# A LIMITED ARITHMETIC ON SIMPLE CONTINUED FRACTIONS - II

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## 1. INTRODUCTION

In the first paper [2] in this series, we developed certain properties of the simple continued fraction expansions of integral multiples of quadratic surds with expansions of the form $[a,\dot{b}]$ or $[a,\dot{b},\dot{c}]$ where the notation is that of Hardy and Wright [1, Chapter 10]. For easy reference, we restate the principle results here.

Theorem 1. Let $\zeta = [a,\dot{b}]$, let n be a positive integer, let $p_k/q_k$ denote the $k^{th}$ convergent to $\zeta$ and let $t_k = q_{k-1} + q_{k+1}$ for $k \geq 0$ where we take $q_{-1} = 0$. Then $n\zeta = [r,\dot{s}]$ if and only if $n = q_{2m-2}$, $r = p_{2m-2}$, and $s = t_{2m-2}$ for some $m \geq 1$.

Theorem 2. Let $\zeta$, n, $p_k/q_k$ and $t_k$ be as in Theorem 1. Then $n\zeta = [u,\dot{v},\dot{w}]$ if and only if $vn = q_{2m-1}$, $vu = p_{2m-1} - 1$, and $vw = t_{2m-1} - 2$ for some integer $m \geq 1$.

Theorem 3. Let $\zeta = [a,\dot{b},\dot{c}]$, let $p_k/q_k$ be the $k^{th}$ convergent to $\zeta$, let $t_k = q_{k-1} + q_{k+1}$ and $s_k = p_{k-1} + p_{k+1}$ for $k \geq 1$. Then, for every integer $r \geq 1$, we have

$$q_{2r} \cdot \zeta = [p_{2r}, \dot{t}_{2r}, c\dot{t}_{2r}/b],$$
$$q_{2r-1} \cdot \zeta = [p_{2r-1} - 1, \dot{1}, \dot{t}_{2r-1} - 2]$$
$$t_{2r-1} \cdot \zeta = [s_{2r-1}, \dot{q}_{2r-1}, (c^2 + 4\dot{c}/b)q_{2r-1}]$$

and

$$t_{2r} \cdot \zeta = [s_{2r} - 1, \dot{1}, q_{2r} - 2, 1, (bc + 4)\dot{q}_{2r} - 2].$$

Of course, for $a = b = c = 1$, the preceding theorems give results involving the golden ratio, $(1 + \sqrt{5})/2$, and the Fibonacci and Lucas numbers since, in that case,

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$\zeta = (1 + \sqrt{5})/2$, $p_k = F_{k+2}$, $q_k = F_{k+1}$, $t_k = L_{k+1}$, and $s_k = L_{k+2}$

where $F_n$ and $L_n$ denote respectively the $n^{th}$ Fibonacci and Lucas numbers.

In the present paper, we devote our attention primarily to the study of the simple continued fraction expansions of positive rational multiples of quadratic surds with expansions of the form $[\dot{a}]$. Again, we note that, for $a = 1$, the theorems specialize to results about the golden ratio and Fibonacci and Lucas numbers.

## 2. PRELIMINARY CONSIDERATIONS

Let the integral sequences $\{f_n\}_{n\geq 0}$ and $\{g_n\}_{n\geq 0}$ be defined as follows:

(1) $$f_0 = 0,\quad f_1 = 1,\quad f_n = af_{n-1} + f_{n-2},\quad n \geq 0\ ,$$

and

(2) $$g_0 = 2,\quad g_1 = a,\quad g_n = ag_{n-1} + g_{n-2},\quad n \geq 0\ ,$$

where $a$ is any positive integer. These difference equations are easily solved to give

(3) $$f_n = \frac{\xi^n - \bar{\xi}^n}{\sqrt{a^2 + 4}}\ ,\quad n \geq 0\ ,$$

and

(4) $$g_n = \xi^n + \bar{\xi}^n\ ,\quad n \geq 0\ ,$$

where

$$\xi = (a + \sqrt{a^2 + 4})/2 \quad \text{and} \quad \bar{\xi} = (a - \sqrt{a^2 + 4})/2$$

are the two irrational roots of the equation

(5) $$x^2 - ax - 1 = 0\ .$$

Incidentally, if $\beta$ is a quadratic surd, we will always denote the conjugate surd by $\bar{\beta}$. The following formulas, of interest in themselves, generalize results for the Fibonacci and Lucas numbers and are easily proved by induction.

