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# General congruences for Bernoulli polynomials <sup>☆</sup>

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## Abstract

In this paper we establish some explicit congruences for Bernoulli polynomials modulo a general positive integer. In particular Voronoi's and Kummer's congruences are vastly extended. © 2002 Elsevier Science B.V. All rights reserved.

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## 1. Introduction

The Bernoulli numbers  $B_0, B_1, B_2, \dots$  are defined by the power series

$$\frac{z}{e^z - 1} = \sum_{k=0}^{\infty} B_k \frac{z^k}{k!} \quad (0 < |z| < 2\pi),$$

they can also be defined recursively:

$$B_0 = 1 \quad \text{and} \quad \sum_{k=0}^n \binom{n+1}{k} B_k = 0 \quad \text{for } n \in \mathbb{Z}^+ = \{1, 2, 3, \dots\}.$$

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Here are some interesting analytic results:

$$\tan x = \sum_{k=1}^{\infty} (-1)^{k-1} \frac{2^{2k}(2^{2k}-1)B_{2k}}{(2k)!} x^{2k-1} \quad \text{for } x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right),$$

$$\zeta(-k) = -\frac{B_{k+1}}{k+1} \quad \text{and} \quad \zeta(2k) = (-1)^{k-1} \frac{(2\pi)^{2k}}{2(2k)!} B_{2k} \quad \text{for } k \in \mathbb{Z}^+,$$

where  $\zeta(s)$  denotes the Riemann zeta function.

Let  $\mathbb{N}$  be the set of non-negative integers. For  $k \in \mathbb{N}$  the  $k$ th Bernoulli polynomial  $B_k(x)$  is given by

$$B_k(x) = \sum_{j=0}^k \binom{k}{j} B_j x^{k-j},$$

therefore  $B_k = B_k(0)$ . It is well known that

$$B_k(x+1) - B_k(x) = kx^{k-1}, \quad (1.1)$$

where we regard  $0x^{-1}$  as 0 even if  $x=0$ ; moreover

$$B_k(x+y) = \sum_{j=0}^k \binom{k}{j} B_j(x) y^{k-j}.$$

A useful theorem of Raabe (see Section 1.13 of [3]) asserts that

$$\sum_{r=0}^{n-1} B_k\left(\frac{x+r}{n}\right) = n^{1-k} B_k(x) \quad \text{for } n=1, 2, 3, \dots \quad (1.2)$$

For a Dirichlet character  $\chi$  modulo a positive integer  $m$ , the generalized Bernoulli polynomial  $B_{k,\chi}(x)$  is defined by

$$B_{k,\chi}(x) = m^{k-1} \sum_{r=0}^{m-1} \chi(r) B_k\left(\frac{x+r}{m}\right); \quad (1.3)$$

by Raabe's theorem  $B_{k,\chi}(x) = B_k(x)$  if  $\chi$  is the principal character  $\chi_0$  with  $\chi_0(a) = 1$  for all  $a \in \mathbb{Z}$ ; we also have

$$B_{k,\chi}(x) = \sum_{j=0}^k \binom{k}{j} B_{j,\chi} x^{k-j},$$

where the generalized Bernoulli number  $B_{j,\chi}$  refers to  $B_{j,\chi}(0)$ .

Bernoulli polynomials are of particular importance in number theory; they have close connections with  $p$ -adic analysis, Dirichlet  $L$ -functions and ideal class groups of

cyclotomic fields (cf. [8, pp. 100–109]; [10, pp. 9–19]; [11, pp. 29–35, 54–63 and 77–86]). A great deal of research on them has been done by many mathematicians. It is recommended that the interested reader consult [2], which contains a complete bibliography of related papers published during the period 1713–1990.

Number-theoretic properties of Bernoulli polynomials are fascinating and quite useful. The classical von Staudt–Clausen theorem (see [5, pp. 233–236]) asserts that

$$B_k + \sum_{p-1|k} \frac{1}{p} \in \mathbb{Z} \quad \text{for all } k \in 2\mathbb{Z}^+ = \{2, 4, 6, \dots\}, \tag{1.4}$$

where the sum is over all primes  $p$  such that  $p - 1 | k$ . In 1889 Voronoi (cf. [5, p. 237]) discovered that if  $k \in 2\mathbb{Z}^+$  and  $B_k = U_k/V_k$  (where  $U_k \in \mathbb{Z}$  and  $V_k \in \mathbb{Z}^+$ ) then

$$(m^k - 1)U_k \equiv km^{k-1}V_k \sum_{j=1}^{q-1} j^{k-1} \left[ \frac{jm}{q} \right] \pmod{q} \tag{1.5}$$

for all relatively prime positive integers  $m$  and  $q$ . (As usual, for each  $c$  in the field  $\mathbb{R}$  of real numbers,  $[c]$  denotes the greatest integer not exceeding  $c$ , and we also set  $\{c\} = c - [c]$ .) Kummer’s approach to Fermat’s last theorem made him essentially obtain the following result in 1851: when  $p$  is a prime and  $k$  is a positive integer with  $p - 1 \nmid k$ ,  $p$  does not divide the denominator of  $B_k/k$ , and

$$(1 - p^{k-1}) \frac{B_k}{k} \pmod{p^\alpha} \text{ only depends on } k \pmod{\varphi(p^\alpha)}, \tag{1.6}$$

where  $\alpha \in \mathbb{Z}^+$  and  $\varphi$  denotes Euler’s totient function. (This is [5, Theorem 5 in Chapter 15]; actually Kummer only handled the case  $\alpha = 1$ .)

Bernoulli polynomials have many applications, they are of independent interest as well. In this paper we aim to give explicit congruences for Bernoulli polynomials modulo a *general* positive integer.

From now on we always let  $q$  be a fixed integer greater than one,  $\mathbb{Q}_q$  the ring of  $q$ -adic numbers and  $\mathbb{Z}_q$  the ring of  $q$ -adic integers. A rational number in  $\mathbb{Z}_q$  is usually called a *q-integer*, and by the von Staudt–Clausen theorem  $qB_k$  is a  $q$ -integer for any  $k \in \mathbb{N}$ . It is well known that  $\mathbb{Q}_p$  forms a field if  $p$  is a prime. A good introduction to  $q$ -adic numbers can be found in [7].

Set  $R(q) = \{0, 1, \dots, q - 1\}$  and  $R_*(q) = \{r \in R(q) : r \text{ is coprime to } q\}$ . For  $x \in \mathbb{Z}_q$ , we let  $\langle x \rangle_q$  denote the unique  $r \in R(q)$  such that  $x - r \in q\mathbb{Z}_q$ , and  $[x]_q$  represent the unique  $y \in \mathbb{Z}_q$  with  $qy - x \in R(q)$ ; clearly  $[x]_q = (x + \langle -x \rangle_q)/q$ .

Let  $n \in \mathbb{N}$ . If  $n > 0$  then  $\mathbb{Q}_{q^n}$  and  $\mathbb{Z}_{q^n}$  can be identified with  $\mathbb{Q}_q$  and  $\mathbb{Z}_q$ , respectively (cf. [7, pp. 40–41]); if  $w_1, w_2 \in \mathbb{Z}_q$  and  $w_1 - w_2 \in q^n\mathbb{Z}_q$  then we say that  $w_1$  is congruent to  $w_2$  modulo  $q^n$  and denote this relation by  $w_1 \equiv w_2 \pmod{q^n}$ ; for polynomials  $P(x), Q(x) \in \mathbb{Z}_q[x]$  we write  $P(x) \equiv Q(x) \pmod{q^n}$  if all corresponding coefficients of  $P(x)$  and  $Q(x)$  are congruent modulo  $q^n$ .

For integers  $a_1, \dots, a_k$ , let  $(a_1, \dots, a_k)$  represent, as usual, the greatest common divisor of  $a_1, \dots, a_k$ . For  $x \in \mathbb{Z}_q$ ,  $(x, q)$  refers to  $(\langle x \rangle_q, q)$ . For a positive integer  $n$  and a prime

$p$ , by  $\text{ord}_p(n)$  we mean the largest  $\alpha \in \mathbb{N}$  such that  $p^\alpha \mid n$ ; if  $\alpha = \text{ord}_p(n)$  then we also write  $p^\alpha \parallel n$ . For  $m, n \in \mathbb{Z}^+$ ,  $m \sim_2 n$  stands for  $\text{ord}_2(m) = \text{ord}_2(n)$ . For convenience we also use the logical notations  $\wedge$  (and),  $\vee$  (or),  $\Leftrightarrow$  (if and only if). For an assertion  $A$ , we set

$$[A] = \begin{cases} 1 & \text{if } A \text{ holds,} \\ 0 & \text{otherwise.} \end{cases} \quad (1.7)$$

Due to their generality, the results in this paper are somewhat complicated. Below we state the main theorems and derive some consequences.

Our generalization of Voronoi's congruences is as follows.

**Theorem 1.1.** *Let  $c \in \mathbb{R}$ ,  $d, k, m \in \mathbb{Z}^+$  and  $d \mid m$ . Then the polynomial*

$$\frac{1}{km/d} \left( m^k B_k \left( \frac{x}{m} \right) - \left( \frac{m}{d} (d, q) \right)^k B_k \left( \frac{d}{(d, q)} \cdot \frac{x}{m} - \left\lfloor \frac{c}{(d, q)} \right\rfloor \right) \right) \quad (1.8)$$

is in  $\mathbb{Z}_q[x]$  and is congruent to

$$\sum_{j=0}^{q-1} (x + jm)^{k-1} \left( \left\lfloor \frac{c + jd}{q} \right\rfloor + \frac{1-d}{2} \right) + \frac{q}{2} [2 \mid q \wedge 2 \mid m \wedge d \sim_2 m] (k-1) x^{k-2}$$

modulo  $q$ .

