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SOME CONGRUENCES FOR THE SECOND-ORDER CATALAN NUMBERS

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Abstract. Let p be any odd prime. We mainly show that

$$\sum_{k=1}^{p-1}\frac{2^k}{k}\binom{3k}{k}\equiv 0 \pmod p$$

and

$$\sum_{k=1}^{p-1}2^{k-1}C_k^{(2)}\equiv(-1)^{(p-1)/2}-1 \pmod p,$$

where $C_k^{(2)}=\binom{3k}{k}/(2k+1)$ is the kth Catalan number of order 2.

1. Introduction

The well-known Catalan numbers are those integers

$$C_n=\frac{1}{n+1}\binom{2n}{n}=\binom{2n}{n}-\binom{2n}{n-1}\quad(n=0,1,2,\ldots).$$

(As usual we regard $\binom{x}{-k}$ as 0 for $k=1,2,\ldots$.) There are many combinatorial interpretations for these important numbers (see, e.g., [St, pp. 219-229]). With the help of a sophisticated binomial identity, H. Pan and Z. W. Sun [PS] obtained some congruences on sums of Catalan numbers; in particular, by [PS, (1.16) and (1.8)], for any prime $p>3$ we have

$$\sum_{k=0}^{p-1}C_k\equiv\frac{3(\frac{p}{3})-1}{2} \pmod p\ \text{ and }\ \sum_{k=1}^{p-1}\frac{C_k}{k}\equiv\frac{3}{2}\left(1-\left(\frac{p}{3}\right)\right) \pmod p,\quad(1.0)$$

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where the Legendre symbol $(\frac{a}{3}) \in \{0, \pm 1\}$ satisfies the congruence $a \equiv (\frac{a}{3}) \pmod 3$. Recently Z. W. Sun and R. Tauraso [ST1, ST2] obtained some further congruences concerning sums involving Catalan numbers.

For $m, n \in \mathbb{N} = \{0, 1, 2, \dots\}$, we define

$$C_n^{(m)} = \frac{1}{mn+1}\binom{mn+n}{n} = \binom{mn+n}{n} - m\binom{mn+n}{n-1}$$

and call it the nth Catalan number of order m. Clearly

$$C_n^{(1)} = C_n \quad \text{and} \quad C_n^{(2)} = \frac{1}{2n+1}\binom{3n}{n}.$$

In contrast with (1.0), we have the following result involving the second-order Catalan numbers.

Theorem 1.1. *Let p be an odd prime. Then*

$$\sum_{k=1}^{p-1} 2^k C_k^{(2)} \equiv 2\left((-1)^{(p-1)/2} - 1\right) \pmod p \tag{1.1}$$

and

$$\sum_{k=1}^{p-1} \frac{2^k C_k^{(2)}}{k} \equiv 4\left(1 - (-1)^{(p-1)/2}\right) \pmod p. \tag{1.2}$$

Actually Theorem 1.1 follows from our next two theorems.

Theorem 1.2. *Let $p > 5$ be a prime. Then*

$$\sum_{k=0}^{p-1} 2^k \binom{3k}{k} \equiv \frac{6(-1)^{(p-1)/2} - 1}{5} \pmod p, \tag{1.3}$$

$$\sum_{k=0}^{p-1} 2^k \binom{3k+1}{k} \equiv \frac{4(-1)^{(p-1)/2} + 1}{5} \pmod p, \tag{1.4}$$

Theorem 1.3. *For any prime p we have*

$$\sum_{k=1}^{p-1} \frac{2^k}{k}\binom{3k}{k} \equiv 0 \pmod p. \tag{1.5}$$

For any odd prime p we can also prove the following congruences:

$$5\sum_{k=1}^{p-1} 2^k \binom{3k+2}{k} \equiv (-1)^{(p-1)/2} - 1 \pmod p,$$

$$\sum_{k=1}^{p-1} \frac{2^{k-1}}{k}\binom{3k+1}{k} \equiv (-1)^{(p-1)/2} - 1 \pmod p,$$

$$\sum_{k=1}^{p-1} \frac{2^{k-1}}{k}\binom{3k+2}{k} \equiv \frac{3}{2}\left((-1)^{(p-1)/2} - 1\right) \pmod p.$$

We omit their proofs which are similar to those of Theorems 1.2-1.3.

With the help of Theorems 1.2 and 1.3, we can easily deduce Theorem 1.1.

