CLASSROOM NOTES

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CONTINUED FRACTIONS AND MATRICES

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1. Introduction. The closest rational approximation P/Q (P, Q integers) to a given positive real number x, subject to the condition that the denominator Q shall not exceed a given positive integer N, is most easily found by means of continued fractions. For example, successive approximations to π obtained by this method are 3, 22/7, 333/106, 355/113, $\cdots$. Continued fractions may also be used to facilitate the solution of diophantine equations.

It is our purpose to show how the principal theorems in the theory of continued fractions, and the actual computation of successive convergents to a continued fraction can be presented quite simply by the use of two-rowed matrices and their determinants. To a student familiar with the multiplication of two-rowed matrices a good introduction to continued fractions can be presented in an hour's lecture. For a student who has just been introduced to matrices, their use in connection with continued fractions provides an easy application at the elementary level.

2. Simple continued fractions. For a given positive real x let the integral part be a_1 and the remainder r_1, with $0 \leqq r_1 < 1$. We define successively the positive integers $a_2, a_3, \cdots, a_n$, (called partial denominators) and the remainders r_2, $r_3, \cdots r_n$ so that

$$(1) \qquad x = a_1 + r_1, \qquad \frac{1}{r_1} = a_2 + r_2, \cdots, \frac{1}{r_{n-1}} = a_n + r_n, \qquad 0 \leqq r_k < 1.$$

and we write

$$(2) \qquad x = a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{\ddots \, \cfrac{}{a_{n-1} + \cfrac{1}{a_n + r_n}}}}}$$

or

$$(2') \qquad x = a_1 + \frac{1}{a_2 +} \quad \frac{1}{a_3 +} \quad \cdots \frac{1}{a_{n-1} +} \quad \frac{1}{a_n + r_n}.$$

After simplifying the complex fraction (2) and collecting coefficients of r_n, we have

$$x = \frac{P_n + r_n S_n}{Q_n + r_n T_n}, \qquad \text{where} \quad M_n = \begin{pmatrix} P_n & S_n \\ Q_n & T_n \end{pmatrix} \tag{3}$$

is an integral matrix, and where, for the case $n=1$, we have from (1)

$$M_1 = \begin{pmatrix} P_1 & S_1 \\ Q_1 & T_1 \end{pmatrix} = \begin{pmatrix} a_1 & 1 \\ 1 & 0 \end{pmatrix}.$$

It will be convenient to denote the nth denominator in (3) by D_n;

$$D_n = Q_n + r_n T_n \tag{4}$$

To obtain a recursion formula for the P's and Q's, we replace n by $n-1$ in equation (3), divide numerator and denominator by r_{n-1}, and apply (1).

$$x = \frac{P_{n-1} + r_{n-1}S_{n-1}}{Q_{n-1} + r_{n-1}T_{n-1}} = \frac{P_{n-1}(a_n + r_n) + S_{n-1}}{Q_{n-1}(a_n + r_n) + T_{n-1}} = \frac{(a_nP_{n-1} + S_{n-1}) + r_nP_{n-1}}{(a_nQ_{n-1} + T_{n-1}) + r_nQ_{n-1}}. \tag{5}$$

On comparing coefficients of r_n in (3) and (5) we have

$$\begin{aligned} S_n &= P_{n-1}, \qquad & P_n &= a_nP_{n-1} + S_{n-1} \\ T_n &= Q_{n-1}, \qquad & Q_n &= a_nQ_{n-1} + T_{n-1}. \end{aligned} \tag{6}$$

Hence,

$$\begin{aligned} P_n &= a_nP_{n-1} + P_{n-2} \\ Q_n &= a_nQ_{n-1} + Q_{n-2}. \end{aligned} \tag{7}$$

Also from (4) and (5) the ratio D_{n-1}/D_n is seen to be r_{n-1}, and $D_1 = 1$, so

$$1/D_n = r_1r_2 \cdots r_{n-1}, \tag{8}$$

provided that none of the r's are 0. The set of equations (7) can most easily be written in matrix form

$$M_n = \begin{pmatrix} P_n & P_{n-1} \\ Q_n & Q_{n-1} \end{pmatrix} = \begin{pmatrix} P_{n-1} & P_{n-2} \\ Q_{n-1} & Q_{n-2} \end{pmatrix}\begin{pmatrix} a_n & 1 \\ 1 & 0 \end{pmatrix} = M_{n-1}\begin{pmatrix} a_n & 1 \\ 1 & 0 \end{pmatrix}. \tag{9}$$

By induction we then obtain the fundamental relation

$$M_n = \begin{pmatrix} P_n & P_{n-1} \\ Q_n & Q_{n-1} \end{pmatrix} = \begin{pmatrix} a_1 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} a_2 & 1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} a_{n-1} & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} a_n & 1 \\ 1 & 0 \end{pmatrix}. \tag{10}$$

