Eqn. (4)], for all $j \geq 1$

$$R_j(x) = \frac{U_{j-1}\left(\frac{1}{2\sqrt{x}}\right)}{\sqrt{x}U_j\left(\frac{1}{2\sqrt{x}}\right)}, \tag{4}$$

where $U_j(\cos\theta) = \sin(j+1)\theta/\sin\theta$ are the Chebyshev polynomials of the second kind. Next, we find an explicit expression for S_j in terms of G and R_j.

LEMMA 2.6. *For any $j \geq 2$ and any $k \geq 2$*

$$S_j(x, y; k) = R_j(x)\frac{1 - xR_{j-1}(x)G_{12\text{-}3\text{-}\ldots\text{-}k}(x, y)}{1 - xR_j(x)G_{12\text{-}3\text{-}\ldots\text{-}k}(x, y)}. \tag{5}$$

PROOF. Indeed, from (2) and $S_0 = G$ we get $S_1 = 1/(1-xG)$. On the other hand, $R_0 = 0$, $R_1 = 1$, so (5) holds for $j = 1$. Now let $j > 1$, then by induction

$$S_j = \frac{1}{1 - xS_{j-1}} = \frac{1}{1 - xR_{j-1}} \cdot \frac{1 - xR_{j-1}G}{1 - \frac{x(1-xR_{j-2})R_{j-1}G}{1-xR_{j-1}}}.$$

Relation (2) for R_j and R_{j-1} yields $(1-xR_{j-2})R_{j-1} = (1-xR_{j-1})R_j = 1$, which together with the above formula gives (5). □

As a corollary from Lemma 2.6 and (3) we get the following expression for the generating function $F_{12\text{-}3\text{-}\ldots\text{-}k}(x, y)$.

COROLLARY 2.7. *For any $k \geq 3$*

$$F_{12\text{-}3\text{-}\ldots\text{-}k}(x, y) = R_k(x) + \big(R_{k-2}(x) - R_{k-3}(x)\big)\sum_{m\geq 1}\big(xR_{k-2}(x)G_{12\text{-}3\text{-}\ldots\text{-}k}(x, y)\big)^m.$$

Now we are ready to express the generating function $F_{12\text{-}3\text{-}\ldots\text{-}k;r}(x)$ where $0 \leq r \leq k-2$, via Chebyshev polynomials.

THEOREM 2.8. *For any $k \geq 3$, $F_{12\text{-}3\text{-}\ldots\text{-}k;r}(x)$ is a rational function given by*

$$F_{12\text{-}3\text{-}\ldots\text{-}k;r}(x) = \frac{x^{r-1}U_{k-2}^{r-1}\left(\frac{1}{2\sqrt{x}}\right)}{(1-x)^r U_k^{r+1}\left(\frac{1}{2\sqrt{x}}\right)}, \qquad 1 \leq r \leq k-2,$$

$$F_{12\text{-}3\text{-}\ldots\text{-}k;0}(x) = \frac{U_{k-1}\left(\frac{1}{2\sqrt{x}}\right)}{\sqrt{x}U_k\left(\frac{1}{2\sqrt{x}}\right)},$$

where U_j is the jth Chebyshev polynomial of the second kind.

PROOF. Observe that $G_{12\text{-}3\text{-}\ldots\text{-}k}(x, y) = \frac{1}{1-x} \cdot \frac{1}{1-\frac{x^2}{(1-x)^2}y} + y^{k-1}P(x, y)$, so from Corollary 2.7 we get

$$F_{12\text{-}3\text{-}\ldots\text{-}k}(x, y) = R_k(x) + \big(R_{k-2}(x) - R_{k-3}(x)\big) \sum_{m=1}^{k} \left(\frac{x}{1-x}R_k(x)\right)^m \sum_{n=1}^{k-2} \binom{m-1+n}{n} \times \frac{x^{2n}}{(1-x)^{2n}} y^n + y^{k-1}P'(x, y),$$

where $P(x, y)$ and $P'(x, y)$ are formal power series. To complete the proof, it suffices to use (4) together with the identity $U_{n-1}^2(z) - U_n(z)U_{n-2}(z) = 1$, which follows easily from the trigonometric identity $\sin^2 n\theta - \sin^2\theta = \sin(n+1)\theta \sin(n-1)\theta$. □

2.2. *Pattern* 21-3-...-k. Our second result is a natural analogue of the main theorems of [9, 12, 15].