Proof of Theorem 1.1 via Theorems 1.2 and 1.3. Clearly (1.1) and (1.2) hold for $p = 3, 5$. Assume $p > 5$. By (1.3) and (1.4),

$$\begin{aligned}\sum_{k=0}^{p-1} \frac{2^k}{2k+1}\binom{3k}{k} =&3\sum_{k=0}^{p-1} 2^k\binom{3k}{k} - 2\sum_{k=0}^{p-1} 2^k\binom{3k+1}{k} \\ \equiv&2(-1)^{(p-1)/2} - 1 \pmod{p}.\end{aligned}$$

This proves (1.1). For (1.2) it suffices to note that

$$\sum_{k=1}^{p-1} \frac{2^k}{k(2k+1)}\binom{3k}{k} = \sum_{k=1}^{p-1} \frac{2^k}{k}\binom{3k}{k} - 2\sum_{k=1}^{p-1} \frac{2^k}{2k+1}\binom{3k}{k}.$$

This concludes the proof. □

We are going to provide two lemmas in the next section. Theorems 1.2 and 1.3 will be proved in Sections 3 and 4 respectively.

2. Some Lemmas

Lemma 2.1. *For $m, n \in \mathbb{N}$ we have*

$$\begin{aligned}&2^n \sum_{k=0}^{\lfloor m/3 \rfloor} (-2)^k \binom{n}{m-3k}\binom{3k-m+n}{k} \\ =&(-1)^m \sum_{j=0}^{n} \binom{n}{j} \sum_{k=0}^{m} (-2)^k \binom{n}{m-k}\binom{2j}{k}.\end{aligned} \tag{2.1}$$

Proof. Let $P(x) = (2 + 2x - 4x^3)^n$, and denote by $[x^k]P(x)$ the coefficient of x^k in the expansion of $P(x)$. Then

$$\begin{aligned}2^{-n}[x^m]P(x) =&[x^m]((1+x) - 2x^3)^n \\ =&\sum_{k=0}^{\lfloor m/3 \rfloor} \binom{n}{k}(-2)^k[x^{m-3k}](1+x)^{n-k} \\ =&\sum_{k=0}^{\lfloor m/3 \rfloor} (-2)^k \binom{n}{k}\binom{n-k}{m-3k} \\ =&\sum_{k=0}^{\lfloor m/3 \rfloor} (-2)^k \binom{n}{m-3k}\binom{3k-m+n}{k}.\end{aligned}$$

Since

$$P(x)=(1-x)^n((2x+1)^2+1)^n=\sum_{j=0}^{n}\binom{n}{j}(1-x)^n(2x+1)^{2j},$$

we also have

$$[x^m]P(x)=\sum_{j=0}^{n}\binom{n}{j}\sum_{k=0}^{m}2^k\binom{2j}{k}(-1)^{m-k}\binom{n}{m-k}.$$

Therefore (2.1) is valid. □

For any prime p, if $n,k\in\mathbb{N}$ and $s,t\in\{0,1,\ldots,p-1\}$ then we have the following well-known Lucas congruence (cf. [Gr] or [HS]): $\binom{pn+s}{pk+t}\equiv\binom{n}{k}\binom{s}{t}\pmod p$. This will be used in the proof of the following lemma.

Lemma 2.2. *Let $p>5$ be a prime. Then we have*

$$\sum_{s=0}^{p-1}(-1)^s\sum_{t=0}^{p-1}2^t\binom{2s}{t}\equiv\frac{3(-1)^{(p-1)/2}+2}{5}\pmod p \tag{2.2}$$

and

$$\sum_{s=0}^{p-1}(-1)^s\sum_{t=0}^{p-1}2^t\binom{2s}{p+t}\equiv\frac{3}{10}\left(1-(-1)^{(p-1)/2}\right)\pmod p. \tag{2.3}$$

Proof. Observe that

$$\begin{aligned}
&\sum_{s=0}^{p-1}(-1)^s\sum_{t=0}^{p-1}2^t\binom{2s}{t}\\
=&\sum_{s=0}^{(p-1)/2}(-1)^s\sum_{t=0}^{2s}2^t\binom{2s}{t}+\sum_{s=(p+1)/2}^{p-1}(-1)^s\sum_{t=0}^{p-1}2^t\binom{2s}{t}\\
=&\sum_{s=0}^{(p-1)/2}(-1)^s3^{2s}+\sum_{s=(p+1)/2}^{p-1}(-1)^s\sum_{t=0}^{p-1}2^t\binom{2s}{t}\\
=&\sum_{s=0}^{(p-1)/2}(-1)^s3^{2s}+\sum_{s=(p+1)/2}^{p-1}(-1)^s\left(\sum_{t=0}^{2s}2^t\binom{2s}{t}-\sum_{t=p}^{2s}2^t\binom{2s}{t}\right)\\
=&\sum_{s=0}^{p-1}(-1)^s3^{2s}-\sum_{s=(p+1)/2}^{p-1}(-1)^s\sum_{t=p}^{2s}2^t\binom{2s}{t}\\
=&\sum_{s=0}^{p-1}(-9)^s-\sum_{s=(p+1)/2}^{p-1}(-1)^s\sum_{r=0}^{2s-p}2^{p+r}\binom{2s}{r+p}.
\end{aligned}$$