**Corollary 1.1.** *Let  $k, m$  be positive integers and  $x$  a  $q$ -integer. Then*

$$\frac{1}{k} \left( m^k B_k \left( \frac{x}{m} \right) - (m, q)^k B_k \left( \left\{ \frac{x}{(m, q)} \right\} \right) \right) \quad (1.9)$$

is a  $q$ -integer congruent to

$$\sum_{j=0}^{q-1} (x + jm)^{k-1} \left( \left\lfloor \frac{x + jm}{q} \right\rfloor + \frac{1-m}{2} \right) + \frac{q}{2} [2 \mid q \wedge 2 \mid k \wedge 2 \mid m \wedge (k=2 \vee x \notin 2\mathbb{Z}_q)]$$

modulo  $q$ .

**Proof.** Apply Theorem 1.1 with  $c=x$  and  $d=m$ .  $\square$

**Corollary 1.2.** *Let  $k, m \in \mathbb{Z}^+$  and  $y \in \mathbb{Q} \cap \mathbb{Z}_q$ . Then*

$$\frac{1}{k(m, q)} \left( m^k B_k \left( \frac{x+y}{m} \right) - (m, q^2)^k B_k \left( \frac{x}{(m, q^2)} + \left\{ \frac{y}{(m, q^2)} \right\} \right) \right) \quad (1.10)$$

is in  $\mathbb{Z}_q[x]$  and is congruent to

$$\sum_{j=0}^{q-1} (x + y + jm)^{k-1} \left( \left\lfloor \frac{y + jm}{(m, q)q} \right\rfloor + \frac{1}{2} \left( 1 - \frac{m}{(m, q)} \right) \right)$$

modulo  $q$ .

**Proof.** Put  $c = y/(m, q)$ ,  $d = m/(m, q)$  and substitute  $x + y$  for  $x$  in Theorem 1.1.  $\square$

**Corollary 1.3.** Let  $a \in \mathbb{Z}$ ,  $k, m \in \mathbb{Z}^+$  and  $(m, q) = 1$ . Then

$$\begin{aligned} & \frac{1}{k} \left( m^k B_k \left( \frac{x + a}{m} \right) - B_k(x) \right) \\ & \equiv \sum_{j=0}^{q-1} \left( \left\lfloor \frac{a + jm}{q} \right\rfloor + \frac{1 - m}{2} \right) (x + a + jm)^{k-1} \pmod{q}. \end{aligned} \tag{1.11}$$

**Proof.** This follows from Corollary 1.2 in the case  $y = a$ .  $\square$

**Remark 1.1.** Under the condition of Corollary 1.3, Sun [9] announced that

$$\begin{aligned} \frac{m^k B_k(a/m) - B_k}{k} & \equiv \sum_{j=0}^{q-1} \left\lfloor \frac{a + jm}{q} \right\rfloor (a + jm)^{k-1} + \left( 1 - \frac{m + 1}{2} m^{k-1} \right) q B_{k-1} \\ & \quad + [k > 1] \left( 1 - \frac{m + 1}{2} m^{k-2} \right) \frac{k - 1}{2} q^2 B_{k-2} \pmod{q}. \end{aligned}$$

The right-hand side of this congruence contains two unpleasant terms involving Bernoulli numbers; our (1.11) seems better.

We call a function  $f : \mathbb{Z}^+ \rightarrow \mathbb{Z}_q$   $q$ -normal if there are constants  $c_r \in \mathbb{Z}_q$  ( $r \in R_*(q)$ ) such that

$$f(k) \equiv \sum_{r \in R_*(q)} c_r r^k \pmod{q} \quad \text{for all } k \in \mathbb{Z}^+. \tag{1.12}$$

Clearly the set of  $q$ -normal functions forms a commutative ring with respect to the functional addition and multiplication.

Our next theorem is a completely new result.

**Theorem 1.2.** Let  $x, y \in \mathbb{Z}_q$ ,  $m \in \mathbb{Z}^+$  and  $(m, q) = 1$ . Set

$$F(k) = \frac{1}{km^{k-1}} \sum_{d|q} \mu(d) d^{k-1} (m^k B_k([x]_d) - B_k([y]_d)) \quad \text{for } k \in \mathbb{Z}^+, \tag{1.13}$$

where  $\mu$  denotes the Möbius function. Then  $F$  is  $q$ -normal, and furthermore

$$F(k) \equiv \sum_{\substack{j=0 \\ (x+j, q)=1}}^{q-1} \left( \left\lfloor \frac{\langle mx - y \rangle_{q^2} + jm}{q} \right\rfloor + \frac{1-m}{2} \right) (x+j)^{k-1} \pmod{q}. \quad (1.14)$$

**Corollary 1.4.** Let  $k, m \in \mathbb{Z}^+$  and  $(m, q) = 1$ . Then

$$\varphi_k(q)(1-m^k) \frac{B_k}{k} \equiv \sum_{\substack{r=1 \\ (r, q)=1}}^{q-1} r^{k-1} \left( A_r(m, q) - \frac{m+1}{2} \right) \pmod{q} \quad (1.15)$$

where  $\varphi_k(q) = \prod_{p|q} (1-p^{k-1})$  and  $A_r(m, q)$  denotes the least positive integer  $x$  such that  $qx - r \in m\mathbb{Z}$ .

**Proof.** Let  $j \in R(q)$ . Denote by  $r$  the least positive residue of  $-jm$  modulo  $q$ . Obviously  $1 \leq r \leq q$ . Since  $\lfloor jm/q \rfloor \in R(m)$  and  $jm = q \lfloor jm/q \rfloor + q - r$ , we must have  $A_r(m, q) = \lfloor jm/q \rfloor + 1$ . Note that  $(j, q) = 1$  if and only if  $(r, q) = 1$ .

In view of the above,

$$\sum_{j \in R_*(q)} (-jm)^{k-1} \left( \left\lfloor \frac{jm}{q} \right\rfloor + \frac{1-m}{2} \right) \equiv \sum_{r \in R_*(q)} r^{k-1} \left( A_r(m, q) - \frac{m+1}{2} \right) \pmod{q}.$$

Clearly  $(-1)^k \varphi_k(q) B_k = \varphi_k(q) B_k$ , for,  $\varphi_1(q) = 0$ , and  $B_k = 0$  if  $k \in \{3, 5, \dots\}$ . So (1.15) follows from Theorem 1.2 in the case  $x = y = 0$ .  $\square$

**Remark 1.2.** Let  $q > 2$ ,  $m \in \mathbb{Z}^+$  and  $(m, q) = 1$ . It is apparent that

$$\sum_{\substack{r=1 \\ (r, q)=1}}^{q-1} \frac{1}{r} = \sum_{\substack{1 \leq r < q/2 \\ (r, q)=1}} \left( \frac{1}{r} + \frac{1}{q-r} \right) \equiv 0 \pmod{q}.$$

As we will see later

$$B_{\varphi(q)} \prod_{p|q} \left( 1 - \frac{p^{\varphi(q)} - 1}{p-1} \right) \equiv 1 \pmod{q}. \quad (1.16)$$

So (1.15) in the case  $k = \varphi(q)$  implies the following celebrated congruence discovered by Baker [14] and Lerch [15] in 1906:

$$\frac{m^{\varphi(q)} - 1}{q} \equiv - \sum_{\substack{r=1 \\ (r, q)=1}}^{q-1} \frac{A_r(m, q)}{r} \pmod{q}. \quad (1.17)$$

Now we turn to congruences of Kummer's type modulo a general positive integer.

**Theorem 1.3.** Let  $x \in \mathbb{Z}_q$  and

$$S_q = \{k \in \mathbb{Z}^+ : k \not\equiv 0 \pmod{p-1} \text{ for any prime divisor } p \text{ of } q\}. \quad (1.18)$$

(i) We can find  $c_r \in \mathbb{Z}_q$  ( $r \in R_*(q)$ ) such that

$$\sum_{d|q} \mu(d) \frac{d^{k-1}}{k} B_k([x]_d) \equiv \sum_{r \in R_*(q)} c_r r^k \pmod{q} \text{ for all } k \in S_q. \quad (1.19)$$

(ii) If  $l \in S_q$ , then

$$\sum_{k=0}^n \binom{n}{k} \frac{(-1)^k}{k\varphi(q)+l} \sum_{d|q} \mu(d) d^{k\varphi(q)+l-1} B_{k\varphi(q)+l}([x]_d) \equiv 0 \pmod{q^n} \quad (1.20)$$

for every  $n=0, 1, 2, \dots$ .

**Remark 1.3.** Let  $q$  be a prime power  $p^\alpha$ . Then the Kummer result follows from Theorem 1.3(i) in the case  $x=0$ , and Theorem 1.3(ii) was recently showed by Young in [12,13] where  $p$ -adic integrals and measures are employed. When  $q$  is a prime, Theorem 1.3(ii) in the case  $x=0$  gives the strong version of Kummer’s congruences (see [1]), and in the case  $x \neq 0$  it was first obtained by Sun [9].

For generalized Bernoulli polynomials, we have

**Theorem 1.4.** Let  $m \in \mathbb{Z}^+$  and  $(m, q)=1$ . Let  $\mathbb{Q}_q^*$  denote the algebraic closure of  $\mathbb{Q}_q$ , and  $\mathbb{Z}_{q,m}$  stand for the ring

$$\left\{ \sum_{\gamma \in \mathbb{Q}_q^*, \gamma^{\varphi(m)}=1} a_\gamma \gamma : a_\gamma \in \mathbb{Z}_q \right\}.$$

Let  $\chi : \mathbb{Z} \rightarrow \mathbb{Z}_{q,m}$  be a Dirichlet character modulo  $m$  and  $r(q)$  be the product of distinct prime divisors of  $q$ . Let  $x \in \mathbb{Z}_q$  and

$$G(k) = \sum_{d|q} \mu(d) \chi(d) \frac{d^{k-1}}{k} \left( B_{k,\chi} \left( \frac{r(q)}{d} x \right) - [\chi = \chi_0] B_k \right) \text{ for } k \in \mathbb{Z}^+. \quad (1.21)$$

Then we can find  $c_r \in \mathbb{Z}_{q,m}$  ( $r \in R_*(q)$ ) such that

$$G(k) - \sum_{r \in R_*(q)} c_r r^k \in q\mathbb{Z}_{q,m} \text{ for all } k=1, 2, 3, \dots \quad (1.22)$$

If  $l \in \mathbb{Z}^+$  then

$$\sum_{k=0}^n \binom{n}{k} (-1)^k G(k\varphi(q)+l) \in q^n \mathbb{Z}_{q,m} \text{ for } n=0, 1, 2, \dots \quad (1.23)$$

**Remark 1.4.** For generalized Bernoulli numbers, the analogue of Kummer's result obtained by Ernvall [4] follows from the first part of Theorem 1.4. When  $q$  is a power of a prime  $p$  and  $\chi$  is non-principal, the second part of Theorem 1.4 was recently obtained by Young [12,13], and independently given by Sun [9] in the case  $x=0$  and  $p-1 \nmid l$ . Theorem 1.4 in the case  $\chi=\chi_0$ , together with Theorem 1.3 in the case  $x=0$ , shows that we can substitute  $(r(q)/d)x$  for  $[x]_d$  in Theorem 1.3.