THEOREM 2.9. *For any* $k \geq 2$,

$$F_{21\text{-}3\text{-}\ldots\text{-}k}(x, y) = 1 - \cfrac{x}{xy^{d_1} - \cfrac{1}{1 - \cfrac{x}{xy^{d_2} - \cfrac{1}{1 - \cfrac{x}{xy^{d_3} - \cfrac{1}{\ddots}}}}}},$$

where $d_i = \binom{i-1}{k-2}$, *and* $\binom{a}{b}$ *is assumed* 0 *whenever* $a < b$ *or* $b < 0$.

PROOF. Following [12] we define $\nu_j(\pi)$, $j \geq 3$, as the number of occurrences of 21-3-...-j in π. Define $\nu_2(\pi)$ for any π, as the number of occurrences of 21 in π, $\nu_1(\pi)$ as the number of letters of ν, and $\nu_0(\pi) = 1$ for any π, which means that the empty pattern occurs exactly once in each permutation. The *weight* of a permutation π is a monomial in k independent variables $q_1, \ldots, q_k$ defined by

$$v_k(\pi) = \prod_{j=1}^{k} q_j^{\nu_j(\pi)}.$$

The *total weight* is a polynomial

$$V_k(q_1, \ldots, q_k) = \sum_{\pi \in \mathcal{S}(1\text{-}3\text{-}2)} v_k(\pi).$$

The following proposition is implied immediately by the definitions.

PROPOSITION 2.10. $F_{21\text{-}3\text{-}\ldots\text{-}k}(x, y) = V_k(x, 1, \ldots, 1, y)$ *for* $k \geq 2$.

We now find a recurrence relation for the numbers $\nu_j(\pi)$. Let $\pi \in S_n$, so that $\pi = (\pi', n, \pi'')$.

PROPOSITION 2.11. *For any nonempty* $\pi \in \mathcal{S}(1\text{-}3\text{-}2)$

$$\nu_j(\pi) = \nu_j(\pi') + \nu_j(\pi'') + \nu_{j-1}(\pi'),$$

where $j \neq 2$. *Besides, if* π'' *is nonempty then*

$$\nu_2(\pi) = \nu_2(\pi') + \nu_2(\pi'') + 1,$$

otherwise

$$\nu_2(\pi) = \nu_2(\pi'').$$

PROOF. Similar to Proposition 2.3 we get π avoids 1-3-2 if and only if π' is a 1-3-2-avoiding permutation of the numbers $\{n-l+1, n-l+2, \dots, n-1\}$, while π'' is a 1-3-2-avoiding permutation of the numbers $\{1, 2, \dots, n-l\}$. Finally, if $(i_1, \dots, i_j)$ is an occurrence of 21-3-…-j in π then either $i_j < l$ and so it is also an occurrence of 21-3-…-j in π', or $i_1 > l$ and so it is also an occurrence of 21-3-…-j in π'', or $i_j = l$ and so $(i_1, \dots, i_{j-1})$ is an occurrence of 21-3-…-$(j-1)$ in π', where $j \neq 2$. For $j = 2$ the proposition is trivial. □

Now we are able to find the recurrence relation for the total weight V. Proposition 2.11 yields

$$\begin{aligned} V_k(q_1, \dots, q_k) &= 1 + \sum_{\varnothing \neq \pi \in \mathcal{S}(1\text{-}3\text{-}2)} \prod_{j=1}^{k} q_j^{\nu_j(\pi)} \\ &= 1 + \sum_{\varnothing \neq \pi'' \in \mathcal{S}(1\text{-}3\text{-}2)} \cdot \\ &\quad \sum_{\pi' \in \mathcal{S}(1\text{-}3\text{-}2)} \prod_{j=1}^{k} q_j^{\nu_j(\pi'')} \cdot q_1^{\nu_1(\pi')+1} q_2 \cdot \prod_{j=2}^{k-1} (q_j q_{j+1})^{\nu_j(\pi')} \cdot q_k^{\nu_k(\pi')} \\ &\quad + \sum_{\pi' \in \mathcal{S}(1\text{-}3\text{-}2)} q_1 q_1^{\nu(\pi')} q_k^{\nu_k(\pi')} \prod_{j=2}^{k-1} (q_j q_{j+1})^{\nu_j(\pi')}. \end{aligned}$$

Hence

$$\begin{aligned} V_k(q_1, \dots, q_k) = 1 &+ q_1 V_k(q_1, q_2 q_3, \dots, q_{k-1} q_k, q_k) \\ &+ q_1 q_2 V_k(q_1, q_2 q_3, \dots, q_{k-1} q_k, q_k)(V_k(q_1, q_2, \dots, q_k) - 1). \end{aligned} \tag{6}$$