(6)
$$f_{2n} = \sum_{i=0}^{n-1} \binom{n+i}{2i+1} a^{2i+1}, \qquad f_{2n+1} = \sum_{i=0}^{n} \binom{n+i}{2i} a^{2i}, \qquad n \ge 0,$$

(7)
$$g_n = f_{n-1} + f_{n+1}, \quad n \ge 1,$$

(8)
$$f_{m+n+1} = f_m f_n + f_{m+1} f_{n+1}, \quad m \ge 0, \quad n \ge 0,$$

(9)
$$g_{m+n+1} = f_m g_n + f_{m+1} g_{n+1}, \quad m \ge 0, \quad n \ge 0,$$

(10)
$$f_m f_n - f_{m-1} f_{n+1} = (-1)^{m-1} f_{m-n+1}, \quad 1 \le m \le n.$$

Also, we obtain in the usual way from (8) the following lemma.

Lemma 4. For the integral sequence $\{f_n\}_{n \ge 0}$ we have that $f_m | f_n$ if and only if $m|n$, where $m$ and $n$ are positive integers and $m > 2$ if $a = 1$.

## 3. PRINCIPAL RESULTS

Our first theorem, together with the results of the first paper in this couplet, yields a series of results concerning the simple continued fraction expansion of multiples of $\xi = [\dot{a}]$ by the reciprocals of positive integers. The theorem is also of some interest in its own right.

Theorem 5. Let $\zeta = (a + b\sqrt{c})/d$ with $a, b, c,$ and $d$ integers, $c$ not a perfect square, and $c$ and $d$ positive. Let $r$ be a positive rational number such that $2ar/d$ is an integer. Let $a^2 + d^2 = b^2c$ and let $1 < \zeta \le r$. Then

$$r\zeta = [a_0, a_1, a_2, \cdots],$$

if and only if

$$\frac{\zeta}{r} = \left[0,\ a_0 - \frac{2ar}{d},\ a_1,\ a_2,\ \cdots\right] .$$

Proof. We note first that $a_0 - 2ar/d$ is positive. This is so since

$$a_0 = \left[\frac{ar + rb\sqrt{c}}{d}\right] > \frac{ra + rb\sqrt{c}}{d} - 1$$

so that

$$a_0 - \frac{2ar}{d} > \frac{-ra + rb\sqrt{c}}{d} - 1$$

$$= \frac{r}{d}\left(\frac{a^2 - b^2c}{-a - b\sqrt{c}}\right) - 1$$

$$= \frac{rd}{a + b\sqrt{c}} - 1$$

$$= \frac{r}{\zeta} - 1 > 0 \ ,$$

by hypothesis. Now let $\mu = [a_1, a_2, a_3, \cdots]$ so that

$$r\zeta = a_0 + \frac{1}{\mu} .$$

Then

$$\left[0,\ a_0 - \frac{2ar}{d},\ a_1,\ a_2,\ \cdots\right] = \frac{1}{a_0 - \frac{2ar}{d} + \frac{1}{\mu}}$$

$$= \frac{1}{r\zeta - \frac{2ar}{d}}$$

$$= \frac{1}{r\left(\frac{a + b\sqrt{c}}{d} - \frac{2a}{d}\right)}$$

$$= \frac{d}{r(-a + b\sqrt{c})}$$

$$= \frac{-d(a + b\sqrt{c})}{r(a^2 - b^2c)}$$

$$= \frac{a + b\sqrt{c}}{dr}$$

$$= \frac{\zeta}{r} \ ,$$

and the proof is complete.

Corollary 6. Let a and n be positive integers. Let $\xi = [\dot{a}]$ and let $n > \xi$. Then

$$n\xi = [a_0, a_1, a_2, \cdots]$$

if and only if

$$\frac{\xi}{n} = [0, a_0, -an, a_1, a_2, \cdots] .$$

Proof. Since

$$\xi = [\dot{a}] = \frac{a + \sqrt{a^2 + 4}}{2} ,$$

we may use the preceding theorem with $a = a$, $b = 1$, $c = a^2 + 4$, $d = 2$, and $r = n$. The result then follows immediately since

$$\frac{2ar}{d} = \frac{2an}{2}$$

is an integer and

$$a^2 + d^2 = a^2 + 4 = b^2c ,$$

as required.

Now for

$$\xi = [\dot{a}] = \frac{a + \sqrt{a^2 + 4}}{2} .$$

The convergents $p_k/q_k$ are given by the equations

$$(11) \qquad \begin{aligned} p_0 &= a, \ p_1 = a^2 + 1, \ p_n = ap_{n-1} + p_{n-2} , \\ q_0 &= 1, \ q_1 = a, \ q_n = aq_{n-1} + q_{n-2} , \end{aligned} \qquad n \geq 2 ,$$

and it is clear that $p_n = f_{n+2}$ and $q_n = f_{n+1}$ for $n \geq 0$. Also, $p'_n = f_n$ and $q'_n = f_{n+1}$ for $n \geq 0$, where $p'_n/q'_n$ is the $n^{th}$ convergent to $1/\xi$. The following results could all be stated in terms of the sequences $\{p_n\}$ and $\{q_n\}$; instead, we use the sequences $\{f_n\}$ and $\{g_n\}$.