3. Theorems on continued fractions. Since the determinant of the matrix M_n is the product of the determinants of its factors, we obtain from (10) the important relations

$$\begin{vmatrix} P_n & P_{n-1} \\ Q_n & Q_{n-1} \end{vmatrix} = (-1)^n \tag{11}$$

$$\frac{P_n}{Q_n} - \frac{P_{n-1}}{Q_{n-1}} = \frac{(-1)^n}{Q_nQ_{n-1}} \tag{12}$$

$$x - \frac{P_n}{Q_n} = \frac{P_n + r_nP_{n-1}}{Q_n + r_nQ_{n-1}} - \frac{P_n}{Q_n} = \frac{r_n(-1)^{n-1}}{Q_nD_n} = \frac{(-1)^{n-1}}{Q_nD_{n+1}}. \tag{13}$$

THEOREM I. *The rational fractions P_n/Q_n, called the "convergents" of x, are alternately less and greater than x, and the difference of successive convergents is the reciprocal of the product of their denominators. Since this product becomes infinite with n, the differences defined by* (12) *approach zero and alternate in sign as n increases, so the sequence P_n/Q_n converges. Its limit is x.*

The proof of these statements follows directly from (13) and (12).

Equation (3) may be replaced by the matrix equation

$$\begin{pmatrix} x \\ 1 \end{pmatrix} = \begin{pmatrix} P_n & P_{n-1} \\ Q_n & Q_{n-1} \end{pmatrix} \begin{pmatrix} 1/D_n \\ 1/D_{n+1} \end{pmatrix}. \tag{14}$$

Solving for r_n, which is the ratio D_n/D_{n+1}, we have

$$r_n = -\frac{P_n - Q_nx}{P_{n-1} - Q_{n-1}x} \tag{15}$$

$$a_{n+1} + r_{n+1} = \frac{Q_{n-1}x - P_{n-1}}{-Q_nx + P_n}. \tag{16}$$

Equation (16) makes it possible to compute a_{n+1} directly if the matrix M_n is known. The next matrix M_{n+1} is then obtained from (10). For example, in the continued fraction expansion for π the partial denominator a_5 is the largest integer in $(106\pi-333)/(-113\pi+355)=292.+$

THEOREM II. *A periodic continued fraction represents a root of a quadratic equation.*

Proof. If two different remainders are equal, equate them, using (15), and solve for x.

THEOREM III. *The rational fraction P_n/Q_n defined by* (1) *and* (10) *differs from x by not more than $1/Q_nQ_{n+1}$, and it approximates the real number x more closely than does any other rational fraction with a denominator not exceeding Q_n.*

Proof. The difference (13) is equal in absolute value to $1/Q_nD_{n+1}$, which by (4) is certainly less than $1/Q_nQ_{n+1}$. Suppose there were two integers A and B, with $0<B<Q_n$, such that the fraction A/B were closer to x than P_n/Q_n. Then we should have

$$-\frac{1}{Q_nD_{n+1}} < (-1)^n\left(x-\frac{A}{B}\right) < \frac{1}{Q_nD_{n+1}}. \tag{17}$$

Adding $1/Q_nD_{n+1}$ to each term and replacing x by its value in (13) gives

$$0 < (-1)^n\left(\frac{P_n}{Q_n}-\frac{A}{B}\right) < \frac{2}{Q_nD_{n+1}} < \frac{2}{Q_nQ_{n+1}}$$

or

$$0 < (-1)^n(BP_n - AQ_n) < 2B/Q_{n+1}. \tag{18}$$

It is easily shown that the inequality (18) cannot be satisfied by integers A, B with $B<Q_n$. For then the right member is less than 2, and the integer in the middle member could only be 1. By (11) the latter condition is satisfied by taking $A=P_{n-1}$, $B=Q_{n-1}$, and this is the only choice for which $0<B<Q_n$, as we shall see in §4 below. However, with $B=Q_{n-1}$, we are led from (18) to the inequality $1<2Q_{n-1}/Q_{n+1}$, which is impossible by (7).

4. Continued fractions for rational numbers. If at some stage in (1) we have $r_n=0$, then x reduces by (3) to the rational number P_n/Q_n. Conversely, every rational number is represented by a terminating continued fraction. The next to the last convergent is important in solving the linear diophantine equation (19).

Given two integers P_n and Q_n without common factor, to find all pairs of integers u and v which satisfy the equation

$$P_nu - Q_nv = 1. \tag{19}$$

The solution is given by

$$\begin{aligned} v &= (-1)^nP_{n-1} + NP_n \\ u &= (-1)^nQ_{n-1} + NQ_n \end{aligned} \qquad N \text{ any integer,} \tag{20}$$

where P_{n-1} and Q_{n-1} are obtained from the continued fraction expansion of P_n/Q_n.