Observe now that $V_k(q_1, \dots, q_k) = V_k(q_1, \mathbf{q}^{0,2}, \dots, \mathbf{q}^{0,k})$ and by (6) and Proposition 2.4 we get

$$\begin{aligned} V_k(q_1, \mathbf{q}^{d,2}, \dots, \mathbf{q}^{d,k}) = 1 &+ q_1 V_k(q_1, \mathbf{q}^{d+1,2}, \dots, \mathbf{q}^{d+1,k}) \\ &+ q_1 q^{d,2} V_k(q_1, q^{d+1,2}, \dots, \mathbf{q}^{d+1,k})(V_k(q_1, \mathbf{q}^{d,2}, \dots, \mathbf{q}^{d,k}) - 1). \end{aligned}$$

To obtain the continued fraction representation for $F_{21\text{-}3\text{-}\dots\text{-}k}(x, y)$ it is sufficient to use Proposition 2.10 and to observe that

$$q_1 \mathbf{q}^{d,2} \Big|_{q_1 = x, q_2 = \dots = q_{k-1} = 1, q_k = y} = x y^{\binom{d}{k-2}}.$$

COROLLARY 2.12.

$$F_{21}(x, y) = \frac{1 - x + xy - \sqrt{(1-x)^2 - 2x(1+x)y + x^2 y^2}}{2xy},$$

in other words, for any $r \geq 1$

$$f_{21;r}(n) = \frac{r+1}{n(n-r)} \binom{n}{r+1}^2.$$

PROOF. For $k = 2$, $q_1 = x$, and $q_2 = y$; Proposition 2.10 and (6) yields $F_{21}(x, y) = 1 + x F_{21}(x, y) + xy F_{21}(x, y)(F_{21}(x, y) - 1)$, which means $F_{21}(x, y) = F_{12}(x, y)$. Using Corollary 2.5 we have the desired result. □

Now, we are ready to find an explicit expression for $F_{21\text{-}3\text{-}\dots\text{-}k;r}(x)$ where $0 \le r \le k-2$.

Consider a recurrence relation

$$T'_j = 1 - \frac{x}{x - \frac{1}{T'_{j-1}(x)}}, \qquad j \ge 1. \tag{7}$$

The solution of (7) with the initial condition $T'_0 = 0$ is given by $R_j(x)$ (Lemma 2.13), and the solution of (7) with the initial condition

$$T'_0 = G_{21\text{-}3\text{-}\dots\text{-}k}(x,y) = \cfrac{1}{xy^{\binom{k-2}{k-2}} - \cfrac{1}{1 - \cfrac{x}{xy^{\binom{k-1}{k-2}} - \cfrac{1}{1 - \cfrac{x}{xy^{\binom{k}{k-2}} - \cfrac{1}{\ddots}}}}}},$$

is denoted by $S'_j(x,y;k)$, or just S'_j when the value of k is clear from the context. Our interest in (7) is stipulated by the following relation, which is an easy consequence of Theorem 2.9:

$$F_{21\text{-}3\text{-}\dots\text{-}k}(x,y) = S'_k(x,y;k). \tag{8}$$

First of all, we find an explicit formula for the functions $T'_j(x)$ in (7).

LEMMA 2.13. *For any $j \ge 1$,*

$$T'_j(x) = R_j(x). \tag{9}$$

PROOF. Indeed, it follows immediately from (7) that $T'_0(x) = 0$ and $T'_1(x) = 1$. Let us induce, we assume $T'_{j-1}(x) = R_{j-1}(x)$, and prove that $T'_j(x) = R_j(x)$. By use of (7)

$$T'_j(x) = 1 - \frac{x}{x - \frac{1}{R_{j-1}(x)}}.$$

On the other hand, following [12], $R_j(x) = \frac{1}{1-xR_{j-1}(x)}$ which means that $R_j(x) = 1 + xR_{j-1}(x)R_j(x)$, hence $T'_j(x) = R_j(x)$. □

Next, we find an explicit expression for S'_j in terms of G and R_j.

LEMMA 2.14. *For any $j \ge 2$ and any $k \ge 2$*

$$S'_j(x,y;k) = R_j(x)\frac{1 - xR_{j-1}(x)G_{21\text{-}3\text{-}\dots\text{-}k}(x,y;k)}{1 - xR_j(x)G_{21\text{-}3\text{-}\dots\text{-}k}(x,y)}. \tag{10}$$

As a corollary from Lemma 2.14 and (6) we get the following expression for the generating function $F_{21\text{-}3\text{-}\dots\text{-}k}(x,y)$.