Let us give one more theorem.

**Theorem 1.5.** Let  $n$  be a positive integer with  $r(n)=r(q)$ . Let  $x \in \mathbb{Z}_q$  and  $(x, q)=1$ . For  $k \in \mathbb{Z}^+$  let  $S_k(n) = \sum_{r \in R_*(n)} r^k$  and

$$H(k) = S_k(n) \left( \varphi(n) \frac{n^{k-1}}{k} B_k \left( \frac{x}{n} \right) - \varphi_k(n) \frac{B_k}{k} \right). \quad (1.24)$$

Then the function  $H$  is  $q$ -normal, and there are  $c_r \in \mathbb{Z}_q$  ( $r \in R_*(q)$ ) such that

$$S_k(n) \frac{n^{k-1}}{k} B_k \left( \frac{x}{n} \right) \equiv \sum_{r \in R_*(n)} c_r r^k \pmod{q} \quad \text{for all } k \in S_q. \quad (1.25)$$

Theorem 1.5 yields the following analogue of Kummer's congruences.

**Corollary 1.5.** Let  $p$  be a fixed prime and  $x$  a fixed  $p$ -adic integer with  $(x, p)=1$ . For  $\alpha, k \in \mathbb{Z}^+$  with  $p-1 \mid k$ ,

$$P_k(x) := \frac{p^{k-1}}{k} B_k \left( \frac{x}{p} \right) + \frac{p^{k-1} - 1}{p-1} \cdot \frac{B_k}{k} \in \mathbb{Z}_p \quad (1.26)$$

and

$$P_k(x) \pmod{p^\alpha} \text{ only depends on } k \pmod{\varphi(p^\alpha)}. \quad (1.27)$$

**Proof.** Applying Theorem 1.5 with  $q=p^\alpha$  and  $n=p$ , we find that  $S_k(p)P_k(x) = H(k)/(p-1) \in \mathbb{Z}_p$  for  $k \in \mathbb{Z}^+$ , and that  $S_k(p)P_k(x) \pmod{p^\alpha}$  only depends on  $k \pmod{\varphi(p^\alpha)}$ . If  $p-1 \mid k$  then  $S_k(p) = \sum_{r=1}^{p-1} r^k \equiv p-1 \not\equiv 0 \pmod{p}$ ; if  $l \in \mathbb{Z}^+$  and  $l \equiv k \pmod{\varphi(p^\alpha)}$  then  $S_l(p) \equiv S_k(p) \pmod{p^\alpha}$ . So the desired result follows.  $\square$

We shall provide auxiliary results in the next section and prove a key theorem in Section 3. In Section 4 we will be able to extend Voronoi's congruences greatly. In the last section we will prove Theorems 1.2–1.5.

## 2. Preliminaries

**Lemma 2.1.** Let  $n$  be a positive integer. Then

- (i)  $q^{n-1}/n! \in \mathbb{Z}_q$ ,  $q^{n-2}/n \in \mathbb{Z}_q$  if  $n > 2$ , and  $q^{n-3}/(n(n-1)) \in \mathbb{Z}_q$  if  $n > 4$ .



- (ii)  $q/n \in \mathbb{Z}_q$  if  $n$  is squarefree, and  $\binom{x}{n} \in \mathbb{Z}_q$  if  $x \in \mathbb{Z}_q$ .
- (iii)  $q^2/20 \equiv q^2/12 \equiv q^2/4 \equiv [2\|q]q/2 \pmod{q}$ , and

$$\frac{q}{2}x^n \equiv \frac{q}{2}x \equiv \frac{q}{2}[x \notin 2\mathbb{Z}_q] \pmod{q} \quad \text{for } x \in \mathbb{Z}_q.$$

**Proof.** (i) Let  $p$  be any prime divisor of  $q$ . Then

$$\text{ord}_p(n!) = \sum_{i=1}^n \left\lfloor \frac{n}{p^i} \right\rfloor < \sum_{i=1}^{\infty} \frac{n}{p^i} = \frac{n}{p-1}$$

and hence  $(p-1)\text{ord}_p(n!) \leq n-1 \leq \text{ord}_p(q^{n-1})$ . If  $p > 2$  and  $n > 1$ , then

$$\text{ord}_p(n) \leq \text{ord}_p(n(n-1)) \leq \text{ord}_p(n!) \leq \frac{n-1}{p-1} \leq \frac{n-1}{2}.$$

Clearly  $(n-1)/2$  is not more than  $n-2$  or  $n-3$  according as  $n > 2$  or  $n > 4$ . If  $n > 2$  then  $\text{ord}_2(n) \leq \text{ord}_2(n!/2) \leq n-1-1$ ; if  $n > 5$  then  $\text{ord}_2(n(n-1)) \leq \text{ord}_2(n!/(2 \cdot 4)) \leq n-1-3$ ; we also have  $\text{ord}_2(5 \cdot (5-1)) = 5-3$ . So part (i) follows, moreover  $q^{n-4}/(n(n-1)) \in \mathbb{Z}_q$  if  $n > 5$ .

(ii) If  $n$  is squarefree, then  $(q, n/(q, n)) = 1$  and hence  $q/n = (q/(q, n))/(n/(q, n)) \in \mathbb{Z}_q$ .

Suppose that the  $q$ -adic expansion of  $x \in \mathbb{Z}_q$  is  $x = a_0 + a_1q + a_2q^2 + \dots$  where the digits  $a_i$  lie in  $R(q)$ . Let  $a = a_0 + a_1q + \dots + a_{n-1}q^{n-1}$ . Then  $x \equiv a \pmod{q^n}$  and hence

$$\binom{x}{n} - \binom{a}{n} = \frac{1}{n!} \left( \prod_{i=0}^{n-1} (x-i) - \prod_{i=0}^{n-1} (a-i) \right) \equiv 0 \pmod{q}$$

since  $q^{n-1}/n! \in \mathbb{Z}_q$  by part (i). Therefore  $\binom{x}{n} \in \mathbb{Z}_q$ .

(iii) By part (ii),  $q^2/m = (q/m)q \equiv 0 \pmod{q}$  for  $m = 5, 6$ . If  $2 \nmid q$  or  $4 \mid q$  then  $q^2/4 = (q/4)q \equiv 0 \pmod{q}$ ; if  $2 \parallel q$  then  $q^2/4 - q/2 = q(q/2 - 1)/2 \equiv 0 \pmod{q}$ . So

$$\frac{q^2}{20} = \frac{q^2}{4} - \frac{q^2}{5} \equiv \frac{q^2}{12} = \frac{q^2}{4} - \frac{q^2}{6} \equiv \frac{q^2}{4} \equiv \frac{q}{2}[2\|q] \pmod{q}.$$

Let  $x \in \mathbb{Z}_q$ . Then  $(x^n - x)/2 = \sum_{0 < i < n} x^{i-1} \binom{x}{2} \in \mathbb{Z}_q$ . If  $x/2 \notin \mathbb{Z}_q$ , then  $q$  is even and the initial digit  $a_0 = \langle x \rangle_q$  of  $x \in \mathbb{Z}_q$  is odd, therefore  $(x-1)/2 \in \mathbb{Z}_q$ . Thus

$$\frac{q}{2}x^n \equiv \frac{q}{2}x = \frac{q}{2} + \frac{q}{2}(x-1) \equiv \frac{q}{2}[x \notin 2\mathbb{Z}_q] \pmod{q}.$$

The proof of Lemma 2.1 is now complete.  $\square$

**Notation 2.1.** For  $k \in \mathbb{N}$  and  $m \in \mathbb{Z} \setminus \{0\}$  we define  $\delta_m^{(k)}(x) \in \mathbb{Q}[x]$  and  $\delta_{m,q}^{(k)}(x) \in \mathbb{Z}_q[x]$  as follows:

$$\delta_m^{(k)}(x) = m^{k-1} \left( B_k \left( \frac{x}{m} \right) - \left( \frac{x}{m} \right)^k \right) = \sum_{0 < l \leq k} \binom{k}{l} B_l m^{l-1} x^{k-l} \tag{2.1}$$

and

$$\delta_{m,q}^{(k)}(x) = q\delta_m^{(k)}(x) + [k > 0] \frac{q^2}{2} k\delta_m^{(k-1)}(x). \quad (2.2)$$

**Lemma 2.2.** *Let  $k \geq 0$  and  $m \neq 0$  be integers.*

(i) *For each  $w \in \mathbb{Z}_q$  we have*

$$\delta_m^{(k)}(x + (m, q)w) - \delta_m^{(k)}(x) \in \mathbb{Z}_q[x]. \quad (2.3)$$

(ii) *If  $k > 0$  and  $y \in \mathbb{Z}_q$  then*

$$\frac{1}{k}(\delta_m^{(k)}(x + qy) - \delta_m^{(k)}(x)) \equiv y\delta_{m,q}^{(k-1)}(x) \pmod{q}. \quad (2.4)$$

**Proof.** The case  $k=0$  is trivial, so we assume  $k > 0$ . Clearly

$$\begin{aligned} \delta_m^{(k)}(x + m) - \delta_m^{(k)}(x) &= m^{k-1} \left( B_k \left( \frac{x}{m} + 1 \right) - B_k \left( \frac{x}{m} \right) - \left( \frac{x}{m} + 1 \right)^k + \left( \frac{x}{m} \right)^k \right) \\ &= m^{k-1} \left( k \left( \frac{x}{m} \right)^{k-1} - \sum_{j=0}^{k-1} \binom{k}{j} \left( \frac{x}{m} \right)^j \right) \in \mathbb{Z}[x]. \end{aligned}$$

For  $w \in \mathbb{Z}_q$  we can choose  $a \in \mathbb{Z}$  and  $n \in \mathbb{Z}^+$  so that  $w \equiv a \pmod{q}$  and  $(m/(m, q))n \equiv a \pmod{q/(m, q)}$ . Then  $(m, q)w \equiv (m, q)a \equiv mn \pmod{q}$ . As  $qB_l \in \mathbb{Z}_q$  for all  $l \in \mathbb{N}$ , we have  $\delta_m^{(k)}(x + (m, q)w) - \delta_m^{(k)}(x + mn) \in \mathbb{Z}_q[x]$ . On the other hand,

$$\delta_m^{(k)}(x + mn) - \delta_m^{(k)}(x) = \sum_{i=0}^{n-1} (\delta_m^{(k)}(x + im + m) - \delta_m^{(k)}(x + im)) \in \mathbb{Z}[x].$$

So (2.3) holds.