Corollary 7. Let r, s, and n be the positive integers with $n > \xi = [\dot{a}]$. Then $\xi/n = [0, r, \dot{s}]$, if and only if, $n = f_{2m-1}$, $r = f_{2m}$, and $s = q_{2m-1}$ for some $m \geq 2$.

Proof. This is an immediate consequence of Theorem 1 with a = b, and Corollary 6.

Corollary 8. Let u, v, w, and n be positive integers with $n > \xi = [\dot{a}]$. Then $\xi/n = [0, u, \dot{v}, \dot{w}]$, if and only if, $vn = f_{2m}$, $vu = f_{2m+1} - 1$, and $vw = g_{2m} - 2$ for some integer $m \geq 2$.

Proof. This is an immediate consequence of Corollary 6 and Theorem 2 with u = v = w = a.

The next corollary results from Theorem 3 and Corollary 6 by taking a = b = c. However, since, in this special case, parts (a) and (b) of Theorem 2 yield results already obtained, we concern ourselves only with parts (c) and (d).

Corollary 9. Let n be a positive integer greater than $\xi$. Then for $r \geq 1$,

$$\frac{\xi}{g_{2r}} = [0, g_{2r-1}, \dot{f}_{2r}, (a^2 + 4)\dot{f}_{2r}]$$

and

$$\frac{\xi}{g_{2r+1}} = [0, g_{2r} - 1, \dot{1}, f_{2r+1} - 2, 1, (a^2 + 4)\dot{f}_{2r+1} - 2].$$

The next theorem shows that the periodic part of the simple continued fraction expansion of n for any positive integer $n > \xi = [\dot{a}]$ is almost symmetric. Of course, by Corollary 6, the same thing is true of $\xi/n$.

Theorem 10. Let a and n be positive integers with $n > \xi = [\dot{a}]$. Then $n\xi = [a_0, \dot{a}_1, \cdots, \dot{a}_r]$ and the vector $(a_1, a_2, \cdots, a_{r-1})$ is symmetric if $r \geq 2$.

Proof. Since $a_0 = [n\xi]$, we have that

$$0 < n\xi - a_0 < 1$$

and

$$\xi_1 = \frac{1}{n\xi - a_0} > 1 \quad ,$$

where $\xi_1$ is the first complete quotient in the expansion of n . Moreover,

$$\overline{\xi_1} = \frac{1}{n\overline{\xi} - a_0} = -\frac{1}{\frac{n}{\xi} + a_0}$$

so that

$$-1 < \overline{\xi_1} < 0 \quad ,$$

since $a_0 + n/\xi$ is clearly greater than one. Thus, $\xi_1$ is a reduced quadratic surd and by the general theory (see, for example [3, Chapter 4]) has a purely periodic simple continued fraction expansion, say

$$\xi_1 = [\dot{a}_1, a_2, \cdots, \dot{a}_r] .$$

Additionally, we also have that $[a_r, a_{r-1}, \cdots, a_1]$ is the expansion of the negative reciprocal of the conjugate of $\xi_1$. Thus,

$$[\dot{a}_r, a_{r-1}, \cdots, \dot{a}_1] = -\frac{1}{\overline{\xi_1}} = \frac{n}{\xi} + a_0$$

so that

(12) $$\frac{\xi}{n} = [0, a_r - a_0, \dot{a}_{r-1}, a_{r-2}, \cdots, a_1, \dot{a}_r] .$$

But, from above,

$$n\xi = [a_0, \dot{a}_1, \cdots, \dot{a}_r] ,$$

and by Corollary 6,

$$(13) \qquad \frac{\xi}{n} = [0, a_0 - an, \dot{a}_1, \dot{a}_2, \cdots, \dot{a}_r] .$$

Thus, comparing (12) and (13), we have that the vector $(a_1, a_2, \cdots, a_{r-1})$ is symmetric.

We now turn our attention to the consideration of more general positive rational multiples of $\xi = [\dot{a}]$.

Theorem 11. Let r be rational with $0 < r < 1$. If the simple continued fraction expansion of $r\xi$ is not purely periodic, then

$$r\xi = [0, a_1, \dot{a}_2, \cdots, \dot{a}_n]$$

and

$$\frac{\xi}{r} = [a_n - a_1, \dot{a}_{n-1}, a_{n-2}, \cdots, a_2, \dot{a}_n]$$

for some $n \geq 2$.