COROLLARY 2.15. *For any $k \ge 3$*

$$F_{21\text{-}3\text{-}\dots\text{-}k}(x,y) = R_k(x) + \big(R_{k-2}(x) - R_{k-3}(x)\big)\sum_{m\ge1}\big(xR_{k-2}(x)G_{21\text{-}3\text{-}\dots\text{-}k}(x,y)\big)^m.$$

Now we are ready to express the generating function $F_{21\text{-}3\text{-}\dots\text{-}k;r}(x)$ where $0 \le r \le k-2$, via Chebyshev polynomials.

THEOREM 2.16. *For any $k \geq 3$, $F_{21\text{-}3\text{-}\dots\text{-}k;r}(x)$ is a rational function given by*

$$F_{21\text{-}3\text{-}\dots\text{-}k;r}(x) = \frac{x^{\frac{r-1}{2}} U_{k-2}^{r-1}\left(\frac{1}{2\sqrt{x}}\right)}{U_k^{r+1}\left(\frac{1}{2\sqrt{x}}\right)}, \quad 1 \leq r \leq k-2,$$
$$F_{21\text{-}3\text{-}\dots\text{-}k;0}(x) = \frac{U_{k-1}\left(\frac{1}{2\sqrt{x}}\right)}{\sqrt{x}U_k\left(\frac{1}{2\sqrt{x}}\right)},$$

where U_j is the jth Chebyshev polynomial of the second kind.

PROOF. Observe that $G_{21\text{-}3\text{-}\dots\text{-}k}(x, y) = 1 + \frac{x}{1-x-xy} + y^{k-1}P(x, y)$, so by Corollary 2.15 we get

$$F_{21\text{-}3\text{-}\dots\text{-}k}(x, y) = R_k(x) + \big(R_{k-2}(x) - R_{k-3}(x)\big)\sum_{m=1}^{k}\left(xR_{k-2}(x)\left(1 + \frac{x}{1-x-xy}\right)\right)^m + y^{k-1}P'(x, y),$$

where $P(x, y)$ and $P'(x, y)$ are formal power series. To complete the proof, it suffices to use (9) together with the identity $U_{n-1}^2(z) - U_n(z)U_{n-2}(z) = 1$. □

REMARK 2.17. Theorem 2.16 and [12] yield the number of 1-3-2-avoiding permutations in S_n such that contain exactly r times the pattern 21-3-…-k is the same number of 1-3-2-avoiding permutations in S_n such that contain exactly r times the pattern 1-2-3$\cdots$-k, for all $r = 0, 1, 2, \dots, k-2$. However, the question is if there exists a natural bijection between the set of 1-3-2-avoiding permutations in S_n such that contain exactly r times the generalized pattern 21-3-…-k, and the set of 1-3-2-avoiding permutations in S_n such that contain exactly r times the classically pattern 1-2-3-…-k.

2.3. Patterns: $\tau = 12\dots k$ and $\tau = k\dots 21$. Let $\pi \in S_n$; we say π has *d-increasing canonical decomposition* if π has the following form

$$\pi = (\pi^1, \pi^2, \dots, \pi^d, a_d, \dots, a_2, a_1, n, \pi^{d+1}),$$

where all the entries of π^i are greater than all the entries of π^{i+1}, and $a_d < a_{d-1} < \cdots < a_1 < n$. We say π has *d-decreasing canonical decomposition* if π has the following form

$$\pi = (\pi^1, n, a_1, \dots, a_d, \pi^{d+1}, \pi^d, \dots, \pi^d),$$

where all the entries of π^i are greater than all the entries of π^{i+1}, and $a_d < a_{d-1} < \cdots < a_1 < n$. The following proposition is the basis of all other results in this section.