For $s=(p+1)/2,\ldots,p-1$, by Lucas' congruence we have

$$\sum_{r=0}^{2s-p}2^r\binom{p+(2s-p)}{p+r}\equiv\sum_{r=0}^{2s-p}2^r\binom{2s-p}{r}=3^{2s-p}\pmod p.$$

Thus, with the help of Fermat's little theorem, we get

$$\begin{aligned}\sum_{s=0}^{p-1}(-1)^s\sum_{t=0}^{p-1}2^t\binom{2s}{t}\equiv&\frac{1-(-9)^p}{10}-\sum_{s=(p+1)/2}^{p-1}(-1)^s\frac{2}{3}\cdot 9^s\\ \equiv&1-\frac{2}{3}(-9)^{\frac{p+1}{2}}\frac{(1-(-9)^{(p-1)/2})}{10}\\ \equiv&\frac{3(-1)^{(p-1)/2}+2}{5}\pmod p.\end{aligned}$$

This proves (2.2).

In view of Lucas' congruence and Fermat's little theorem, we also have

$$\begin{aligned}&\sum_{s=0}^{p-1}(-1)^s\sum_{t=0}^{p-1}2^t\binom{2s}{p+t}\\ \equiv&\sum_{s=(p+1)/2}^{p-1}(-1)^s\sum_{t=0}^{p-1}2^t\binom{2s-p}{t}=\sum_{s=(p+1)/2}^{p-1}(-1)^s3^{2s-p}\\ =&3^{-p}(-9)^{(p+1)/2}\frac{1-(-9)^{(p-1)/2}}{10}=(-1)^{(p+1)/2}\frac{3}{10}\left(1+(-1)^{(p+1)/2}3^{p-1}\right)\\ \equiv&\frac{3}{10}\left(1-(-1)^{(p-1)/2}\right)\pmod p.\end{aligned}$$

So (2.3) is also valid. We are done. $\square$

3. Proof of Theorem 1.2

In order to prove Theorem 1.2, we first present an auxiliary result.

Theorem 3.1. *Let $p>5$ be a prime, and let $d,\delta\in\{0,1\}$. Then*

$$\begin{aligned}&\frac{(-1)^{d+\delta}}{2^\delta}\sum_{\delta p-d\leqslant 3k\leqslant\delta p+p-1-d}2^k\binom{3k+d}{k}\\ \equiv&\frac{4-\delta}{10}+\frac{(3\delta-2)(5d-3)}{10}(-1)^{(p-1)/2}\pmod p.\end{aligned}\tag{3.1}$$

Proof. Applying (2.1) with $n=p-1$ and $m=\delta p+p-1-d$, we get

$$\begin{aligned}&2^{p-1}\sum_{k=0}^{\lfloor(\delta p+p-1-d)/3\rfloor}(-2)^k\binom{p-1}{\delta p+p-1-d-3k}\binom{3k+d-\delta p}{k}\\ =&(-1)^{\delta p+p-1-d}\sum_{j=0}^{p-1}\binom{p-1}{j}\sum_{k=0}^{\delta p+p-1-d}(-2)^k\binom{p-1}{\delta p+p-1-d-k}\binom{2j}{k}.\end{aligned}$$