Proof. If $r\xi$ had a purely periodic simple continued fraction expansion, then $r\xi$ would have to be a reduced quadratic surd so that $r\xi > 1$ and $-1 < \overline{r\xi} < 0$. But the first of these inequalities implies that

$$(14) \qquad \frac{1}{\xi} < r < 1 ,$$

and, since $\overline{\xi} = -1/\xi$, the second implies that

$$(15) \qquad \xi > r > 0 ,$$

which is already implied by (14). Therefore, since $r\xi$ is not purely periodic, we have

$$0 < r < \frac{1}{\xi} \quad , \tag{16}$$

so that $0 = [r\xi] = a_0$. Now consider

$$\xi_1 = \frac{1}{r\xi} > 1 \quad ,$$

and set $a_1 = [\xi_1] \geq 1$. Again,

$$\frac{1}{\frac{1}{r\xi} - a_1} = \frac{1}{\xi_1 - a_1} = \xi_2 > 1$$

and

$$\bar{\xi}_2 = \frac{1}{\frac{1}{r\bar{\xi}} - a_1} = -\frac{1}{\frac{\xi}{r} + a_1} \quad ,$$

since $\xi\bar{\xi} = -1$. Therefore, $-1 < \bar{\xi}_2 < 0$ and $\xi_2$ is reduced. Thus, $\xi_2$ has a purely periodic simple continued fraction expansion,

$$\xi_2 = [\dot{a}_2, a_3, \cdots, \dot{a}_n] \; ,$$

and

$$r\xi = [0, a_1, \dot{a}_2, \cdots, \dot{a}_n] \; ,$$

as claimed. Also,

$$\frac{\xi}{r} + a_1 = -\frac{1}{\bar{\xi}_2} = [\dot{a}_n, a_{n-1}, \cdots, \dot{a}_2] \quad ,$$

so that

$$\frac{\xi}{r} = [a_n - a_1, \dot{a}_{n-1}, a_{r-2}, \cdots, a_2, \dot{a}_n]$$

and the proof is complete.

Theorem 12. Let r be rational with $0 < r < 1$. If the simple continued fraction expansion of $r\xi$ is purely periodic, then

$$r\xi = [\dot{a}_0, a_1, \cdots, \dot{a}_n]$$

and

$$\frac{\xi}{r} = [\dot{a}_n, a_{n-1}, \cdots, \dot{a}_0]$$

for some $n \geq 0$.

Proof. Since the simple continued fraction expansion of $r\xi$ is purely periodic, it is reduced and we have by the preceding proof that

$$\frac{1}{\xi} < r < 1 .$$

Since we also have

$$\frac{\xi}{r} = -\frac{1}{r\bar{\xi}} = [\dot{a}_n, a_{n-1}, \cdots, \dot{a}_0] ,$$

the proof is complete.

In passing, we note that the periodic part of the expansions of $r\xi$ need not exhibit any symmetry or even near symmetry. For example, for

$$\alpha = [\dot{1}] = \frac{1 + \sqrt{5}}{2} ,$$

we have that

$$\frac{2}{9}\alpha = [0, 2, \dot{1}, 3, 1, 1, 3, \dot{9}]$$

and

$$\frac{3}{4}\alpha = [\dot{1}, 4, 1, 2, 6, \dot{2}] .$$

Also, it is easy to find rational numbers r with $0 < r < 1$ such that the surds $r\alpha$ and $\alpha/r$ are not equivalent where we recall that two real numbers $\mu$ and $\nu$ are said to be equivalent if and only if there exist integers a, b, c, and d with $|ad - bc| = 1$ and such that

$$\mu = \frac{a\nu + b}{c\nu + d} .$$

However, as the following theorems show, there exist interesting examples, where near symmetry of the periodic part of the expansions of $r\xi$ and $\xi/r$ and equivalence of $r\xi$ and $r/\xi$ both hold. We will indicate that $r\xi$ and $\xi/r$ are equivalent by the notation

$$r\xi \sim \frac{\xi}{r} .$$

<u>Theorem 13.</u> Let a be a positive integer, let $\xi = [\dot{a}]$, and let the sequences $\{f_n\}_{n\geq 0}$ and $\{g_n\}_{n\geq 0}$ be as defined above. Then, for $n \geq 1$,

$$\frac{f_{2n+1}}{f_{2n+2}} \cdot \xi = [\dot{a}_0, a_1, \cdots, \dot{a}_r]$$

and

$$\frac{f_{2n+2}}{f_{2n+1}} \cdot \xi = [\dot{a}_r, a_{r-1}, \cdots, \dot{a}_0]$$

where the vector $(a_2, a_3, \cdots, a_r, a_0)$ is symmetric, $a_0 = a_2 = 1$, and $a_1 = f_{4n+3} - 1$.