PROPOSITION 2.18. *Let $\pi \in S_n$(1-3-2). Then there exists unique $d \geq 0$ and $e \geq 0$ such that π has a d-increasing canonical decomposition, and has e-decreasing canonical decomposition.*

PROOF. Let $\pi \in S_n$(1-3-2), and let $a_d, a_{d-1}, \dots, a_1, n$ a maximal increasing subsequence of π such that $\pi = (\pi', a_d, \dots, a_1, n, \pi'')$. Since π avoids 1-3-2 there exists d subsequences π^j such that $\pi = (\pi^1, \dots, \pi^d, a_d, \dots, a_1, n, \pi'')$, and all the entries of π^i are greater than all the entries of π^{i+1}, and all the entries of π^d are greater than all entries of π''. Hence, π has d-increasing canonical decomposition. Similarly, there exist e unique such that π is e-decreasing canonical decomposition. □

Let us define $I_\tau(x,y;d)$ (respectively, $J_\tau(x,y;e)$) as the generating function for all d-increasing (respectively, e-decreasing) canonical decomposition of permutations in $S_n(1\text{-}3\text{-}2)$ with exactly r occurrences of τ. The following proposition is implied immediately by the definitions.

PROPOSITION 2.19.

$$F_\tau(x,y) = 1 + \sum_{d\geq 0} I_\tau(x,y;d) = 1 + \sum_{e\geq 0} J_\tau(x,y;e).$$

PROOF. Immediately, by definitions of the generating functions and Proposition 2.18 (1 for the empty permutation). □

Now, we present examples for Propositions 2.18 and 2.19.

First example

THEOREM 2.20. $F_{k\dots 21}(x,y) = F_{12\dots k}(x,y)$, *such that*

$$F_{12\dots k}(x,y) = \sum_{n=0}^{k-2} x^n F^n_{12\dots k}(x,y) + \frac{x^{k-1}F^{k-1}_{12\dots k}(x,y)}{1 - xyF_{12\dots k}(x,y)}.$$

PROOF. By Proposition 2.18 and definitions it is easy to obtain for all $d \geq 0$

$$I_{12\dots k}(x,y;d) = x^{d+1}y^{s_d}F^{d+1}_{12\dots k}(x,y),$$

where $s_d = d+1-k$ for $d \geq k-1$, and otherwise $s_d = 0$. So by Proposition 2.19 the theorem holds.

Similarly, we obtain the same result for $F_{k\dots 21}(x,y)$. □

As a remark, by the above theorem, it is easy to obtain the same results for Corollaries 2.5 and 2.12.

Second example

THEOREM 2.21. *Let* $1 \leq l \leq k-1$. *Then* $F_{1\text{-}2\text{-}\dots\text{-}(l-1)\text{-}l(l+1)\dots k}(x,y) = U_l(x,1,\dots,1,y)$ *where*

$$U_l(q_1,\dots,q_l) = 1 + \sum_{d\geq 0}\left(q_l^{\binom{d+1+l-k}{l}} \prod_{j=1}^{l-1} q_j^{\binom{d+1}{j}} \prod_{j=0}^{d} U_l(p_{1;j},\dots,p_{l;j})\right),$$

and for $i = 1,2,\dots,l$, $p_{i;j} = \prod_{m=1}^{l-1} q_j^{\binom{j}{m-i}}$, $p_{l,j} = q_l$ *for all* $0 \leq j \leq k-l$, *and* $p_{i;j} = \prod_{m=1}^{l} p_{i;k-l}^{\binom{j-k+l}{l-i}}$ *for all* $j \geq k-l+1$.

PROOF. Following [12] we define $\gamma_j(\pi)$, $j \leq l-1$, as the number of occurrences of $1\text{-}2\text{-}\dots\text{-}j$ in π. Define $\gamma_l(\pi)$ for any π, as the number of occurrences of $1\text{-}2\text{-}\dots\text{-}(l-1)\text{-}l(l+1)\dots k$ in π, and $\gamma_0(\pi) = 1$ for any π, which means that the empty pattern occurs exactly once in each permutation. The *weight* of a permutation π is a monomial in l independent variables $q_1,\dots,q_l$ defined by

$$u_l(\pi) = \prod_{j=1}^{l} q_j^{\gamma_j(\pi)}.$$

The *total weight* is a polynomial

$$U_l(q_1,\dots,q_l)=\sum_{\pi\in\mathcal{S}(1\text{-}3\text{-}2)} u_l(\pi).$$

The following proposition is implied immediately by the definitions and Proposition 2.18. □

PROPOSITION 2.22. *$F_{1\text{-}2\text{-}\dots\text{-}(l-1)\text{-}l(l+1)\dots k}(x,y)=U_k(x,1,\dots,1,y)$ for $k>l\geq 1$, and $U_l(q_1,\dots,q_l)=1+\sum_{d\geq 0}\sum_{\pi\in A_d}u_l(\pi)$, where A_d is the set of all d-increasing canonical decomposition permutations in $\mathcal{S}(1\text{-}3\text{-}2)$.*

Let us denote $U_{l;d}(q_1,\dots,q_l)=\sum_{\pi\in A_d}u_l(\pi)$.

