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2-Adic Congruences of Nörlund Numbers and of Bernoulli Numbers of the Second Kind

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In this paper, we find simple 2-adic congruences mod $2^{[n/2]+1}$ for the Nörlund numbers $B_n^{(n)}$ and for the Bernoulli numbers of the second kind b_n. These congruences improve F. T. Howard's mod 8 congruences (*in* "Applications of Fibonacci Numbers," Vol. 5, pp. 355–366, Kluwer Academic, Dordrecht, 1993). We use recurrence relations to determine when our congruences are best possible and to obtain further information about the 2-adic expansions of $B_n^{(n)}/n!$ and of b_n.

1. INTRODUCTION

The *higher order Bernoulli numbers* $B_n^{(I)}$ are defined by [2, 9]

$$\sum_{n=0}^{\infty} \frac{B_n^{(I)}}{n!} t^n = \left(\frac{t}{e^t - 1}\right)^I. \tag{1}$$

If $n = I$, the rational numbers $B_n^{(n)}$ are called *Nörlund numbers*. They have the expansion [2, 5]

$$\sum_{n=0}^{\infty} \frac{B_n^{(n)}}{n!} t^n = \frac{t}{(1+t)\log(1+t)}. \tag{2}$$

The *Bernoulli numbers of the second kind* b_n are defined by [6]

$$\sum_{n=0}^{\infty} b_n t^n = \frac{t}{\log(1+t)}. \tag{3}$$

These expansions should be considered as formal power series (although (1) converges in the complex domain if $|t| < 2\pi$, while (2) and (3) converge if $|t| < 1$), since a feature of our approach is the interplay between formal and 2-adic convergent power series.

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In Section 3 (Theorem 2), we prove the 2-adic congruences

$$(-1)^n 2^n B_n^{(n)}/n! = -1 \pmod{2^{[n/2]+1}} \tag{4}$$

and

$$(-1)^n 2^n b_n \equiv 1 \pmod{2^{[n/2]+1}}, \tag{5}$$

where $[x]$ is the integer part of x.

These congruences improve Howard's congruences [5, Theorem 5.3, (6.6)], that $(-1)^n 2^n B_n^{(n)}/n! \equiv -1 \pmod 8$ if $n \geqslant 4$, and that $2^n b_n \equiv (-1)^n \pmod 8$ if $n \geqslant 5$. It does not appear to us that Howard's methods alone are sufficient to prove our congruences, but that some p-adic analysis is necessary.

Letting v be the 2-adic exponential valuation on $\mathbf{Q}$, we in fact prove that $v((-1)^n 2^n B_n^{(n)}/n! + 1) \geqslant [n/2] + 1$, with inequality iff $n \equiv 2 \pmod 4$. Similarly $v((-1)^n 2^n b_n - 1) \geqslant [n/2] + 1$, with inequality iff $4 \mid n$. Thus congruence (4) is best possible unless $n \equiv 2 \pmod 4$, and congruence (5) is best possible unless $4 \mid n$ (see Section 3, Corollaries 1, 2).

Let $C_n = (-1)^n 2^n B_n^{(n)}/n! + 1$ and $c_n = (-1)^n 2^n b_n - 1$. We establish, in Section 2, convergence of the pair of 2-adic series

$$\sum_{n=0}^{\infty} C_n = 0 \qquad \text{and} \qquad \sum_{n=0}^{\infty} c_n = -2. \tag{6}$$

It follows that $(C_n) \to 0$ and $(c_n) \to 0$ for the 2-adic metric. We then use descent to deduce the congruences (4) and (5). Finally, we employ recursions for the (C_n) sequence and for the (c_n) sequence to get information about the 2-adic expansions of $B_n^{(n)}/n!$ and of b_n.