Proof. We first demonstrate the purely periodic nature of the expansions in question. From the definition, it is clear that $f_n$ is strictly increasing for $n \geq 2$. Also, $f_n / f_{n+1}$ is the $n^{th}$ convergent to $1/\xi$. Therefore,

$$\frac{f_{2n}}{f_{2n+1}} < \frac{1}{\xi} < \frac{f_{2n+1}}{f_{2n+2}} < 1 \quad , \tag{17}$$

and it follows from the proof of Theorem 11 that $\xi f_{2n+1}/f_{2n+2}$ has a purely periodic expansion. Also, from Theorem 12, $\xi f_{2n+2}/f_{2n+1}$ has a purely periodic expansion whose period is the reverse of that for $\xi f_{2n+1}/f_{2n+2}$.

Additionally, from (17), it follows that

$$0 < \frac{f_{2n+1}}{f_{2n+2}} - \frac{1}{\xi} < \frac{1}{2}\left(\frac{f_{2n+1}}{f_{2n+2}} - \frac{f_{2n}}{f_{2n+1}}\right) = \frac{1}{2f_{2n+1}f_{2n+2}}$$

so that

$$1 < \frac{f_{2n+1}}{f_{2n+2}} \cdot \xi < \frac{\xi}{2f_{2n+1}f_{2n+2}} + 1$$

$$< \frac{f_{2n+1}}{2f_{2n+1}f_{2n+2}} + 1 < 2 \, .$$

Thus, $a_0 = [\xi f_{2n+1}/f_{2n+2}] = 1$. Now

$$\xi_1 = \frac{1}{\dfrac{f_{2n+1}}{f_{2n+2}} \cdot \xi - 1} \quad ,$$

and we claim that

$$f_{4n+3} - 1 < \xi_1 < f_{4n+3} \quad ,$$

so that $a_1 = f_{4n+3} - 1$. To see that this is so, we note that, since $f_{n+2}/f_{n+1}$ is the $n^{th}$ convergent to $\xi$,

$$\frac{f_{2n+2}}{f_{2n+1}} < \frac{f_{4n+4}}{f_{4n+3}} < \xi$$

and

$$1 < \frac{f_{2n+1}f_{4n+4}}{f_{2n+2}f_{4n+3}} < \frac{f_{2n+1}}{f_{2n+2}} \cdot \xi .$$

But this gives, using (10),

$$\frac{f_{2n+1}}{f_{2n+2}} \cdot \xi - 1 > \frac{f_{2n+1}f_{4n+4} - f_{2n+2}f_{4n+3}}{f_{2n+2}f_{4n+3}}$$

$$= \frac{f_{2n+2}}{f_{2n+2}f_{4n+3}} = \frac{1}{f_{4n+3}}$$

or

(18) $$\xi_1 < f_{4n+3}$$

as desired. Also, we have that

$$\frac{f_{2n+2}}{f_{2n+1}} < \xi < \frac{f_{4n+3}}{f_{4n+2}}$$

so that, again by (10),

$$0 < \frac{f_{2n+1}}{f_{2n+2}} \cdot \xi - 1$$

$$< \frac{f_{2n+1}f_{4n+3} - f_{2n+2}f_{4n+2}}{f_{2n+2}f_{4n+2}}$$

$$= \frac{f_{2n+1}}{f_{2n+2}f_{4n+2}}$$

$$= \frac{f_{2n+1}}{f_{2n+1}f_{4n+3} - f_{2n+1}}$$

and

$$\xi_1 > f_{4n+3} - 1 .$$

Thus, $a_1 = [\xi_1] = f_{4n+3} - 1$ as claimed. Finally, to show the symmetry of the vector $(a_2, a_3, \cdots, a_r, a_0)$, it suffices to show that

$$\frac{1}{\dfrac{1}{\dfrac{1}{\dfrac{f_{2n+1}}{f_{2n+2}} \cdot \xi - a_0} - a_1} - a_2} = \frac{f_{2n+2}}{f_{2n+1}} \cdot \xi . \tag{19}$$