PROPOSITION 2.23. *For any $d\geq 0$,*

$$U_{l;d}(q_1,\dots q_l)=q_l^{\binom{d+1+l-k}{l}}\prod_{j=1}^{l-1}q_j^{\binom{d+1}{j}}\prod_{j=0}^{d}U_l(p_{1;j},\dots,p_{l;j}).$$

PROOF. Let π be d-increasing canonical decomposition, that is,

$$\pi=(\pi^1,\pi^2,\dots,\pi^d,a_d,\dots,a_2,a_1,n,\pi^{d+1}),$$

where the numbers $a_d<a_{d-1}<\cdots<a_1<n$ appear as consecutive numbers in π, all entries of π^j are greater than all the entries of π^{j+1}, and all entries of π^d are greater than a_d. So, by calculating $u_l(\pi)$ and summing over all $\pi\in A_d$ we have that

$$U_{l;d}(q_1,\dots,q_d)=q_l^{\binom{d+1+l-k}{l}}\cdot\prod_{j=1}^{l-1}q_j^{\binom{d+1}{j}}\cdot\prod_{j=0}^{d}U_l(p_{1;j},\dots,p_{l;j}).$$

□

Therefore, Theorem 2.21 holds, by using Propositions 2.22 and 2.23. □

Now, let $l=k-1$ and by using Theorem 2.21, it is easy to obtain the following.

COROLLARY 2.24. *For $k\geq 3$,*

$$F_{1\text{-}2\text{-}\dots\text{-}(k-2)\text{-}(k-1)k}(x,y)=\sum_{j=0}^{k-1}(xF_{1\text{-}2\text{-}\dots\text{-}(k-2)\text{-}(k-1)k}(x,y))^j.$$

REMARK 2.25. Similarly, the argument of d-increasing canonical decomposition, or the argument d-decreasing canonical decomposition yields other formulae, for example, the formula for $F_{12\text{-}3\text{-}45}(x,y)$.

3. THREE LETTERS PATTERN WITHOUT INTERNAL DASHES

In this section, we give a complete answer for $F_\tau(x,y)$ where τ is a generalized pattern without internal dashes; that is, τ is 123, 213, 231, 312, and 321, by the following four subsections.

3.1. Patterns 123 *and* 321.

THEOREM 3.1.

$$F_{123}(x,y) = F_{321}(x,y) = \frac{1 + xy - x - \sqrt{1 - 2x - 3x^2 - xy(2 - 2x - xy)}}{2x(x + y - xy)}.$$

PROOF. Theorem 2.20 yields, $F_{123}(x,y) = F_{321}(x,y) = H$ where

$$H = 1 + xH + \frac{x^2H^2}{1 - xyH},$$

so the theorem holds. □

3.2. Pattern 231.

THEOREM 3.2.

$$F_{231}(x,y) = \frac{1 - 2x + 2xy - \sqrt{1 - 4x + 4x^2 - 4x^2y}}{2xy},$$

that is, for all $r, n \geq 0$

$$F_{231;r}(x) = \frac{1}{r+1}\binom{2r}{r}\frac{x^{2r+1}}{(1-2x)^{2r+1}}, \qquad f_{231;r}(n) = \frac{2^{n-2r-1}}{r+1}\binom{n-1}{2r}\binom{2r}{r}.$$

PROOF. Let $l = \pi^{-1}(n)$. Since π avoids 1-3-2, each number in π' is greater than any of the numbers in π''. Therefore, π' is a 1-3-2-avoiding permutation of the numbers $\{n-l+1, n-l+2, \ldots, n-1\}$, while π'' is a 1-3-2-avoiding permutation of the numbers $\{1, 2, \ldots, n-l\}$. On the other hand, if π' is an arbitrary 1-3-2-avoiding permutation of the numbers $\{n-l+1, n-l+2, \ldots, n-1\}$ and π'' is an arbitrary 1-3-2-avoiding permutation of the numbers $\{1, 2, \ldots, n-l\}$, then $\pi = (\pi', n, \pi'')$ is 1-3-2-avoiding.