2. *P*-ADIC PRELIMINARIES

If p is a prime (eventually $p = 2$), $v = v_p$ is the *p-adic exponential valuation* on $\mathbf{Q}$, i.e., if $r \in \mathbf{Q}$ then $r = p^{v(r)} m/n$ where m and n are prime to p, with $v(0) = \infty$; $|r| = p^{-v(r)}$ defines the *p-adic absolute value*; and $d(r, s) = |r - s|$ defines the *p-adic metric*. Then the *field of p-adic numbers* $\mathbf{Q}_p$ is the completion of $\mathbf{Q}$, and the *ring of p-adic integers* $\mathbf{Z}_p$ is the completion of $\mathbf{Z}$. The basic valuation properties are [4, 7]

$$v(ab) = v(a) + v(b) \qquad \text{and} \qquad v(a+b) \geqslant \min\{v(a), v(b)\},$$

i.e.,

$$|ab| = |a|\,|b| \qquad \text{and} \qquad |a+b| \leqslant \max\{|a|, |b|\}. \tag{7}$$

For the special case $p=2$, we have

$$v(a+b)=\min\{v(a), v(b)\} \qquad \text{iff} \quad v(a)\neq v(b),$$

i.e.,

$$|a+b|=\max\{|a|, |b|\} \qquad \text{iff} \quad |a|\neq |b|. \tag{8}$$

In what follows, we will always assume the p-adic metric on $\mathbf{Q}_p$, and divisibility and congruences will always refer to $\mathbf{Z}_p$, so $a\equiv b \pmod{p^k}$ iff $p^k \mid (a-b)$ in $\mathbf{Z}_p$ iff $v(a-b)\geqslant k$.

A power series $f(t)=\sum_{n=0}^{\infty} a_n t^n \in \mathbf{Q}_p[[t]]$ is called *analytic* if it converges on $\mathbf{Z}_p$. (We use t both as a variable and as a general element of $\mathbf{Z}_p$, depending on the context.) If A is the set of all analytic functions, the following are well known and easy to prove [4, 7].

(i) $\sum a_n t^n \in A$ iff $\sum a_n$ converges iff $(a_n)\to 0$.

(ii) A is a subalgebra of $\mathbf{Q}_p[[t]]$.

(iii) If $f(t)=\sum a_n t^n \in A$ and $c\in \mathbf{Z}_p$, then $f(t)=\sum b_n(t-c)^n$, where $\sum b_n t^n \in A$ and $b_n=f^{(n)}(c)/n!$.

(iv) If $f(t)\in A$ and $c\in \mathbf{Z_p}$, then $f(t)/(t-c)\in A$ iff $f(c)=0$; if $f(c)=0$ and $g(t)=f(t)/(t-c)$, then $g(c)=f'(c)$, and if $h(t)=(g(t)-f'(c))/(t-c)$, then $h(t)\in A$ and $h(c)=g'(c)=f''(c)/2$.

(v) The usual rules for differentiation hold, namely an analytic function has analytic derivatives of all orders, and the series can be differentiated termwise; the rules for sums, products, and quotients all apply. (These differentiability rules are of course *local*, unlike the *global* concept of analyticity.)

The following result, which is useful for finding units in A, is not quite as easy, and we will give a proof. The proof is often buried in the Weierstrass Preparation Theorem [7, Theorem 14].

Let

$$S=\left\{\sum_{n=0}^{\infty} a_n t^n \;\middle|\; |a_0|=1,\ |a_n|<1 \text{ for } n>0, \text{ and } (a_n)\to 0\right\}. \tag{9}$$

LEMMA 1. *S is a multiplicative subgroup of A.*

Proof. Obviously $1\in S$, and as noted above, $S\subset A$. Let $f(t)=\sum a_n t^n$ and $g(t)=\sum b_n t^n \in S$. Then $f(t)\, g(t)=\sum c_n t^n$, where $c_n=\sum_j a_j b_{n-j}$. Thus $|c_0|=1$, while if $n>0$, then inductively $|a_j b_{n-j}|<1$, so $|c_n|<1$. If $\varepsilon>0$, there exists N such that $|a_j|$ and $|b_j|<\varepsilon$ if $j>N$, so $|c_n|<\varepsilon$ if $n>2N$. (This is essentially the proof of the multiplicative closure of A.)