Observe that

$$\begin{aligned}
&\sum_{k=0}^{\lfloor(\delta p+p-1-d)/3\rfloor}(-2)^k\binom{p-1}{\delta p+p-1-d-3k}\binom{3k+d-\delta p}{k}\\
=&\sum_{\delta p-d\leqslant 3k\leqslant\delta p+p-1-d}(-2)^k\binom{p-1}{p+\delta p-1-d-3k}\binom{3k+d-\delta p}{k}\\
\equiv&\sum_{\delta p-d\leqslant 3k\leqslant\delta p+p-1-d}(-2)^k(-1)^{\delta p+p-1-d-3k}\binom{3k+d}{k}\\
\equiv&(-1)^{d+\delta}\sum_{\delta p-d\leqslant 3k\leqslant\delta p+p-1-d}2^k\binom{3k+d}{k}\pmod p
\end{aligned}$$

and

$$\begin{aligned}
&(-1)^{\delta p+p-1-d}\sum_{j=0}^{p-1}\binom{p-1}{j}\sum_{k=0}^{\delta p+p-1-d}(-2)^k\binom{p-1}{\delta p+p-1-d-k}\binom{2j}{k}\\
\equiv&\sum_{j=0}^{p-1}(-1)^j\sum_{\delta p-d\leqslant k<\delta p+p-d}2^k\binom{2j}{k}=\sum_{j=0}^{p-1}(-1)^j\sum_{t=0}^{p-1}2^{\delta p-d+t}\binom{2j}{\delta p-d+t}\\
\equiv&2^{\delta-d}\sum_{s=0}^{p-1}(-1)^s\sum_{t=0}^{p-1}2^t\binom{2s}{\delta p-d+t}\pmod p.
\end{aligned}$$

Therefore

$$\begin{aligned}
&\sum_{\delta p-d\leqslant 3k\leqslant\delta p+p-1-d}2^k\binom{3k+d}{k}\\
\equiv&(-2)^{\delta-d}\sum_{s=0}^{p-1}(-1)^s\sum_{t=0}^{p-1}2^t\binom{2s}{\delta p-d+t}\pmod p.
\end{aligned}$$

Recall that $d\in\{0,1\}$. We have

$$\begin{aligned}
&\sum_{s=0}^{p-1}(-1)^s\sum_{t=0}^{p-1}2^t\binom{2s}{\delta p-d+t}\\
=&\sum_{s=0}^{p-1}(-1)^s\sum_{t=-d}^{p-1-d}2^{d+t}\binom{2s}{\delta p+t}\\
=&\sum_{s=0}^{p-1}(-1)^s\left(\sum_{t=0}^{p-1}2^{d+t}\binom{2s}{\delta p+t}+d\left(\binom{2s}{\delta p-1}-2^p\binom{2s}{\delta p+p-1}\right)\right)\\
=&2^d\sum_{s=0}^{p-1}\sum_{t=0}^{p-1}(-1)^s2^t\binom{2s}{\delta p+t}+d\sum_{s=0}^{p-1}(-1)^s\left(\binom{2s}{\delta p-1}-2^p\binom{2s}{\delta p+p-1}\right)
\end{aligned}$$

and hence

$$\begin{aligned}&(-1)^{d+\delta}\sum_{\delta p-d\leqslant 3k\leqslant \delta p+p-1-d}2^k\binom{3k+d}{k}-2^\delta\sum_{s=0}^{p-1}\sum_{t=0}^{p-1}(-1)^s2^t\binom{2s}{\delta p+t}\\ \equiv& d2^{\delta-d}\sum_{s=0}^{p-1}(-1)^s\left(\binom{2s}{\delta p-1}-2\binom{2s}{\delta p+p-1}\right)\\ \equiv& d2^{\delta-1}(3\delta-2)\sum_{s=0}^{p-1}(-1)^s\binom{2s}{p-1}\equiv d(3\delta-2)2^{\delta-1}(-1)^{(p-1)/2}\pmod p.\end{aligned}$$

Since

$$\sum_{s=0}^{p-1}(-1)^s\sum_{t=0}^{p-1}2^t\binom{2s}{\delta p+t}\equiv\frac{4-\delta}{10}+\frac{3}{10}(2-3\delta)(-1)^{(p-1)/2}\pmod p$$

by Lemma 2.2, we finally get

$$\begin{aligned}&\frac{(-1)^{d+\delta}}{2^\delta}\sum_{\delta p-d\leqslant 3k\leqslant \delta p+p-1-d}2^k\binom{3k+d}{k}\\ \equiv&\frac{4-\delta}{10}+\frac{3}{10}(2-3\delta)(-1)^{(p-1)/2}+\frac{d}{2}(3\delta-2)(-1)^{(p-1)/2}\\ \equiv&\frac{4-\delta}{10}+\frac{(3\delta-2)(5d-3)}{10}(-1)^{(p-1)/2}\pmod p.\end{aligned}$$