Let  $y \in \mathbb{Z}_q$  and  $D = (\delta_m^{(k)}(x + qy) - \delta_m^{(k)}(x))/k$ . Then

$$\begin{aligned} D &= \frac{m^{k-1}}{k} \sum_{l=1}^k \binom{k}{l} \left( B_{k-l} \left( \frac{x}{m} \right) - \left( \frac{x}{m} \right)^{k-l} \right) \left( \frac{qy}{m} \right)^l \\ &= \sum_{l=1}^k \binom{k-1}{l-1} q\delta_m^{(k-l)}(x) \frac{q^{l-1}}{l} y^l \equiv \delta_{m,q}^{(k-1)}(x)y \pmod{q}, \end{aligned}$$

where in the last step we note that  $(q/2)y^2 \equiv (q/2)y \pmod{q}$  and  $q^{l-1}/l = qq^{l-2}/l \equiv 0 \pmod{q}$  for  $l=3, 4, \dots$ . This proves (2.4).  $\square$

**Lemma 2.3.** *Let  $m, n \in \mathbb{Z}^+$ ,  $k \in \mathbb{N}$ ,  $w_0, \dots, w_{n-1}, y \in \mathbb{Z}_q$  and  $w = qny/m \in \mathbb{Z}_q$ . Then*

$$\frac{1}{m} \sum_{i=0}^{n-1} \frac{(x_i + iqy)^{k+1} - x_i^{k+1}}{k+1} - \frac{n-1}{2} w R_{q,n}^{(k)}(x, y) \equiv 0 \pmod{q}, \tag{2.5}$$

where  $x_i = x + (m, qn)w_i$  and

$$R_{q,n}^{(k)}(x, y) = x^k + \frac{q}{6} (2n-1) k x^{k-1} y + [2\|q] q \binom{k}{3} x^{k-3} y^3. \tag{2.6}$$

**Proof.** For  $t=0, 1, 2, \dots$  we set

$$S(t) = \frac{q^t}{(t+1)(t+2)} \sum_{s=0}^{t+1} \binom{t+2}{s+1} B_{t+1-s} n^s.$$

By simple calculations

$$S(0) = \frac{n-1}{2}, \quad S(1) = \frac{q}{12} (n-1)(2n-1), \quad S(2) = \frac{q^2}{6} (n-1) \binom{n}{2} \equiv 0 \pmod{q}$$

and

$$\begin{aligned} S(3) &= \frac{q^3}{20} \left( n^4 - \frac{5}{2} n^3 + \frac{5}{3} n^2 - \frac{1}{6} \right) \equiv -\frac{q}{2} n^3 \frac{q^2}{4} + q n^2 \frac{q^2}{12} - \frac{q}{10} \cdot \frac{q^2}{12} \\ &\equiv -\frac{q}{2} n \frac{q}{2} [2\|q] - \frac{q}{10} \cdot \frac{q}{2} [2\|q] \equiv \frac{q}{2} [2\|q] (n-1) \pmod{q}. \end{aligned}$$

For any integer  $t > 3$ ,  $q^{t-2}/((t+1)(t+2)) \in \mathbb{Z}_q$  by the proof of Lemma 2.1(i), and so

$$S(t) = q \frac{q^{t-2}}{(t+1)(t+2)} \sum_{s=0}^{t+1} \binom{t+2}{s+1} (q B_{t+1-s}) n^s \equiv 0 \pmod{q}.$$

Notice that

$$\begin{aligned} &\frac{1}{n} \sum_{t=0}^k \binom{k}{t} \frac{q^t}{t+1} x^{k-t} y^t \sum_{i=0}^{n-1} i^{t+1} - \frac{n-1}{2} R_{q,n}^{(k)}(x, y) \\ &= \frac{1}{n} \sum_{t=0}^k \binom{k}{t} x^{k-t} y^t \frac{q^t}{t+1} \cdot \frac{B_{t+2}(n) - B_{t+2}}{t+2} - \frac{n-1}{2} R_{q,n}^{(k)}(x, y) \\ &= \sum_{t=0}^k \binom{k}{t} x^{k-t} y^t S(t) - \frac{n-1}{2} R_{q,n}^{(k)}(x, y) \equiv 0 \pmod{q}. \end{aligned}$$

Let  $u$  and  $v$  be integers such that  $mu + qnv = (m, qn)$ . Then  $(qy/m)(m, qn) = q(uy + vw) \equiv 0 \pmod{q}$ . Thus, for the left-hand side  $L$  of (2.5) we have

$$\begin{aligned} L &= \frac{1}{m} \sum_{i=0}^{n-1} \frac{1}{k+1} \sum_{j=1}^{k+1} \binom{k+1}{j} x_i^{k+1-j} (iqy)^j - \frac{n-1}{2} wR_{q,n}^{(k)}(x, y) \\ &= \frac{qy}{m} \sum_{j=1}^{k+1} \binom{k}{j-1} \frac{q^{j-1}}{j} y^{j-1} \sum_{i=0}^{n-1} i^j x_i^{k+1-j} - \frac{n-1}{2} wR_{q,n}^{(k)}(x, y) \\ &\equiv \frac{w}{n} \sum_{t=0}^k \binom{k}{t} \frac{q^t}{t+1} x^{k-t} y^t \sum_{i=0}^{n-1} i^{t+1} - \frac{n-1}{2} wR_{q,n}^{(k)}(x, y) \equiv 0 \pmod{q}. \end{aligned}$$

This concludes the proof.  $\square$

**Proof of (1.16).** Let  $p$  be any prime divisor of  $q > 2$ . Set  $k = \varphi(q)$  and  $\alpha = \text{ord}_p(q)$ . Then

$$\begin{aligned} \sum_{r=1}^{p-1} r^k &= \frac{1}{k+1} \sum_{l=0}^k \binom{k+1}{k-l} B_{k-l} p^{l+1} \\ &= pB_k + pk \sum_{l=1}^k \binom{k-1}{l-1} (pB_{k-l}) \frac{p^{l-1}}{l(l+1)}. \end{aligned}$$

As  $p^\alpha \mid pk$  and  $k > \alpha$ , we have  $p-1 \equiv pB_k \pmod{p^\alpha}$  and  $(1-p^{k-1})pB_k/(p-1) \equiv 1 \pmod{p^\alpha}$ . If  $p_* \neq p$  is another prime divisor of  $q$  and  $\beta = \text{ord}_{p_*}(p_* - 1)$ , then  $\varphi(p_*^{\alpha+\beta}) \mid \varphi(q)$  and so  $p^\alpha$  divides  $(p_*^{\varphi(q)} - 1)/(p_* - 1)$ . Clearly (1.16) follows from the above.  $\square$

### 3. A crucial theorem

**Lemma 3.1.** Let  $k \in \mathbb{N}$ ,  $m \in \mathbb{Z}^+$  and  $r \in \mathbb{Z}$ . Then

$$\frac{k+1}{m^k} \sum_{\substack{0 \leq j < q \\ m \mid j-r}} (x+j)^k = B_{k+1} \left( \frac{x+q}{m} + \left\{ \frac{r-q}{m} \right\} \right) - B_{k+1} \left( \frac{x}{m} + \left\{ \frac{r}{m} \right\} \right). \quad (3.1)$$

**Proof.** Let  $f(y) = (k+1)^{-1} B((x+r)/m + y)$  for  $y \in \mathbb{Z}$ . Then  $\Delta f(y) = f(y+1) - f(y) = ((x+r)/m + y)^k$ . Note that  $1 + \lfloor (-1-r)/m \rfloor = \lfloor (m-1)/m \rfloor - \{r/m\} - \lfloor r/m \rfloor = -\lfloor r/m \rfloor$ . Therefore

$$\begin{aligned} \frac{1}{m^k} \sum_{\substack{0 \leq j < q \\ m \mid j-r}} (x+j)^k &= \sum_{\substack{0 \leq j < q \\ m \mid j-r}} \Delta f \left( \frac{j-r}{m} \right) = \sum_{(-1-r)/m < i \leq (q-1-r)/m} \Delta f(i) \\ &= f \left( \left\lfloor \frac{q-1-r}{m} \right\rfloor + 1 \right) - f \left( \left\lfloor \frac{-1-r}{m} \right\rfloor + 1 \right) \end{aligned}$$

$$\begin{aligned}
 &= f\left(-\left\lfloor\frac{r-q}{m}\right\rfloor\right) - f\left(-\left\lfloor\frac{r}{m}\right\rfloor\right) \\
 &= \frac{1}{k+1}\left(B_{k+1}\left(\frac{x+q}{m} + \left\{\frac{r-q}{m}\right\}\right) - B_{k+1}\left(\frac{x}{m} + \left\{\frac{r}{m}\right\}\right)\right).
 \end{aligned}$$

This ends the proof.  $\square$

**Remark 3.1.** Since (3.1) is our starting point, we would better give some historical remarks. It was first observed by Lehmer [6] in the case  $x=0$  and  $r=q$ . In 1991 the author obtained a congruence version of (3.1), then his brother Sun derived (3.1) in the case  $x=0$  by a complicated method.