The main thing to show is that if $f(t) \in S$ as above, and $g(t) = 1/f(t)$ is the formal inverse, then $g(t) \in S$. If $g(t) = \sum b_n t^n$, then clearly $\sum_j a_j b_{n-j} = \delta_{n0}$, the Kronecker delta.

Since $b_0 = 1/a_0$, we have $|b_0| = 1$. But $a_0 b_n = -\sum_{j=1}^{n} a_j b_{n-j}$ for $n > 0$, whence inductively $|b_n| < 1$. It remains to show that $(b_n) \to 0$. If $\varepsilon > 0$, take N such that $|a_n| \leqslant \varepsilon$ if $n > N$, and let $B = \max\{|a_1|, ..., |a_N|\}$. Then $B < 1$.

Prove inductively that if $n > kN$, then $|b_n| \leqslant \max\{B^k, \varepsilon\}$. This is clear if $k = 0$. Assume true for $k-1$. Then

$$a_0 b_n = -\sum_{j=1}^{N} a_j b_{n-j} - \sum_{j=N+1}^{n} a_j b_{n-j} = \Sigma_1 + \Sigma_2.$$

For Σ_2, we have $|a_j| \leqslant \varepsilon$ and $|b_{n-j}| \leqslant 1$, so $|\Sigma_2| \leqslant \varepsilon$. For Σ_1, if $n > kN$ and $j \leqslant N$, then $n - j > (k-1) N$, so the induction hypothesis shows that $|a_j b_{n-j}| \leqslant B \max\{B^{k-1}, \varepsilon\} \leqslant \max\{B^k, \varepsilon\}$.

Now choose k such that $B^k < \varepsilon$. If $n > kN$, then $|b_n| \leqslant \varepsilon$. ∎

For the remainder of this paper, take $p = 2$.

EXAMPLE. It is well known [4, 7] that $\log(1+t) = \sum_{n=0}^{\infty} (-1)^n t^{n+1}/(n+1)$ converges iff $|t| < 1$. Thus

$$f(t) = \frac{\log(1-2t)}{-2t} = \sum_{n=0}^{\infty} a_n t^n \in A, \qquad \text{where} \quad a_n = \frac{2^n}{n+1}. \tag{10}$$

Since it is known and easy to see that $\log(-1) = 0$ for the 2-adic logarithm, we have $f(1) = 0$. Thus if $g(t) = f(t)/(t-1)$, then $g(t) \in A$. Note that $g(t) = -\sum_{n=0}^{\infty} S_n t^n$, where $S_n = \sum_{j=0}^{n} a_j$, so $(S_n) \to 0$. Also $g(1) = f'(1)$. Since $(\log(1-2t))' = -2/(1-2t)$, a straightforward differentiation calculation then shows that $g(1) = -1$.

Furthermore, $a_0 = a_1 = 1$, and it is easy to see that $v(a_n) > 0$ if $n > 1$. (A simple convexity argument shows that $x > \log_2(x+1)$ for $x > 1$, and $v(a_n) = n - v(n+1) \geqslant n - \log_2(n+1)$.) Hence $g(t) \in S$, so also $h(t) = 1/g(t) \in S$ by the lemma, so that $h(t) \in A$.

Recall that $C_n = (-1)^n 2^n B_n^{(n)}/n! + 1$ and $c_n = (-1)^n 2^n b_n - 1$.

From (2) and (3) we get $b_n = B_n^{(n)}/n! + B_{n-1}^{(n-1)}/(n-1)!$ so

$$c_n = C_n - 2C_{n-1} \qquad \text{for} \quad n > 0. \tag{11}$$

The generating functions of C_n and c_n are

$$\sum_{n=0}^{\infty} C_n t^n = \frac{-2t}{(1-2t)\log(1-2t)} + \frac{1}{1-t} \tag{12}$$

and

$$\sum_{n=0}^{\infty} c_n t^n = \frac{-2t}{\log(1-2t)} - \frac{1}{1-t}. \tag{13}$$

THEOREM 1. $\sum C_n t^n \in A$ *and* $\sum c_n t^n \in A$.