5. General continued fractions. If the simple continued fraction (2) is replaced by the more general form

$$x = a_1 + \cfrac{b_2}{a_2 + \cfrac{b_3}{a_3 + \cfrac{b_4}{\ddots\, \cfrac{}{a_{n-1} + \cfrac{b_n}{a_n + r_n}}}}} \tag{21}$$

in which r_n is not necessarily less than 1, and a_k and b_k are not necessarily positive integers, then it is readily shown that equations (7) become

$$P_n = a_n P_{n-1} + b_n P_{n-2}$$
$$Q_n = a_n Q_{n-1} + b_n Q_{n-2} \tag{22}$$

and the fundamental relation (10) may be written

$$M_n = \begin{pmatrix} P_n & P_{n-1} \\ Q_n & Q_{n-1} \end{pmatrix} = \begin{pmatrix} a_1 & 1 \\ b_1 & 0 \end{pmatrix}\begin{pmatrix} a_2 & 1 \\ b_2 & 0 \end{pmatrix} \cdots \begin{pmatrix} a_n & 1 \\ b_n & 0 \end{pmatrix}, \tag{23}$$

where we define $b_1 = Q_1 = P_0 = 1$, $Q_0 = 0$.

By taking determinants in (23), the difference between successive convergents to x is seen to equal $(-1)^n b_1 b_2 \cdots b_n / Q_n Q_{n-1}$. Since the sum of these terms may not always converge, the question of convergence in this case is more complicated than that for the simple continued fractions, and we shall not discuss it at this time.

It may be of interest to include without proof the expansion

$$\frac{x}{\tan^{-1} x} = 1 + \cfrac{x^2}{3 + \cfrac{(2x)^2}{5 + \cfrac{(3x)^2}{7 + \cdots}}} \tag{24}$$

which is typical of many continued fraction expansions for analytic functions obtained from hypergeometric series.

Setting $x = 1$ in (24) we see that the value of π can be expressed as a continued fraction in which the coefficients are given by a simple law

$$\frac{4}{\pi} = 1 + \cfrac{1}{3 + \cfrac{4}{5 + \cfrac{9}{7 + \cdots}}}. \tag{25}$$

This expansion is of interest because of its regularity, but it does not enjoy the rapidity of convergence which characterizes the simple continued fractions. Successive convergents in (25) are given by

$$\begin{pmatrix} P_{n+1} & P_n \\ Q_{n+1} & Q_n \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 3 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 5 & 1 \\ 4 & 0 \end{pmatrix}\begin{pmatrix} 7 & 1 \\ 9 & 0 \end{pmatrix}\begin{pmatrix} 9 & 1 \\ 16 & 0 \end{pmatrix} \cdots \begin{pmatrix} 2n+1 & 1 \\ n^2 & 0 \end{pmatrix}. \tag{26}$$

We may derive from (26) an alternative expansion for the convergents to $4/\pi$, in which the partial numerators b_i are all 1 but the partial denominators are not all integers. This is as follows:

$$(27)\quad \begin{pmatrix} P_{n+1} & P_n \\ Q_{n+1} & Q_n \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 3 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 5/4 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 28/9 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 81/64 & 1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} a_{n+1} & 1 \\ 1 & 0 \end{pmatrix}$$

where

$$a_{2n} = (4n-1)\left(\frac{2\cdot 4 \cdots (2n-2)}{3\cdot 5 \cdots (2n-1)}\right)^2,$$

$$a_{2n+1} = (4n+1)\left(\frac{1\cdot 3 \cdots (2n-1)}{2\cdot 4 \cdots 2n}\right)^2$$

$$(28)\qquad \lim_{n=\infty} a_{2n} = \pi$$

$$\lim a_{2n+1} = \frac{4}{\pi}\cdot$$

Since each matrix in (27) has the determinant (-1), formulas (12) and (13) are valid in this case. The limits in (28) are easily obtained from the Wallis product formula for $\pi/2$. A close estimate for the error of P_n/Q_n is $(\sqrt{2}-1)^{2n-1}$.

DERIVATION OF THE TANGENT HALF-ANGLE FORMULA

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The following derivation of the formula for $\tan \theta/2$ appears to be an improvement over standard derivations, for it gives the result directly without a complicated discussion of the appropriate algebraic sign.

From the equation

$$\sin\frac{\theta}{2} = \sin\left(\theta - \frac{\theta}{2}\right) = \sin\theta\cos\frac{\theta}{2} - \cos\theta\sin\frac{\theta}{2}$$

one obtains

$$(1+\cos\theta)\sin\frac{\theta}{2} = \sin\theta\cos\frac{\theta}{2}\cdot$$

Consequently:

$$\tan\frac{\theta}{2} = \frac{\sin\theta/2}{\cos\theta/2} = \frac{\sin\theta}{1+\cos\theta}\cdot$$