Making use of the determined values of $a_0$ and $a_1$ and setting $a_2 = 1$, this means that we must show that

$$\frac{1}{\dfrac{1}{\dfrac{1}{\dfrac{f_{2n+1}}{f_{2n+2}} \cdot \xi - 1} - (f_{4n+3} - 1)} - 1} = \frac{f_{2n+2}}{f_{2n+1}} \cdot \xi \tag{20}$$

which will also, of course, confirm the fact that $a_2 = 1$. Now (20) is true if and only if

$$\frac{1}{\qquad\qquad\qquad} - 1 = \frac{f_{2n+1}}{\xi f_{2n+2}}$$

$$\frac{1}{\dfrac{f_{2n+1}}{f_{2n+2}} \cdot \xi - 1} - (f_{4n+3} - 1) ,$$

which is true if and only if

$$\frac{1}{\dfrac{f_{2n+1}}{f_{2n+2}} \cdot \xi - 1} - (f_{4n+3} - 1) = \frac{\xi f_{2n+2}}{f_{2n+1} + \xi f_{2n+2}}$$

which, in turn, is true if and only if

$$\frac{f_{2n+2}}{\xi f_{2n+1} - f_{2n+2}} - \frac{\xi f_{2n+2}}{f_{2n+1} + \xi f_{2n+2}} = f_{4n+3} - 1 .$$

To see that this last equation is true we make use of (8), (10), and the fact that $\xi^2 = a\xi + 1$ to obtain

$$\frac{f_{2n+2}}{\xi f_{2n+1} - f_{2n+2}} - \frac{\xi f_{2n+2}}{f_{2n+1} + \xi f_{2n+2}} = \frac{f_{2n+2}f_{2n+1} + \xi f^2_{2n+2} - \xi^2 f_{2n+1}f_{2n+2} + \xi f^2_{2n+2}}{\xi f^2_{2n+1} + \xi^2 f_{2n+1}f_{2n+2} - f_{2n+1}f_{2n+2} - \xi f^2_{2n+2}}$$

$$= \frac{2\xi f^2_{2n+2} - a\xi f_{2n+1}f_{2n+2}}{\xi f^2_{2n+1} + a\xi f_{2n+1}f_{2n+2} - \xi f^2_{2n+2}}$$

$$= \frac{f_{2n+2}(f_{2n+2} + f_{2n})}{f_{2n+1}f_{2n+3} - f^2_{2n+2}}$$

$$= f^2_{2n+2} + f_{2n+2}f_{2n} + f_{2n+1}f_{2n+3} - f^2_{2n+2} - 1$$

$$= f_{4n+3} - 1 .$$

This completes the proof.

Because of the similarity of method, the following theorems are stated without proof. The notation is as before.

Theorem 14. For $n \geq 2$,

$$\frac{f_{2n}}{f_{2n+1}} \cdot \xi = [a_0, a_1, \dot{a}_2, \cdots, \dot{a}_r]$$

and

$$\frac{f_{2n+1}}{f_{2n}} \cdot \xi = [a_3 - 1, \dot{a}_4, a_5, \cdots, a_r, a_2, \dot{a}_3] \quad ,$$

where the vector $(a_3, a_4, \cdots, a_r)$ is symmetric, $a_0 = 0$, $a_1 = 1$, $a_2 = f_{4n+1} - 1$, and $a_3 = f_3 + 1$.

Theorem 15. Let $n \geq 2$ be an integer. Then

$$\frac{f_{n+2}}{g_n} \cdot \xi = [\dot{a}_0, a_1, \cdots, \dot{a}_r] \; ,$$

$$\frac{g_n}{f_{n+2}} \cdot \xi = [\dot{a}_r, a_{r-1}, \cdots, \dot{a}_0],$$

$$\frac{f_{n+1}}{g_n} \cdot \xi = [\dot{b}_0, b_1, \cdots, \dot{b}_s] \; ,$$

and

$$\frac{g_n}{f_{n+1}} \cdot \xi = [\dot{b}_s, b_{s-1}, \cdots, \dot{b}_0] \; .$$

Theorem 16. Let $n$ be a positive integer. Then

$$\frac{g_{2n+1}}{g_{2n+2}} \cdot \xi = [a_0, a_1, \dot{a}_2, \cdots, \dot{a}_r] \; ,$$

and

$$\frac{g_{2n+2}}{g_{2n+1}} \cdot \xi = [2, \dot{a}_4, a_5, \cdots, a_r, a_2, \dot{a}_3]$$

and the vector $(a_3, a_4, \cdots, a_r)$ is symmetric with $a_0 = 0$, $a_1 = 1$, $a_2 = f_{4n+3} - 1$, and $a_3 = 3$.

Theorem 17. Let $n$ be a positive integer. Then

$$\frac{g_{2n}}{g_{2n+1}} \cdot \xi = [\dot{a}_0, a_1, \cdots, \dot{a}_r]$$

and

$$\frac{g_{2n+1}}{g_{2n}} \cdot \xi = [\dot{a}_r, a_{r-1}, \cdots, a_1, \dot{a}_0] ,$$

where the vector $(a_2, a_3, \cdots, a_r, a_0)$ is symmetric with $a_0 - a_2 = 1$, and $a_1 = f_{4n+1} - 1$.