Proof. With the notations of the preceding example

$$\sum c_n t^n = \frac{1}{t-1}\left(\frac{-2t(t-1)}{\log(1-2t)} + 1\right) = \frac{1}{t-1}\left(\frac{1}{g(t)} + 1\right) = \frac{1}{t-1}(h(t)+1).$$

But we have seen above that $h(t) \in A$ and $h(1) = -1$, so $\sum c_n t^n \in A$.
Similarly

$$\sum C_n t^n = \frac{1}{t-1}\left(\frac{-2t(t-1)}{(1-2t)\log(1-2t)} - 1\right) = \frac{1}{t-1}\left(\frac{h(t)}{1-2t} - 1\right).$$

If $k(t) = h(t)/(1-2t)$ then again by the lemma, $k(t) \in A$. Since $k(1) = -h(1) = 1$, we have $\sum C_n t^n \in A$. ▮

COROLLARY 1. $(C_n) \to 0$ *and* $(c_n) \to 0$.

Proof. These sequences are the terms of convergent series. ▮

COROLLARY 2. $\sum_{n=0}^{\infty} C_n = 0$ *and* $\sum_{n=0}^{\infty} c_n = -2$.

Proof. With the notations of the preceding proof, $\sum c_n = h'(1)$ and $\sum C_n = k'(1)$. But $h'(1) = -g'(1)/(g(1))^2 = -g'(1) = -f''(1)/2$, by (iv) above. A straightforward calculation gives $f''(1) = 4$, so $h'(1) = -2 = \sum_{n=0}^{\infty} c_n$. We can find $k'(1)$ similarly, or use (11) to deduce $\sum_{n=0}^{\infty} C_n = 0$. ▮

COROLLARY 3.

$$\sum_{n=0}^{\infty} (-1)^n C_n = \frac{2}{3\log 3} + \frac{1}{2} \qquad \textit{and} \qquad \sum_{n=0}^{\infty} (-1)^n c_n = \frac{2}{\log 3} - \frac{1}{2}.$$

Proof. Simply evaluate the generating functions at -1. ▮

3. SHARP CONGRUENCES

With the notations of the preceding section, where $f(t)$ and a_n are as in the example, $f(t) \sum C_n t^n = 1/(1-2t) + f(t)/(1-t)$, so

$$C_n = -\sum_{j=1}^{n} a_j C_{n-j} + S_n + 2^n, \qquad \text{where} \quad a_n = \frac{2^n}{n+1} \qquad \text{and}$$
$$S_n = \sum_{j=0}^{n} a_j. \tag{14}$$

Similarly $f(t) \sum c_n t^n = 1 - f(t)/(1-t)$, so

$$c_n = -\sum_{j=1}^{n} a_j c_{n-j} + \delta_{n0} - S_n. \tag{15}$$

Note that if $R_n = \sum_{j=n+1}^{\infty} a_j$, then $(S_n) \to 0$, so $S_n + R_n = 0$.

These notations are now fixed for the remainder of the paper.

LEMMA 2. *$v(a_n) \geqslant [n/2]$ for all n; $v(a_n) = [n/2]$ iff $n = 0, 1, 3$; $v(a_n) = [n/2] + 1$ iff $n = 2, 7$; $v(a_n) > [n/2] + 3$ if $n > 7$; for every c, there exists N such that $v(a_n) > [n/2] + c$ if $n > N$; if $n \geqslant 4u$ and $u > 1$, then $v(a_n) > [n/2] + u$.*

Proof. Since $v(a_n) = n - v(n+1)$ and $v(n+1) \leqslant \log_2(n+1)$, with equality iff $n+1$ is a 2-power, these results are an immediate consequence of the convexity and slow growth of $\log_2(x+1)$. For the last assertion, the critical case is where $n = 4u + 3$, and what is needed here is $u > v(u+1)$ if $u > 1$, which is clear. ∎

The following lemma is entirely elementary, but very useful.