Now let us observe all the possibilities that π' and π'' is empty or not. This yields

$$F_{231}(x,y) = 1 + x + 2x(F_{231}(x,y) - 1) + xy(F_{231}(x,y) - 1)^2,$$

hence the theorem holds. □

3.3. Pattern 213.

THEOREM 3.3.

$$F_{213}(x,y) = \frac{1 - x^2 + x^2y - \sqrt{1 + 2x^2 - 2x^2y + x^4 - 2x^4y + x^4y^2 - 4x}}{2x(1 + xy - x)}.$$

PROOF. Let $D(x,y)$ be the generating function of all 1-3-2-avoiding permutations $(\alpha', n) \in S_n$ such that contain 213 exactly r times. Let $\alpha = (\alpha', n, \alpha'')$; if we consider the two cases α' empty or not we have $F_{213}(x,y) = 1 + D(x,y)F_{213}(x,y)$. Let $\alpha = (\alpha', n)$; if we observe the two cases α' empty or not, then (similarly)

$$D(x,y) = x + x^2 + x^2y(F_{213}(x,y) - 1) + x^2(D(x,y) - 1) + x^2(D(x,y) - 1)(F_{213}(x,y) - 1).$$

However,

$$F_{213}(x,y) = 1 + xF_{213}(x,y)\frac{1 + x - xy + x(y-1)F_{213}(x,y)}{1 - xF_{213}(x,y)},$$

hence, the theorem holds. □

3.4. *Pattern* 312.

THEOREM 3.4.

$$F_{312}(x, y) = \frac{1 - x^2 + x^2y - \sqrt{1 + 2x^2 - 2x^2y + x^4 - 2x^4y + x^4y^2 - 4x}}{2x(1 + xy - x)}.$$

PROOF. Let $\alpha \in \mathcal{S}(1\text{-}3\text{-}2)$; if $\alpha = \varnothing$, then there is one permutation, otherwise by Proposition 2.18 we can write $\alpha = (\alpha^1, n, a_1, a_2, \dots, a_d, \alpha^{d+1}, \alpha^d, \dots, \alpha^2)$ where all the entries of α^j are greater than all the entries of α^{j+1}, and $n > a_1 > a_2 > \cdots > a_d$. Hence, for any $d = 0, 1$ the generating function of these permutations in these cases is $x^{d+1}F_{312}(x, y)$. Let $d \geq 2$; if $\alpha^{d+1} = \varnothing$, then the generating function of these permutations in this case is $x^{d+1}F_{312}^d(x, y)$, otherwise the generating function is $x^{d+1}yF_{312}^d(x, y)(F_{312}(x, y) - 1)$. Hence

$$F_{312}(x, y) = 1 + (x + x^2)F_{312}(x, y) + \sum_{d\geq 2} x^{d+1}F_{312}^d(x, y) + \sum_{d\geq 2} x^{d+1}yF_{312}^d(x, y)(F_{312}(x, y) - 1),$$

which means that

$$F_{312}(x, y) = 1 + xF_{312}(x, y) + \frac{x^2F_{312}(x, y)}{1 - xF_{312}(x, y)} + \frac{x^2yF_{312}(x, y)(F_{312}(x, y) - 1)}{1 - xF_{312}(x, y)},$$

so the rest is easy to see. □

4. THREE LETTERS PATTERN WITH ONE DASH

In this section, we present examples $F_\tau(x, y)$ where τ is a generalized pattern with one dash. Theorem 2.1 yields

THEOREM 4.1. *The generating function $F_{12\text{-}3}(x, y)$ is given by the continued fraction*

$$\cfrac{1}{1 - \cfrac{x}{1-x+xy - \cfrac{xy}{1-x+xy^2 - \cfrac{xy^2}{\ddots}}}}.$$

Theorem 2.9 yields

THEOREM 4.2. *For any $k \geq 2$,*

$$F_{21\text{-}3}(x, y) = 1 - \cfrac{x}{x - \cfrac{1}{1 - \cfrac{x}{xy - \cfrac{1}{1 - \cfrac{x}{xy^2 - \cfrac{1}{\ddots}}}}}}.$$

For $k = 3$ and $l = 2$ Theorem 2.24 yields

THEOREM 4.3.

$$F_{1\text{-}23}(x,y)=1+xF_{1\text{-}23}(x,y)+\sum_{d\geq1}x^{d+1}y^{\binom{d}{2}}F_{1\text{-}23}(x,y)\prod_{j=0}^{d-1}F_{1\text{-}23}(xy^j,y).$$

COROLLARY 4.4.

$$F_{1\text{-}23;0}(x)=\frac{1-x-\sqrt{1-2x-3x^2}}{2x^2};$$
$$F_{1\text{-}23;1}(x)=\frac{x-1}{2x}+\frac{1-2x-x^2}{2x\sqrt{1-2x-3x^2}};$$
$$F_{1\text{-}23;2}(x)=\frac{x^4}{(1-2x-3x^2)^{3/2}};$$
$$F_{1\text{-}23;3}(x)=x^2-1+\frac{11x^7+43x^6+41x^5-7x^4-25x^3+x^2+5x-1}{(1-2x-3x^2)^{5/2}}.$$

PROOF. By Theorem 4.3 and by $F_{1-23}(x,0)=F_{1-23;0}(x)$ we get

$$F_{1\text{-}23;0}(x)=1+xF_{1\text{-}23;0}(x)+x^2F^2_{1\text{-}23;0}(x),$$

which means the first formula holds.