This proves (3.1). $\square$

Proof of Theorem 1.2. Let $d\in\{0,1\}$. If $(2p-d)/3\leqslant k\leqslant p-1$, then $2k+d+1\leqslant 2k+2\leqslant 2p\leqslant 3k+d$ and hence

$$\binom{3k+d}{k}=\frac{(3k+d)\cdots(2k+d+1)}{k!}\equiv 0\pmod p.$$

Therefore

$$\sum_{2p-d\leqslant 3k\leqslant 3p-3}2^k\binom{3k+d}{k}\equiv 0\pmod p.$$

With the help of Theorem 3.1, we have

$$\begin{aligned}\sum_{k=0}^{p-1}2^k\binom{3k+d}{k}\equiv&\sum_{-d\leqslant 3k\leqslant 2p-1-d}2^k\binom{3k+d}{k}\\ \equiv&\sum_{\delta=0}^{1}\sum_{\delta p-d\leqslant 3k\leqslant \delta p+p-1-d}2^k\binom{3k+d}{k}\\ \equiv&\sum_{\delta=0}^{1}(-1)^d(-2)^\delta\left(\frac{4-\delta}{10}+\frac{(3\delta-2)(5d-3)}{10}(-1)^{(p-1)/2}\right)\\ \equiv&\frac{(-1)^{d-1}}{5}\left(1+(10d-6)(-1)^{(p-1)/2}\right)\pmod p.\end{aligned}$$

This yields (1.3) and (1.4). We are done. □

4. Proof of Theorem 1.3

Proof of Theorem 1.3. Obviously (1.5) holds for $p = 2, 3$. Below we assume $p > 3$.

Let $\delta \in \{0, 1\}$. Applying (2.1) with $m = p + \delta p$ and $n = p$ we get

$$\begin{aligned}&2^p \sum_{k=0}^{p} (-2)^k \binom{p}{p+\delta p-3k}\binom{3k-\delta p}{k}\\ =&(-1)^{\delta+1}\sum_{j=0}^{p}\binom{p}{j}\sum_{k=0}^{p+\delta p}(-2)^k\binom{p}{p+\delta p-k}\binom{2j}{k}.\end{aligned} \tag{4.1}$$

Observe that

$$\begin{aligned}&\sum_{k=0}^{p}(-2)^k\binom{p}{p+\delta p-3k}\binom{3k-\delta p}{k}\\ =&\sum_{\delta p\leqslant 3k\leqslant p+\delta p-1}(-2)^k\binom{p}{3k-\delta p}\binom{3k-\delta p}{k}\\ =&1-\delta+\sum_{\delta p<3k<p+\delta p}(-2)^k\binom{p}{3k-\delta p}\binom{3k-\delta p}{k}.\end{aligned}$$

For $j = 1, \ldots, p-1$ clearly

$$\binom{p}{j} = \frac{p}{j}\binom{p-1}{j-1} \equiv p\frac{(-1)^{j-1}}{j} \pmod{p^2}.$$

Thus

$$\begin{aligned}&\sum_{\delta p<3k<p+\delta p}(-2)^k\binom{p}{3k-\delta p}\binom{3k-\delta p}{k}\\ \equiv&\sum_{\delta p<3k<p+\delta p}(-2)^k p\frac{(-1)^{3k-\delta p-1}}{3k-\delta p}\binom{3k-\delta p}{k}\\ \equiv&(-1)^{\delta+1}\sum_{\delta p<3k<p+\delta p}(-2)^k p\frac{(-1)^k}{3k}\binom{(3k-\delta p)+\delta p}{k}\\ &\qquad\qquad\qquad\text{(by Lucas' congruence)}\\ \equiv&(-1)^{\delta+1}\ \frac{p}{3}\sum_{\delta p<3k<p+\delta p}\frac{2^k}{k}\binom{3k}{k} \pmod{p^2}.\end{aligned}$$

Notice that

$$\begin{aligned}&\sum_{j=0}^{p}\binom{p}{j}\sum_{k=0}^{p+\delta p}(-2)^k\binom{p}{p+\delta p-k}\binom{2j}{k}\\
=&\sum_{\delta p\leqslant 2j\leqslant 2p}\binom{p}{j}\sum_{k=\delta p}^{p+\delta p}(-2)^k\binom{p}{k-\delta p}\binom{2j}{k}\\
=&\sum_{\delta p<2j<2p}\binom{p}{j}\sum_{k=\delta p}^{p+\delta p}(-2)^k\binom{p}{k-\delta p}\binom{2j}{k}\\
&+\sum_{2j\in\{\delta p,2p\}}\binom{p}{j}\sum_{k=\delta p}^{p+\delta p}(-2)^k\binom{p}{k-\delta p}\binom{2j}{k}.\end{aligned}$$