LEMMA 3. *$[j/2] + [(n-j)/2] = [n/2]$ unless n is even and j is odd, in which case $[j/2] + [(n-j)/2] = [n/2] - 1$.*

Proof. Just consider the four parity cases for n and j. Equivalently $[n/2] + [m/2] = [(n+m)/2]$ unless n and m are both odd, in which case $[n/2] + [m/2] = [(n+m)/2] - 1$. ∎

We are now ready to prove our main theorem, that $v(C_n)$ and $v(c_n)$ are $\geqslant [n/2] + 1$ for all n. It appears that the recurrences (14), (15) and the preceding lemmas should provide an inductive proof. This does work if n is odd, but there is a major problem if n is even, namely in this case, the 2-power in the modulus must increase. We would have to know a priori that if n is odd then $v(C_n) = [n/2] + 1$. Fortunately Theorem 1 enables us to give a descent proof, namely we can show that if the inequality is false for some n, then we encounter an untenable situation for all larger integers.

THEOREM 2. *$v(C_n) \geqslant [n/2] + 1$ and $v(c_n) \geqslant [n/2] + 1$ for all n.*

Proof. By (11), the second inequality follows immediately from the first. Assume the first inequality is false, and let n be the first counter example. Then $S_n = -R_n$ implies $v(S_n + 2^n) \geqslant [n/2] + 1$. Since $v(a_j C_{n-j}) \geqslant [j/2] + [(n-j)/2] + 1$ if $1 \leqslant j \leqslant n$, by the induction hypothesis and Lemma 2, we have n is even by (14) and Lemma 3. But now $C_n \equiv -(C_{n-1} + 2C_{n-3})$ (mod $2^{[n/2]+1}$) by the induction, (14), and the lemmas. Since $v(C_{n-1}) \geqslant [n/2]$ and $v(C_{n-3}) \geqslant [n/2] - 1$, it follows that $v(C_n) = [n/2]$.

Since $n+1$ is odd, it follows from the induction that for $2 \leqslant j \leqslant n+1$ we have $v(a_j C_{n+1-j}) \geqslant [n/2] + 1$, so by (14), we have $v(C_{n+1}) = [n/2]$.

We now proceed to prove by induction that $v(C_m) = [n/2]$ if $m \geqslant n$. Suppose that $v(C_{n+j}) = [n/2]$ for $0 \leqslant j \leqslant N$, with $N \geqslant 1$. Then

$$C_{n+N+1} \equiv -\left(C_{n+N} + \sum_{j=2}^{N+1} a_j C_{n+N+1-j} + \sum_{j > N+1} a_j C_{n+N+1-j}\right) \pmod{2^{[n/2]+1}}.$$

For the first sum $\sum_1$, we have $v(a_j C_{n+N+1-j}) > [n/2]$ by the above induction hypothesis, while for the second sum $\sum_2$, we have $v(a_j C_{n+N+1-j}) \geqslant [j/2] + [(n+N+1-j)/2] + 1$ by the overall induction. Since $[j/2] + [(n+N+1-j)/2] + 1 \geqslant [(n+N+1)/2] \geqslant [n/2] + 1$, we have $v(\sum_1) > [n/2]$ and $v(\sum_2) > [n/2]$, so we have $v(C_{n+N+1}) = [n/2]$.

The proof is now completed by observing that $v(C_m) = [n/2]$ if $m \geqslant n$ is impossible by Theorem 1 (or by its first corollary). ∎

Henceforth let $K_n = C_n / 2^{[n/2]+1}$ and $k_n = c_n / 2^{[n/2]+1}$, so Theorem 2 says that K_n and k_n are (2-adic) integers. The remainder of the paper is devoted to their analysis. It follows from (11), that $k_n = K_n - 2K_{n-1}$ if n is odd, and $k_n = K_n - K_{n-1}$ if n is even. Hence we will devote most of our attention to K_n. If we can show that for some n, $K_n \equiv a \pmod{2^c}$ with $a = 0$ or ± 1, then $(-1)^n B_n^{(n)}/n! \equiv -1 + a 2^{[n/2]+1} \pmod{2^{[n/2]+1+c}}$, which gives a big piece of the 2-adic expansion.