In view of the preceding results, one would expect an interesting theorem concerning the simple continued fraction expansion of

$$\frac{f_n}{g_n} \cdot \xi \quad \text{and} \quad \frac{g_n}{f_n} \cdot \xi$$

but we were not able to make a general assertion value for all $a$. To illustrate the difficulty, note that, when $a = 2$ and $\xi = 1 + \sqrt{2}$, we have

$$\frac{f_4}{g_4} \cdot \xi = [0, 1, \dot{5}, 1, 3, 5, 1, \dot{7}] ,$$

$$\frac{f_5}{g_5} \cdot \xi = [0, 1, \dot{5}, 1, 5, 3, 1, 4, 1, \dot{7}] ,$$

and

$$\frac{f_6}{g_6} \cdot \xi = [0, 1, \dot{5}, 1, 4, 1, 3, 5, 1, 4, 1, \dot{7}] .$$

However, for

$$\xi = \alpha = [\dot{1}] = \frac{1 + \sqrt{5}}{2} ,$$

we obtain the following rather elegant result:

Theorem 18. Let $\alpha = (1 + \sqrt{5})/2$ and let $F_n$ and $L_n$ denote the $n^{th}$ Fibonacci and Lucas numbers, respectively. Then, for $n \geq 4$,

$$\frac{F_n}{L_n} \cdot \alpha = [0, 1, \dot{2}, 1, \cdots, 1, 3, 1, \cdots, 1, \dot{4}] \tag{21}$$

and

$$\frac{L_n}{F_n} \cdot \alpha = [3, \dot{1}, \cdots, 1, 3, 1, \cdots, 1, 2, \dot{4}] , \tag{22}$$

where, in (21), there are $n - 4$ ones in the first group and $n - 3$ ones in the second group and just the reverse in (22).

Proof. Set

$$\begin{aligned} x_n &= [\dot{2}, \dot{1}, \cdots, 1, 3, 1, \cdots, 1, \dot{4}] \\ &= [2, 1, \cdots, 1, 3, 1, \cdots, 1, 4, x_n] . \end{aligned}$$

Then it is easy to see by direct computation as on computes convergents, that

$$x_n = \frac{a_n x_n + b_n}{c_n x_n + d_n} , \tag{23}$$

where

$$
\begin{aligned}
a_n &= 4(L_{n-1}F_{n-1} + F_{n-2})^2 + (L_{n-2}F_{n-1} + F_{n-3}F_{n-2}) \\
&= 4F_n^2 + F_nF_{n-1} + (-1)^n, \\
b_n &= L_{n-1}F_{n-1} + F_{n-2}^2 = F_n^2, \\
c_n &= 4(L_{n-1}F_{n-3} + F_{n-2}F_{n-4}) + (L_{n-2}F_{n-3} + F_{n-3}F_{n-4}) \\
&= 4F_{n-1}^2 + F_nF_{n-3},
\end{aligned}
$$

and

$$d_n = L_{n-1}F_{n-3} + F_{n-2}F_{n-4} = F_{n-1}^2 .$$

Moreover, from (23),

$$x_n = \frac{(a_n - d_n) + \sqrt{(a_n - d_n)^2 + 4b_nc_n}}{2c_n}$$

and

$$
\begin{aligned}
y_n &= [0,\ 1,\ x_n] \\
&= \frac{x_n}{x_n + 1} \\
&= \frac{(a_n - d_n) + \sqrt{(a_n - d_n)^2 + 4b_nc_n}}{(a_n - d_n + 2c_n) + \sqrt{(a_n - d_n)^2 + 4b_nc_n}} \\
&= \frac{(a_n - d_n - 2b_n) + \sqrt{(a_n - d_n)^2 + 4b_nc_n}}{2(a_n - b_n + c_n - d_n)} .
\end{aligned}
$$