By Theorem 4.3 we get

$$\frac{d}{dy}F_{1\text{-}23}(x,0)=x\frac{d}{dy}F_{1\text{-}23}(x,0)+2x^2F_{1\text{-}23}(x,0)\frac{d}{dy}F_{1\text{-}23}(x,0)+x^3F_{1\text{-}23}(x,0)^2F_{1\text{-}23}(0,0),$$

and by $F_{1\text{-}23;1}(x)=\frac{d}{dy}F_{1\text{-}23}(x,y)\big|_{y=0}$ and the first formula, we get the second formula.

Similarly, by Theorem 4.3 and by $F_{1\text{-}23;r}(x)=\frac{1}{r!}\frac{d^r}{dy^r}F_{1\text{-}23}(x,y)\big|_{y=0}$ the other formulae holds. □

THEOREM 4.5.

$$F_{2\text{-}13}(x,y)=\cfrac{1}{1-\cfrac{x}{1-\cfrac{x}{1-\cfrac{xy}{1-\cfrac{xy}{1-\cfrac{xy^2}{1-\cfrac{xy^2}{1-\ddots}}}}}}}.$$

PROOF. By Propositions 2.18 and 2.19, we obtain

$$F_{2\text{-}13}(x,y)=1+xF_{2\text{-}13}(x,y)\sum_{d\geq0}x^dF^d_{2\text{-}13}(xy,y),$$

and the rest is easy to see. □

5. FURTHER RESULTS

First of all, let us denote by $G_{\tau;\phi}(x,y)$ the generating function for the number of permutations in $S_n(1\text{-}3\text{-}2,\tau)$ such that contain ϕ exactly r times; that is

$$G_{\tau;\phi}(x,z)=\sum_{n\geq0}x^n\sum_{\pi\in S_n(1\text{-}3\text{-}2,\tau)}y^{a_\phi(\pi)},$$

where $a_\phi(\pi)$ is the number of occurrences of ϕ in π. In this section, (similar to previous sections) we find $G_{\tau;\phi}(x, y)$ in terms of continued fractions or by explicit formulae, for some cases of τ and ϕ.

THEOREM 5.1. *The generating functions $G_{123;213}(x, y)$ and $G_{321;312}(x, y)$ are given by*

$$\cfrac{1}{1-x-x^2(1-y)-\cfrac{x^2y}{1-x-x^2(1-y)-\cfrac{x^2y}{1-x-x^2(1-y)-\cfrac{x^2y}{\ddots}}}},$$

equivalently,

$$\frac{1-x-x^2+x^2y-\sqrt{(1-x-x^2)^2-2yx^2(1+x+x^2)+x^4y^2}}{2x^2y}.$$

THEOREM 5.2.

$$G_{123;231}(x, y) = H(x, y) + x^2(1-y)H(x, y)^2,$$

where $H(x, y) = \frac{1}{1-x-x^2yH(x,y)}$, which means the number of permutations in S_n(1-3-2, 123) such that contain 231 exactly $r \geq 0$ times is given by

$$(C_{r+1} - C_r)\binom{n-1}{2r+1} + C_r\binom{n}{2r+1},$$

where C_m is the mth Catalan number.

THEOREM 5.3. *The generating functions $G_{213;123}(x, y)$ and $G_{312;321}(x, y)$ are given by*

$$\frac{1-x-x^2+xy-\sqrt{(1-x-x^2)^2-2xy(1-x+x^2)+x^2y^2}}{2xy(1-x)}.$$

As a concluding remark we note that there are many questions left to answer such as: if there exists a bijection between, for example, the set of 1-3-2-avoiding permutations in S_n such that contain exactly r times the generalized pattern 21-3-...-k, and the set of 1-3-2-avoiding permutations in S_n such that contain exactly r times the classical pattern 1-2-3-...-k, where $r = 0, 1, \ldots, k-2$.

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