**Lemma 3.2.** Let  $k \geq 0$ ,  $m > 0$ ,  $n > 0$  and  $r$  be integers, and  $\lambda: \mathbb{N} \rightarrow \mathbb{Z}_q$  be a function with  $\lambda(0)=0$ . Then

$$\begin{aligned}
 &\sum_{s=1}^n \lambda(s) \sum_{\substack{0 \leq j < q \\ m|j-r-qs}} (x+j)^k \\
 &\equiv \frac{m^k}{k+1} \sum_{t=0}^{n-1} \Delta \lambda(t) B_{k+1}\left(\frac{x-qt}{m} + \left\{\frac{r+qt}{m}\right\}\right) \\
 &\quad - \frac{m^k}{k+1} \lambda(n) B_{k+1}\left(\frac{x-qn}{m} + \left\{\frac{r+qn}{m}\right\}\right) \pmod{q}.
 \end{aligned}$$

**Proof.** By Lemma 3.1 we have

$$\begin{aligned}
 &\frac{k+1}{m^k} \sum_{s=1}^n \lambda(s) \sum_{\substack{0 \leq j < q \\ m|j-r-qs}} (x-qs+j)^k \\
 &= \sum_{s=1}^n \lambda(s) B_{k+1}\left(\frac{x-q(s-1)}{m} + \left\{\frac{r+q(s-1)}{m}\right\}\right) \\
 &\quad - \sum_{s=1}^n \lambda(s) B_{k+1}\left(\frac{x-qs}{m} + \left\{\frac{r+qs}{m}\right\}\right),
 \end{aligned}$$

so the desired result follows by Abel’s partial summation identity.  $\square$

**Lemma 3.3.** Let  $k \in \mathbb{N}$ ,  $m, n \in \mathbb{Z}^+$  and  $y \in \mathbb{R}$ . Then

$$n^k \sum_{r=0}^{n-1} B_{k+1}\left(\frac{x}{n} + \left\{\frac{mr+y}{n}\right\}\right) = (m, n)^{k+1} B_{k+1}\left(\frac{x}{(m, n)} + \left\{\frac{y}{(m, n)}\right\}\right).$$

**Proof.** Set  $\bar{w} = w/(m, n)$  for  $w \in \{m, n, x, y\}$ . Then

$$\begin{aligned} \sum_{r=0}^{n-1} B_{k+1} \left( \frac{x}{n} + \left\{ \frac{mr + y}{n} \right\} \right) &= \sum_{u=0}^{\bar{n}-1} \sum_{v=0}^{(m,n)-1} B_{k+1} \left( \frac{x}{n} + \left\{ \frac{\bar{m}(u + \bar{n}v) + \bar{y}}{\bar{n}} \right\} \right) \\ &= (m, n) \sum_{u=0}^{\bar{n}-1} B_{k+1} \left( \frac{x}{n} + \left\{ \frac{\bar{m}u + \lfloor \bar{y} \rfloor + \{ \bar{y} \}}{\bar{n}} \right\} \right) \\ &= (m, n) \sum_{t=0}^{\bar{n}-1} B_{k+1} \left( \frac{\bar{x}}{\bar{n}} + \frac{t + \{ \bar{y} \}}{\bar{n}} \right). \end{aligned}$$

Applying Raabe's theorem we then obtain the desired identity.  $\square$

Now we are ready to give

**Theorem 3.1.** Let  $d, m$  and  $n$  be positive integers for which  $d | n$  and  $m | qn$ . Let  $\bar{d} = (d, qn/m)$ ,  $\bar{m} = (m, qn/d)$ ,  $k \in \mathbb{N}$  and  $r \in \mathbb{Z}$ . Suppose that  $2 \nmid d$  or  $2 \nmid q$  or  $2m | qn$ . Then

$$\begin{aligned} &\frac{1}{k+1} \left( dm^k B_{k+1} \left( \frac{x}{m} + \left\{ \frac{r}{m} \right\} \right) - \bar{d} \bar{m}^k B_{k+1} \left( \frac{x}{\bar{m}} + \left\{ \frac{r}{\bar{m}} \right\} \right) \right) \\ &\equiv \sum_{s=1}^n \left( \left[ -\frac{ds}{n} \right] + \frac{d+1}{2} \right) \sum_{\substack{0 \leq j < q \\ m|j-r-qs}} (x+j)^k \\ &\quad + \left( \frac{q}{4} [4|d] - \frac{q}{3} [3|d] \right) \frac{qn}{m} \cdot \frac{n}{d} k(x+r)^{k-1} \pmod{q}. \end{aligned} \quad (3.2)$$

**Proof.** Clearly we may assume  $r \in R(m)$ . Let  $y = n/d$  and  $\lambda_d(s) = \lfloor -s/y \rfloor$  for  $s \in \mathbb{N}$ . Then  $\lambda_d(0) = 0$ ,  $\lambda_d(n) = -d$ ,  $\Delta \lambda_d(s) = 0$  if  $y \nmid s$ , and  $\Delta \lambda_d(ty) = -1$  for  $t \in \mathbb{N}$ . Set

$$x_t = x - qty + m \left\{ \frac{r + qty}{m} \right\} \quad \text{for } t \in R(d).$$

Obviously  $x_0 = x + r$  and  $x_t - x_0 \in m\mathbb{Z}$ .

Let  $S(a, x) = \sum_{\substack{0 \leq j < q \\ m|j-a}} (x+j)^k$  for  $a \in \mathbb{Z}$ . By Lemma 3.2,  $\sum_{s=1}^n \lambda_d(s) S(r + qs, x)$  is congruent to

$$\begin{aligned} &\frac{m^k}{k+1} \sum_{s=0}^{n-1} \Delta \lambda_d(s) B_{k+1} \left( \frac{x - qs}{m} + \left\{ \frac{r + qs}{m} \right\} \right) \\ &\quad - \frac{m^k}{k+1} \lambda_d(n) B_{k+1} \left( \frac{x - qn}{m} + \left\{ \frac{r + qn}{m} \right\} \right) \\ &= -\frac{m^k}{k+1} \left( \sum_{t=0}^{d-1} B_{k+1} \left( \frac{x_t}{m} \right) - dB_{k+1} \left( \frac{x_0 - qn}{m} \right) \right) \end{aligned}$$

modulo  $q$ .

In view of the above and Lemmas 2.2 and 2.3

$$\begin{aligned}
 - \sum_{s=1}^n S(r + qs, x) &= \sum_{s=1}^n \lambda_1(s) S(r + qs, x) \\
 &\equiv \frac{m^k}{k+1} \left( B_{k+1} \left( \frac{x_0 - qn}{m} \right) - B_{k+1} \left( \frac{x_0}{m} \right) \right) \\
 &= \frac{\delta_m^{(k+1)}(x_0 - qn) - \delta_m^{(k+1)}(x_0)}{k+1} + \frac{(x_0 - qn)^{k+1} - x_0^{k+1}}{(k+1)m} \\
 &\equiv -n\delta_{m,q}^{(k)}(x_0) + \frac{2-1}{2} \cdot \frac{q2(-n)}{m} R_{q,2}^{(k)}(x_0, -n) \\
 &\equiv -n\delta_{m,q}^{(k)}(x_0) - \frac{qn}{m} \left( x_0^k + \frac{q}{6} (2 \cdot 2 - 1) kx_0^{k-1}(-n) \right) \pmod{q}.
 \end{aligned}$$

If  $2 \mid (d, q)$ , then  $m \mid q(n/2)$  and hence

$$\begin{aligned}
 \frac{1}{2} \sum_{s=1}^n S(r + qs, x) &= \sum_{s=1}^{n/2} S(r + qs, x) \\
 &\equiv \frac{n}{2} \delta_{m,q}^{(k)}(x_0) + \frac{qn/2}{m} \left( x_0^k - \frac{q}{2} \cdot \frac{n}{2} kx_0^{k-1} \right) \pmod{q}
 \end{aligned}$$

in a similar way. If  $2 \nmid d$  or  $2 \nmid q$ , then  $(d+1)/2 \in \mathbb{Z}_q$ . Therefore the right-hand side of the congruence (3.2) belongs to  $\mathbb{Z}_q[x]$ , and

$$\begin{aligned}
 S &:= \frac{d+1}{2} \sum_{s=1}^n S(r + qs, x) + \frac{dm^k}{k+1} \left( B_{k+1} \left( \frac{x_0 - pn}{m} \right) - B_{k+1} \left( \frac{x_0}{m} \right) \right) \\
 &\equiv \frac{d+1}{2} n\delta_{m,q}^{(k)}(x_0) + \frac{d+1}{2} \cdot \frac{qn}{m} \left( x_0^k - \frac{q}{2} \cdot \frac{n}{(2,d,q)} kx_0^{k-1} \right) \\
 &\quad - dn\delta_{m,q}^{(k)}(x_0) - d \frac{qn}{m} \left( x_0^k - \frac{q}{2} n kx_0^{k-1} \right) \\
 &\equiv \frac{1-d}{2} n\delta_{m,q}^{(k)}(x_0) + \frac{1-d}{2} \cdot \frac{qn}{m} \left( x_0^k + \frac{q}{2} \cdot \frac{n}{(2,d,q)} kx_0^{k-1} \right) \pmod{q}.
 \end{aligned}$$