Now

$$a_n - d_n - 2b_n = 4F_n^2 + F_nF_{n-1} + (-1)^n - F_{n-1}^2 - 2F_n^2$$

$$= 2F_n^2 + F_nF_{n-1} - F_nF_{n-2}$$

(25) $$= F_n(2F_n + F_{n-1} - F_{n-2})$$

$$= F_n(F_{n+2} - F_{n-2})$$

$$= F_nL_n \;,$$

$$a_n - b_n + c_n - d_n = (a_n - d_n - 2b_n) + b_n + c_n$$

$$= F_nL_n + F_n^2 + 4F_{n-1}^2 + F_nF_{n-3}$$

(26) $$= F_nL_n + 2F_{n-1}L_n$$

$$= L_n^2$$

and

$$(a_n - d_n)^2 + 4b_nc_n = (4F_n^2 + F_nF_{n-1} + (-1)^n - F_{n-1}^2)^2$$

$$+ 4F_n^2(4F_{n-1}^2 + F_nF_{n-3})$$

(27) $$= F_n^2F_{n+3}^2 + 4F_n^2(4F_{n-1}^2 + F_nF_{n-3})$$

$$= F_n^2(F_{n+3}^2 + 16F_{n-1}^2 + 4F_nF_{n-3})$$

$$= 5F_n^2L_n^2 \quad .$$

Thus, using (25), (26), and (27), in (24), we obtain

$$y_n = \frac{F_nL_n + F_nL_n\sqrt{5}}{2L_n^2} = \frac{F_n}{L_n} \cdot \frac{1 + \sqrt{5}}{2} \quad ,$$

as claimed. The other part of the proof is an immediate consequence of Theorem 11.

Finally, we comment on the question of the equivalence of $r\xi$ and $\xi/r$. If $r = g_m/f_n$ or $r = g_m/g_n$, where m and n are nonnegative integers, it frequently turns out to be the case that $r\xi \sim \xi/r$. However, this is not necessarily the case and hence, a fortiori, it is not necessarily the case for more general r. For example, for $\alpha = (1+\sqrt{5})/2 = [\dot{1}]$,

$$\frac{3}{7} \cdot \alpha = [0, 1, \dot{2}, 3, 1, \dot{4}]$$

and

$$\frac{7}{3} \cdot \alpha = [3, \dot{1}, 3, 2, \dot{4}]$$

where $3 = f_4 = g_2$ and $7 = g_4$; and other examples are easily found. However, if $r = f_m$ and $s = f_n$ for nonnegative integers m and n then we always have

$$\frac{r}{s} \cdot \xi \sim \frac{s}{r} \cdot \xi \quad ,$$

as the following theorem shows.

Theorem 18. If m and n are nonnegative integers, then

$$\frac{f_m}{f_n} \cdot \xi \sim \frac{f_n}{f_m} \cdot \xi \quad .$$

Proof. Without loss of generality, we may assume that $0 < m < n$ and that $(m,n) = 1$. We let

$$a = \frac{f_m f_{2qm+2}}{f_n}, \quad b = c = f_{2qm+1}, \quad d = \frac{f_n f_{2qm}}{f_m},$$

where $q$ is chosen so that

$$2q + 2 \equiv 0 \pmod{n},$$

as may easily be done since $(m,n) = 1$. With this choice for $q$ it follows from Lemma 4 that $f_n | f_{2qm+2}$ and $f_m | f_{2qm}$ so that $a$, $b$, $c$, and $d$ are all integers. Also, by (10),

$$\begin{aligned} ad - bc &= \frac{f_m f_{2qm+2}}{f_n} \cdot \frac{f_n f_{2qm}}{f_m} - f_{2qm+1}^2 \\ &= f_{2qm+2} f_{2qm} - f_{2qm+1}^2 \\ &= -1 . \end{aligned}$$

Finally, we show that

$$\frac{f_m}{f_n} \cdot \xi = \frac{a\left(\frac{f_n}{f_m} \cdot \xi\right) + b}{c\left(\frac{f_n}{f_m} \cdot \xi\right) + c} \tag{28}$$

for this choice of $a$, $b$, $c$, and $d$. Making the indicated substitutions, we have that (28) holds if and only if

$$\frac{f_m}{f_n} \cdot \xi = \frac{\frac{f_m f_{2qm+2}}{f_n}\left(\frac{f_n}{f_m} \cdot \xi\right) + f_{2qm+1}}{f_{2qm+1}\left(\frac{f_n}{f_m} \cdot \xi\right) + \frac{f_n f_{2qm}}{f_m}},$$

and this is true if and only if

$$\xi^2 f_{2qm+1} + \xi f_{2qm} = \xi f_{2qm+2} + f_{2qm+1} .$$

But this is clearly true since $a\xi^2 = a\xi + 1$ and $af_{2qm+1} + f_{2qm} = f_{2qm+2}$ and the proof is complete.

Finally, we note that the list of stated theorems is not exhaustive. One could no doubt prove theorems concerning

$$\frac{f_n}{f_{n+2}}\xi, \quad \frac{f_n}{f_{n+4}}\xi, \quad \frac{f_n}{f_{n+5}}\xi \quad ,$$

and so on. However, we were not able to arrive at general formulations of the expansions of

$$\frac{f_m}{f_n}\cdot\xi, \quad \frac{f_m}{g_n}\cdot\xi, \quad \text{or} \quad \frac{g_m}{g_n}\cdot\xi \quad ,$$

for arbitrary positive integers m and n. The results stated seem to be the most interesting.

## REFERENCES

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