Clearly

$$\begin{aligned}&\sum_{\delta p<2j<2p}\binom{p}{j}\sum_{k=\delta p}^{p+\delta p}(-2)^k\binom{p}{k-\delta p}\binom{2j}{k}\\
\equiv&\sum_{\delta p<2j<2p}\binom{p}{j}\left((-2)^{\delta p}\binom{p}{0}\binom{2j}{\delta p}+(-2)^{p+\delta p}\binom{p}{p}\binom{2j}{p+\delta p}\right)\\
\equiv&\sum_{\delta p<2j<2p}\binom{p}{j}(-2)^{\delta p}\binom{2j-\delta p}{0}\\
&+(1-\delta)\sum_{p<2j<2p}\binom{p}{j}(-2)^{p+\delta p}\binom{2j-p}{p-p}\ \text{(by Lucas' congruence)}\\
\equiv&(-2)^{\delta}2^{1-\delta}(2^{p-1}-1)+(1-\delta)(-2)^{1+\delta}(2^{p-1}-1)\\
\equiv&(-1)^{\delta}\delta(2^p-2)=-\delta(2^p-2)\pmod{p^2}.\end{aligned}$$

(Note that $\delta\in\{0,1\}$ and $2\sum_{p/2<j<p}\binom{p}{j}=\sum_{j=1}^{p-1}\binom{p}{j}=2^p-2$.) Also,

$$\sum_{2j=\delta p}\binom{p}{j}\sum_{k=\delta p}^{p+\delta p}(-2)^k\binom{p}{k-\delta p}\binom{2j}{k}=(1-\delta)\sum_{k=0}^{p}(-2)^k\binom{p}{k}\binom{0}{k}=1-\delta$$

and

$$\begin{aligned}&\sum_{2j=2p}\binom{p}{j}\sum_{k=\delta p}^{p+\delta p}(-2)^k\binom{p}{k-\delta p}\binom{2j}{k}\\
\equiv&\sum_{k\in\{\delta p,p+\delta p\}}(-2)^k\binom{p}{k-\delta p}\binom{2p}{k}\\
\equiv&(-2)^{\delta p}\binom{2}{\delta}+(-2)^{p+\delta p}\binom{2}{1+\delta}=4^{\delta p}-2^{p+1}\pmod{p^2}.\end{aligned}$$

(Recall that $\frac{1}{2}\binom{2p}{p} = \binom{2p-1}{p-1} \equiv 1 \pmod{p^3}$ by the Wolstenholme congruence (cf. [Gr] or [HT]).)

Combining the above with (4.1), we have

$$\begin{aligned}&2^p\left(1-\delta+(-1)^{\delta+1}\,\frac{p}{3}\sum_{\delta p<3k<p+\delta p}\frac{2^k}{k}\binom{3k}{k}\right)\\ \equiv&(-1)^{\delta+1}\left(\delta(2-2^p)+1-\delta+4^{\delta p}-2^{p+1}\right) \pmod{p^2}.\end{aligned}$$

Setting $\delta = 0$ and $\delta = 1$ respectively, we obtain

$$2^p - 2^p\frac{p}{3}\sum_{0<3k<p}\frac{2^k}{k}\binom{3k}{k} \equiv 2^{p+1}-2 \pmod{p^2}$$

and

$$2^p\frac{p}{3}\sum_{p<3k<2p}\frac{2^k}{k}\binom{3k}{k} \equiv 2-2^p+4^p-2^{p+1} \pmod{p^2}.$$

It follows that

$$\frac{2}{3}p\sum_{0<3k<2p}\frac{2^k}{k}\binom{3k}{k} \equiv 4^p-4\cdot 2^p+4=(2^p-2)^2\equiv 0 \pmod{p^2}.$$

If $2p \leqslant 3k < 3p$, then

$$\binom{3k}{k} = \frac{3k\cdots(2k+1)}{k!} \equiv 0 \pmod{p}.$$

Therefore

$$\sum_{k=1}^{p-1}\frac{2^k}{k}\binom{3k}{k} = \sum_{0<3k<2p}\frac{2^k}{k}\binom{3k}{k} + \sum_{2p\leqslant 3k<3p}\frac{2^k}{k}\binom{3k}{k} \equiv 0 \pmod{p}.$$

This completes the proof of Theorem 1.3. □

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