We also have

$$\begin{aligned}
 &\sum_{s=1}^n \left( \left\lfloor -\frac{ds}{n} \right\rfloor + \frac{d+1}{2} \right) S(r + qs, x) \\
 &= \sum_{s=1}^n \lambda_d(s) S(r + qs, x) + \frac{d+1}{2} \sum_{s=1}^n S(r + qs, x) \\
 &\equiv \frac{m^k}{k+1} \left( dB_{k+1} \left( \frac{x_0}{m} \right) - \sum_{t=0}^{d-1} B_{k+1} \left( \frac{x_t}{m} \right) \right) + S = W + R + S \pmod{q}
 \end{aligned}$$

where

$$W = \frac{m^k}{k+1} \left( dB_{k+1} \left( \frac{x_0}{m} \right) - \sum_{t=0}^{d-1} B_{k+1} \left( \frac{x_t + tqy}{m} \right) \right)$$

and

$$R = \frac{m^k}{k+1} \sum_{t=0}^{d-1} \left( B_{k+1} \left( \frac{x_t + tqy}{m} \right) - B_{k+1} \left( \frac{x_t}{m} \right) \right).$$

In light of Lemma 3.3 and the equality  $d\bar{m} = \bar{d}m$ ,

$$\begin{aligned} -(k+1)W + dm^k B_{k+1} \left( \frac{x}{m} + \left\{ \frac{r}{m} \right\} \right) &= m^k \sum_{t=0}^{d-1} B_{k+1} \left( \frac{x}{m} + \left\{ \frac{r}{m} + \frac{qn}{m} \frac{t}{d} \right\} \right) \\ &= m^k \frac{\bar{d}^{k+1}}{d^k} B_{k+1} \left( \frac{dx/m}{\bar{d}} + \left\{ \frac{dr/m}{\bar{d}} \right\} \right) \\ &= \bar{d}\bar{m}^k B_{k+1} \left( \frac{x}{\bar{m}} + \left\{ \frac{r}{\bar{m}} \right\} \right). \end{aligned}$$

Let  $D = ((k+1)m)^{-1} \sum_{t=0}^{d-1} ((x_t + tqy)^{k+1} - x_t^{k+1})$ . By Lemma 2.2

$$R - D \equiv \sum_{t=0}^{d-1} ty \delta_{m,q}^{(k)}(x_t) \equiv \frac{d(d-1)}{2} y \delta_{m,q}^{(k)}(x_0) \pmod{q}.$$

Since  $((d-1)/2)qdy/m = ((d-1)/2)qn/m \in \mathbb{Z}_q$ , applying Lemma 2.3 we get

$$D \equiv \frac{d-1}{2} \cdot \frac{qdy}{m} R_{q,d}^{(k)}(x_0, y) \equiv \frac{d-1}{2} \cdot \frac{qn}{m} \left( x_0^k + \frac{q}{6} (2d-1) kx_0^{k-1} y \right) \pmod{q}.$$

$$\begin{aligned} (R-D) + D + S &\equiv \frac{d-1}{2} \cdot \frac{qn}{m} \cdot \frac{n}{d} kx_0^{k-1} \left( (2d-1) \left( \frac{q}{2} - \frac{q}{3} \right) - \frac{q}{2} \cdot \frac{d}{(2,d,q)} \right) \\ &\equiv \left( \frac{q}{3} [3|d] - \frac{q}{4} [4|d] \right) \frac{qn}{m} \cdot \frac{n}{d} kx_0^{k-1} \pmod{q}. \end{aligned}$$

Combining the above we obtain (3.2).  $\square$



**4. General version of Voronoi’s congruences**

**Lemma 4.1.** *Let  $k \in \mathbb{N}$ ,  $d, m, n \in \mathbb{Z}^+$ ,  $d \mid n$ ,  $m \mid qn$  and  $r \in R(m)$ . Let  $f$  be any function from  $\mathbb{Z}$  to  $\mathbb{Z}_q$ . Then*

$$\begin{aligned} & \sum_{s=1}^n f\left(\left\lfloor -\frac{ds}{n} \right\rfloor\right) \sum_{\substack{0 \leq j < q \\ m \mid j-r-qs}} (x+j)^k \\ & \equiv \sum_{j=0}^{qn/m-1} (x+r+jm)^k f\left(\left\lfloor \frac{r+jm}{qn/d} \right\rfloor - d\right) \pmod{q}. \end{aligned} \tag{4.1}$$

**Proof.** Let  $R$  denote the right-hand side of the congruence (4.1). Then

$$\begin{aligned} R &= \sum_{\substack{0 \leq a < qn \\ a \equiv r \pmod{m}}} (x+a)^k f\left(\left\lfloor \frac{ad}{qn} \right\rfloor - d\right) \\ &= \sum_{\substack{1 \leq b \leq qn \\ m \mid qn-b-r}} (x+qn-b)^k f\left(\left\lfloor \frac{-bd}{qn} \right\rfloor\right) \\ &\equiv \sum_{\substack{1 \leq b \leq qn \\ m \mid b+r}} (x-b)^k f\left(\left\lfloor \frac{-b/q}{n/d} \right\rfloor\right) \\ &= \sum_{s=1}^n \sum_{\substack{0 \leq j < q \\ m \mid qs-j+r}} (x+j-qs)^k f\left(\left\lfloor \frac{(j-qs)/q}{n/d} \right\rfloor\right) \\ &\equiv \sum_{s=1}^n \sum_{\substack{0 \leq j < q \\ m \mid j-r-qs}} (x+j)^k f\left(\left\lfloor \frac{-s}{n/d} \right\rfloor\right) \pmod{q}. \end{aligned}$$

This completes the proof.  $\square$

**Lemma 4.2.** *Let  $k \in \mathbb{N}$ ,  $m, n \in \mathbb{Z}^+$ ,  $2 \mid n$  and  $2m \mid qn$ . Then*

$$\begin{aligned} & \sum_{j=0}^{qn/(2m)-1} (x+jm)^k - \frac{1}{2} \sum_{j=0}^{qn/m-1} (x+jm)^k \\ & \equiv \frac{q}{2} k [2 \parallel n] ([q \sim_2 m] x^{k-1} + [2 \parallel q \wedge 2 \nmid m] \Delta(x^{k-1})) \pmod{q}. \end{aligned} \tag{4.2}$$

**Proof.** Since  $(qn/2)^2/2 = q(q/2)(n/2)^2 \equiv 0 \pmod{q}$ , if  $j \in \mathbb{Z}$  then

$$\begin{aligned} d_j &:= \frac{1}{2} \left( (x+jm)^k - \left( x+jm + q\frac{n}{2} \right)^k \right) \\ &\equiv -\frac{k}{2} (x+jm)^{k-1} q \frac{n}{2} = -\frac{q}{2} k \frac{n}{2} \left( x^{k-1} + \sum_{0 < l < k} \binom{k-1}{l} (jm)^l x^{k-1-l} \right) \\ &\equiv \frac{q}{2} k [2||n] (x^{k-1} + jm((x+1)^{k-1} - x^{k-1})) \pmod{q}. \end{aligned}$$

As the left-hand side  $d$  of the congruence (4.2) equals  $\sum_{0 \leq j < qn/(2m)} d_j$ , we have

$$\begin{aligned} d &\equiv \frac{q}{2} k [2||n] \left( \frac{qn}{2m} x^{k-1} + \frac{m}{2} \cdot \frac{qn}{2m} \left( \frac{qn}{2m} - 1 \right) \Delta(x^{k-1}) \right) \\ &\equiv \frac{q}{2} k [2||n] \left( \left[ 2 \uparrow \frac{qn}{2m} \right] x^{k-1} + \left[ 2 \uparrow m \wedge 2 \left\| \frac{qn}{2m} \right\| \right] \Delta(x^{k-1}) \right) \pmod{q} \end{aligned}$$

and this concludes the proof.  $\square$

Now we are able to give

**Theorem 4.1.** Let  $k \in \mathbb{N}$ ,  $d, m, n \in \mathbb{Z}^+$ ,  $d | n$ ,  $m | qn$ , and  $2 \uparrow d$  or  $2 \uparrow q$  or  $2m | qn$ . Put  $\bar{d} = (d, qn/m)$  and  $\bar{m} = (m, qn/d)$ . Then for any  $y \in \mathbb{R}$  the polynomial

$$L(x, y) = \frac{1}{k+1} \left( dm^k B_{k+1} \left( \frac{x}{m} \right) - \bar{d} \bar{m}^k B_{k+1} \left( \frac{x}{\bar{m}} - \left\lfloor \frac{y}{\bar{m}} \right\rfloor \right) \right) \quad (4.3)$$

is in  $\mathbb{Z}_q[x]$  and is congruent to

$$\begin{aligned} R(x, y) &= \sum_{j=0}^{qn/m-1} (x+jm)^k \left( \left\lfloor \frac{y+jm}{qn/d} \right\rfloor + \frac{1-d}{2} \right) - \frac{q}{3} [3 | d] \frac{n}{d} \cdot \frac{qn}{m} kx^{k-1} \\ &\quad + \frac{q}{2} k [d \sim_2 n] \left( \left[ 2 | n \wedge 2 \left\| \frac{qn}{m} \right\| \right] x^{k-1} + [2 \uparrow m \wedge 2 || n \wedge 2 || q] \Delta(x^{k-1}) \right) \end{aligned}$$

modulo  $q$ .

**Proof.** Clearly

$$L(x+m, y+m) - L(x, y) = \frac{dm^k}{k+1} \left( B_{k+1} \left( \frac{x}{m} + 1 \right) - B_{k+1} \left( \frac{x}{m} \right) \right) = dx^k.$$

Since  $(q/2)[2 | n \wedge 2 || (qn/m)]m \equiv 0 \pmod{q}$  and  $qn(d-1)/2 = q(n/d) \binom{d}{2} \equiv 0 \pmod{q}$ , we also have

$$\begin{aligned} R(x+m, y+m) - R(x, y) &\equiv \sum_{i=1}^{qn/m} (x+im)^k \left( \left\lfloor \frac{y+im}{qn/d} \right\rfloor + \frac{1-d}{2} \right) \end{aligned}$$

$$\begin{aligned}
 & - \sum_{j=0}^{qn/m-1} (x+jm)^k \left( \left\lfloor \frac{y+jm}{qn/d} \right\rfloor + \frac{1-d}{2} \right) \\
 &= (x+qn)^k \left( \left\lfloor \frac{y+qn}{qn/d} \right\rfloor + \frac{1-d}{2} \right) - x^k \left( \left\lfloor \frac{y}{qn/d} \right\rfloor + \frac{1-d}{2} \right) \\
 &\equiv dx^k \pmod{q}.
 \end{aligned}$$

So we can assume  $y \in [0, m)$  without any loss of generality.

Let  $r = \lfloor y \rfloor$ . Then  $L(x, y) = L(x, r)$  and  $R(x, y) = R(x, r)$ . By Theorem 3.1,  $L(x, r)$  is in  $\mathbb{Z}_q[x]$  and is congruent to

$$\begin{aligned}
 & \sum_{s=1}^n \left( \left\lfloor -\frac{ds}{n} \right\rfloor + \frac{d+1}{2} \right) \sum_{\substack{0 \leq j < q \\ m|j-r-qs}} (x-r+j)^k \\
 & + \left( \frac{q}{4} [4|d] - \frac{q}{3} [3|d] \right) \frac{qn}{m} \cdot \frac{n}{d} kx^{k-1}
 \end{aligned}$$

modulo  $q$ . Set

$$P(x) = \sum_{j=0}^{qn/m-1} (x+jm)^k \left( \left\lfloor \frac{r+jm}{qn/d} \right\rfloor + \frac{1-d}{2} \right).$$

If  $2 \nmid (d, q)$ , then  $(d+1)/2 \in \mathbb{Z}_q$  and hence

$$L(x, r) \equiv P(x) - \frac{q}{3} [3|d] \frac{qn}{m} \cdot \frac{n}{d} kx^{k-1} \equiv R(x, r) \pmod{q}$$

with the help of Lemma 4.1. In the case  $2 | (d, q)$ , by Lemmas 4.1 and 4.2 we have

$$\begin{aligned}
 & L(x, r) + \frac{q}{3} [3|d] \frac{n}{d} \cdot \frac{qn}{m} kx^{k-1} - \frac{q}{2} [4|d] \frac{n}{d} \cdot \frac{qn}{2m} kx^{k-1} \\
 &\equiv \sum_{s=1}^n \left( \left\lfloor -\frac{ds}{n} \right\rfloor + \frac{d}{2} \right) \sum_{\substack{0 \leq j < q \\ m|j-r-qs}} (x-r+j)^k + \sum_{s=1}^{n/2} \sum_{\substack{0 \leq j < q \\ m|j-r-qs}} (x-r+j)^k \\
 &\equiv \sum_{j=0}^{qn/m-1} (x+jm)^k \left( \left\lfloor \frac{r+jm}{qn/d} \right\rfloor - \frac{d}{2} \right) + \sum_{j=0}^{qn/(2m)-1} (x+jm)^k \\
 &\equiv P(x) + \frac{q}{2} k [2|n] ([q \sim_2 m] x^{k-1} + [2|q \wedge 2 \nmid m] \Delta(x^{k-1})) \pmod{q},
 \end{aligned}$$

therefore  $L(x, r) \equiv R(x, r) \pmod{q}$ . We are done.  $\square$

**Proof of Theorem 1.1.** Simply apply Theorem 4.1 with  $n=m$  and  $y=cm/d$ .  $\square$

## 5. Proofs of Theorems 1.2–1.5

**Lemma 5.1.** Let  $d, k, m, n$  be positive integers with  $(m, q) = 1$  and  $d \mid (n, q)$ , and  $x, y$  be  $q$ -adic numbers with  $nx \in \mathbb{Z}_q$  and  $mx - y \in a + (q^2/d)\mathbb{Z}_q$  where  $a \in \mathbb{Z}$ . Then

$$\frac{m^k B_k(x) - B_k(y)}{km^{k-1}} - \sum_{j=0}^{q/d-1} \left( \left\lfloor \frac{a+jm}{q/d} \right\rfloor + \frac{1-m}{2} \right) (x+j)^{k-1} \in \frac{q}{n^{k-1}} \mathbb{Z}_q. \quad (5.1)$$

**Proof.** Write  $mx - y = a + (q^2/d)z$  where  $z \in \mathbb{Z}_q$ . For  $1 \leq l \leq k$ ,  $(q^2/d)^{l-1}/l = (q/d)^{l-1} q^{l-1}/l \in \mathbb{Z}_q$  by Lemma 2.1, and

$$\frac{q}{d} B_{k-l}(y) n^{k-l} = \frac{q}{d} (ny)^{k-l} + \sum_{0 < h \leq k-l} \binom{k-l}{h} (qB_h)(ny)^{k-l-h} \frac{n^h}{d} \in \mathbb{Z}_q$$

since  $ny = m(nx) - n(mx - y) \in \mathbb{Z}_q$ . Thus

$$\begin{aligned} \frac{n^{k-1}}{k} (B_k(mx - a) - B_k(y)) &= \frac{n^{k-1}}{k} \sum_{l=1}^k \binom{k}{l} \left( \frac{q^2}{d} z \right)^l B_{k-l}(y) \\ &= q \sum_{l=1}^k \binom{k-1}{l-1} \frac{(q^2/d)^{l-1}}{l} z^l \frac{q}{d} B_{k-l}(y) n^{k-1} \equiv 0 \pmod{q}. \end{aligned}$$

Let  $d' = m$ ,  $n' = mn/d$ ,  $m' = mn$ ,  $x' = m'x \in \mathbb{Z}_q$  and  $y' = an \in \mathbb{Z}$ . Clearly  $d' \mid n'$ ,  $m' \mid qn'$ ,  $(d', q) = 1$ ,  $(d', qn'/m') = (m, q/d) = 1$  and  $(m', qn'/d') = (mn, qn/d) = (m, q/d)n = n$ . By Theorem 4.1,

$$\begin{aligned} &\frac{1}{k} \left( d'(m')^{k-1} B_k \left( \frac{x'}{m'} \right) - n^{k-1} B_k \left( \frac{x'}{n} - \left\lfloor \frac{y'}{n} \right\rfloor \right) \right) \\ &\equiv \sum_{j=0}^{qn'/m'-1} \left( \left\lfloor \frac{y'+jm'}{qn'/d'} \right\rfloor + \frac{1-d'}{2} \right) (x'+jm')^{k-1} \pmod{q}. \end{aligned}$$

As  $n^{k-1} k^{-1} (m^k B_k(x) - B_k(y)) \equiv k^{-1} (d'(m')^{k-1} B_k(x) - n^{k-1} B_k(mx - a)) \pmod{q}$ , the desired result follows.  $\square$

**Proof of Theorem 1.2.** For  $d \in \mathbb{Z}^+$  with  $d \mid q$ , let  $a_d = \langle m[x]_d - [y]_d \rangle_{q^2}$  and  $\psi(d) = \langle mx - y \rangle_{q^2} + m \langle -x \rangle_d / d$ . Then  $\lfloor \psi(d) \rfloor \in a_d + (q^2/d)\mathbb{Z}_q$  because

$$d \lfloor \psi(d) \rfloor \equiv mx - y + m \langle -x \rangle_d - \langle -y \rangle_d = d(m[x]_d - [y]_d) \equiv da_d \pmod{q^2}.$$

Since  $[(x + j, q) = 1] = \sum_{d|(x+j, q)} \mu(d)$ , the right-hand side of the congruence (1.14) equals  $\sum_{d|q} \mu(d)\Psi(d)$  where

$$\begin{aligned} \Psi(d) &= \sum_{\substack{j=0 \\ d|j-\langle -x \rangle_d}}^{q-1} \left( \left\lfloor \frac{\langle mx - y \rangle_{q^2} + jm}{q} \right\rfloor + \frac{1-m}{2} \right) (x+j)^{k-1} \\ &= \sum_{i=0}^{q/d-1} \left( \left\lfloor \frac{\langle mx - y \rangle_{q^2} + m(\langle -x \rangle_d + id)}{q} \right\rfloor + \frac{1-m}{2} \right) (x + \langle -x \rangle_d + id)^{k-1} \\ &= d^{k-1} \sum_{i=0}^{q/d-1} \left( \left\lfloor \frac{\psi(d) + im}{q/d} \right\rfloor + \frac{1-m}{2} \right) ([x]_d + i)^{k-1}. \end{aligned}$$

By Lemma 5.1,  $\Psi(d) \equiv d^{k-1}/(km^{k-1})^{-1}(m^k B_k([x]_d) - B_k([y]_d)) \pmod{q}$ . We are done.  $\square$

**Lemma 5.2.** For some  $m \in \mathbb{Z}^+$  with  $(m, q) = 1$ , we can find  $c_j \in \mathbb{Z}_q$  ( $j \in R_*(q)$ ) such that

$$\frac{1}{m^k - 1} \equiv \sum_{j \in R_*(q)} c_j j^k \pmod{q} \quad \text{for all } k \in S_q. \tag{5.2}$$

**Proof.** Write  $q = p_1^{\alpha_1} \cdots p_r^{\alpha_r}$  where  $p_1, \dots, p_r$  are distinct primes and  $\alpha_1, \dots, \alpha_r \in \mathbb{Z}^+$ . For each  $i = 1, \dots, r$  let  $g_i$  be a primitive root modulo  $p_i$ . By the Chinese remainder theorem, there exists an integer  $\delta_i \equiv 1 \pmod{p_i^{\alpha_i}}$  divisible by  $q/p_i^{\alpha_i}$ , also there is a positive integer  $m$  such that  $m \equiv m_i \pmod{p_i^{\alpha_i}}$  for all  $i = 1, \dots, r$  where  $m_i = g_i^{p_i^{\alpha_i-1}}$ . Clearly  $(m, q) = 1$  since  $p_i \nmid m_i$ .

Let  $k \in S_q$ . Then  $m_i^k \not\equiv 1 \pmod{p_i}$ ,  $\sum_{j=0}^{p_i-2} m_i^{jk} = (m_i^{(p_i-1)k} - 1)/(m_i^k - 1) \equiv 0 \pmod{p_i^{\alpha_i}}$  and so

$$\begin{aligned} (m_i^k - 1) \sum_{j=0}^{p_i-2} j m_i^{jk} &= (p_i - 2) m_i^{(p_i-1)k} + \sum_{0 < j \leq p_i-2} (j - 1 - j) m_i^{jk} \\ &\equiv p_i - 2 + 1 - \sum_{j=0}^{p_i-2} m_i^{jk} \equiv p_i - 1 \pmod{p_i^{\alpha_i}}. \end{aligned}$$

Therefore

$$\frac{1}{m^k - 1} \equiv - \sum_{i=1}^r \delta_i \frac{p_i^{\alpha_i} - 1}{p_i - 1} \sum_{j=0}^{p_i-2} j m^{jk} \pmod{q},$$

which concludes the proof.  $\square$

**Proof of Theorem 1.3.** Choose an  $m \in \mathbb{Z}^+$  with  $(m, q) = 1$  as in Lemma 5.2. By Theorem 1.2 the function

$$f(k) = (m^k - 1) \sum_{d|q} \mu(d) \frac{d^{k-1}}{k} B_k([x]_d) \quad (k \in \mathbb{Z}^+)$$

is  $q$ -normal. So the first part of Theorem 1.3 follows.

Let  $l \in S_q$ . By part (i), (1.20) holds if  $n = 0$ . Let  $n \in \mathbb{Z}^+$ . Then there are  $c_r \in \mathbb{Z}_q$  ( $r \in R_*(q^n)$ ) such that

$$\sum_{d|q} \mu(d) \frac{d^{k-1}}{k} B_k([x]_d) = \sum_{d|q^n} \mu(d) \frac{d^{k-1}}{k} B_k([x]_d) \equiv \sum_{r \in R_*(q^n)} c_r r^k \pmod{q^n}$$

for all  $k \in \mathbb{Z}^+$  with  $k \equiv l \pmod{\varphi(q)}$ . This implies (1.20) because

$$\sum_{k=0}^n \binom{n}{k} (-1)^k r^{k\varphi(q)+l} = r^l (1 - r^{\varphi(q)})^n \equiv 0 \pmod{q^n} \quad \text{for all } r \in R_*(q^n).$$

We are done.  $\square$

**Proof of Theorem 1.4.** Let  $q'$  be any positive integer dividing  $q$  and divisible by  $r(q)$ . Let  $d \in \mathbb{Z}^+$ ,  $d|q$  and  $\mu(d) \neq 0$ . Obviously  $d|q'$ . If  $a \in R(m)$ , then  $m[q'x/m + \{da/m\}]_d = (q'/d)x + a$  because

$$d \frac{(q'/d)x + a}{m} - \left( \frac{q'x}{m} + \left\{ \frac{da}{m} \right\} \right) = \left[ \frac{da}{m} \right] \in R(d).$$

Thus, for any  $k \in \mathbb{Z}^+$  we have

$$\begin{aligned} m^{1-k} \chi(d) B_{k,\chi} \left( \frac{q'}{d} x \right) &= \chi(d) \sum_{a=0}^{m-1} \chi(a) B_k \left( \frac{(q'/d)x + a}{m} \right) \\ &= \sum_{a=0}^{m-1} \chi(da) B_k \left( \left[ \frac{q'x}{m} + \left\{ \frac{da}{m} \right\} \right]_d \right) = \sum_{r=0}^{m-1} \chi(r) B_k \left( \left[ \frac{q'x + r}{m} \right]_d \right). \end{aligned}$$

It is well known that  $\sum_{r=0}^{m-1} \chi(r) = 0$  if  $\chi \neq \chi_0$ . By the above

$$\sum_{d|q} \mu(d) \chi(d) \frac{d^{k-1}}{k} \left( B_{k,\chi} \left( \frac{q'}{d} x \right) - [\chi = \chi_0] B_k \right) = \frac{1}{m} \sum_{r=0}^{m-1} \chi(r) \Delta_r,$$

where

$$\begin{aligned} \Delta_r &= \sum_{d|q} \mu(d) \frac{d^{k-1}}{k} \left( m^k B_k \left( \left[ \frac{q'x+r}{m} \right]_d \right) - B_k \right) \\ &\equiv m^{k-1} \sum_{\substack{0 \leq j < q \\ ((q'x+r)/m+j, q)=1}} \left( \left\lfloor \frac{\langle q'x+r \rangle_{q^2} + jm}{q} \right\rfloor + \frac{1-m}{2} \right) \\ &\quad \times \left( \frac{q'x+r}{m} + j \right)^{k-1} \\ &\equiv \sum_{\substack{0 \leq j < q \\ (r+jm, q)=1}} \left( \left\lfloor \frac{\langle q'x \rangle_{q^2} + r + jm}{q} \right\rfloor + \frac{1-m}{2} \right) \\ &\quad \times (q'x+r+jm)^{k-1} \pmod{q}. \end{aligned}$$

by Theorem 1.2. Taking  $q'=r(q)$  we then obtain the first part of Theorem 1.4.

The second part follows from the first part as in the proof of Theorem 1.3.  $\square$

**Remark 5.1.** By the proofs of Theorems 1.3 and 1.4, we can replace  $\varphi(q)$  in (1.20) and (1.23) by any (positive) multiple of  $\varphi(q)$ .

**Proof of Theorem 1.5.** Let  $m$  be any positive integer with  $(m, q)=1$ . Since  $n|q^h$  for sufficiently large  $h$ ,  $mx \in \mathbb{Z} + n\mathbb{Z}_q$  and hence we can let  $\langle mx \rangle_n$  denote the unique  $a \in R(n)$  such that  $mx \in a + n\mathbb{Z}_q$ . Let

$$H_m(k) = \frac{n^{k-1}}{k} \left( m^k B_k \left( \frac{x}{n} \right) - B_k \left( \frac{\langle mx \rangle_n}{n} \right) \right) \quad \text{for } k \in \mathbb{Z}^+.$$

As  $((x/n)n + jn, r(q)) = (x, r(q)) = 1$  for  $j \in R(q)$ , the function  $H_m$  is  $q$ -normal by Lemma 5.1.

Observe that

$$\sum_{m \in R_*(n)} H_m(k) = \frac{n^{k-1}}{k} \left( S_k(n) B_k \left( \frac{x}{n} \right) - \sum_{r \in R_*(n)} B_k \left( \frac{r}{n} \right) \right).$$

By (1.2),  $\sum_{d|l} \sum_{c \in R_*(d)} B_k(c/d) = l^{1-k} B_k$  for all  $l \in \mathbb{Z}^+$ ; applying the Möbius inversion formula we get that

$$\sum_{r \in R_*(n)} B_k \left( \frac{r}{n} \right) = \sum_{d|n} \mu(d) \left( \frac{n}{d} \right)^{1-k} B_k = n^{1-k} \varphi_k(n) B_k.$$

So the function  $H^*(k) = S_k(n)(n^{k-1}/k)B_k(x/n) - \varphi_k(q)B_k/k$  is  $q$ -normal. By Corollary 1.4, for any  $m \in \mathbb{Z}^+$  with  $(m, q)=1$ , the function  $f_m(k) = (m^k - 1)\varphi_k(q)B_k/k$  is also

$q$ -normal. Let  $H_*(k) = \sum_{m \in R_*(n)} f_m(k) = (S_k(n) - \varphi(n))\varphi_k(q)B_k/k$ . Then  $H(k) = \varphi(n)H^*(k) - H_*(k)$  is  $q$ -normal. In light of Lemma 5.2 and the above, we also have the second part of Theorem 1.5.  $\square$

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## References

- [1] L. Carlitz, Some congruences for the Bernoulli numbers, *Amer. J. Math.* 75 (1953) 163–172.
- [2] K. Dilcher, L. Skula, I.S. Slavutskii, Bernoulli numbers—Bibliography (1713–1990), *Queen's Papers, Pure Appl. Math.* 87 (1991) 71–175. The web site of the on-line version is <http://www.mathstat.dal.ca/~dilcher/bernoulli.html>.
- [3] A. Erdélyi, et al., *Higher Transcendental Functions*, McGraw-Hill, New York, 1953 (Chapter 1).
- [4] R. Ernvall, Generalized irregular primes, *Mathematika* 30 (1983) 67–73.
- [5] K. Ireland, M. Rosen, *A Classical Introduction to Modern Number Theory*, Graduate Texts in Mathematics, Vol. 84, 2nd Edition, Springer, New York, 1990 (Chapter 15).
- [6] E. Lehmer, On congruences involving Bernoulli numbers and the quotients of Fermat and Wilson, *Ann. Math.* 39 (1938) 350–360.
- [7] K. Mahler, *Introduction to  $p$ -adic Numbers and their Functions*, Cambridge University Press, Cambridge, 1973.
- [8] P. Ribenboim, *Thirteen Lectures on Fermat's Last Theorem*, Springer, New York, 1979 (Sections 2–4 in Chapter VI).
- [9] Z.H. Sun, Congruences concerning Bernoulli numbers and Bernoulli polynomials, *Discrete Appl. Math.* 105 (2000) 193–223.
- [10] J. Urbanowicz, K.S. Williams, *Congruences for  $L$ -Functions*, Kluwer, Dordrecht, 2000 (Chapter 1).
- [11] L.C. Washington, *Introduction to Cyclotomic Fields*, Springer, New York, 1982 (Chapter 4 and Sections 5.2, 5.3, 5.6).
- [12] P.T. Young, Congruences for Bernoulli, Euler, and Stirling numbers, *J. Number Theory* 78 (1999) 204–227.
- [13] P.T. Young, Kummer congruences for values of Bernoulli and Euler polynomials, *Acta Arith.* 99 (2001) 277–288.
- [14] H.F. Baker, Remark on the Eisenstein–Sylvester extension of Fermat's theorem, *Proc. London Math. Soc.* 4 (1906) 131–135.
- [15] M. Lerch, Sur les théorèmes de Sylvester concernant le quotient de Fermat, *C.R. Acad. Sci. Paris* 142 (1906) 35–38.