Suppose that $N \geqslant 2(c+1)$ and if $n > N$ then $v(a_n) > [n/2] + c$. For example, by Lemma 2, if $c > 1$, we can take $N = 4c - 1$. Then by (14) and Lemma 3, if $n \geqslant N$ then

for n odd,
$$\sum_{j=0}^{N} \left(\frac{a_j}{2^{[j/2]}}\right) K_{n-j} \equiv 0 \pmod{2^c}; \qquad (16)$$

for n even,

$$\sum_{\text{even } j}^{N} \left(\frac{a_j}{2^{[j/2]}}\right) K_{n-j} + \sum_{\text{odd } j}^{N} \left(\frac{a_j}{2^{[j/2]+1}}\right) K_{n-j} \equiv 0 \pmod{2^c}. \qquad (17)$$

Observe that all coefficients in these recursions are (2-adic) integers except for $j=1$ and 3 in (17), where the coefficients are 1/2. Multiply (17) by 2 to get for even $n \geqslant N$

$$\sum_{\text{even } j}^{N} \left(\frac{a_j}{2^{[j/2]-1}}\right) K_{n-j} + \sum_{\text{odd } j>3}^{N} \left(\frac{a_j}{2^{[j/2]}}\right) K_{n-j} + K_{n-1} + K_{n-3} \equiv 0 \pmod{2^{c+1}}. \tag{18}$$

By (16) and (18), $K_n \pmod{2^c}$, for n odd, inductively determines the $K_n \pmod{2^c}$ for n even, while $K_n \pmod{2^c}$, for n even, inductively determines the $K_n \pmod{2^{c+1}}$ for n odd. Thus we can spiral up, and eventually get K_n mod any power of 2 for all n, odd or even.

Observe that in (16) we only require that $N \geqslant 2c+1$, while in (16)–(18), if we sum from 1 to n, then only $v(S_n + 2^n) > [n/2] + c$ is required. An example is given by $n=3$ and $c=1$.

Remarks. The following corollary is equivalent to the assertion that the congruence $(-1)^n B_n^{(n)}/n! \equiv -1 \pmod{2^{[n/2]+1}}$ is best possible unless $n \equiv 2 \pmod 4$, i.e., $v(C_n) = [n/2]+1$ iff $n \equiv 2 \pmod 4$ is false.

COROLLARY 1. (i) *If n is odd, then K_n is odd.*

(ii) *If n is even, then K_n is odd iff $4 \mid n$.*

Proof. Consider congruence (18), with $c=0$, so that $N=3$ works. We get $2K_n \equiv -(a_2/2^0) K_{n-2} - K_{n-1} - K_{n-3} \pmod 2$ for n even $\geqslant 4$. But these are all (2-adic) integers, and $a_2 = 4/3$, so K_{n-1} and K_{n-3} have the same parity for all even $n \geqslant 4$. Since $K_1 = 1$, this proves (i).

Next consider congruence (16) with $c=1$, so we can take $N=7$. Thus $K_n \equiv -\sum_{j=1}^{7} (a_j/2^{[j/2]}) K_{n-j} \pmod 2$ for odd $n \geqslant 7$, so $K_n \equiv K_{n-1} + K_{n-3} \pmod 2$. But we know that K_n is odd for n odd. Since a direct computation shows that K_0 and K_4 are odd and K_2 is even, it follows that the K_n for n even alternate parity, proving (ii). (Taking the comments following (18) into account, we only need K_0 odd.) ∎

Remarks. The following corollary is equivalent to proving that the congruence $(-1)^n b_n \equiv 1 \pmod{2^{[n/2]+1}}$ is best possible iff $4 \nmid n$.

COROLLARY 2. (i) *k_n is odd if n is odd.*

(ii) *If n is even, then k_n is even iff $4 \mid n$.*

Proof. This follows immediately because of the relations between the k_n and the K_n. ∎

Our computations all satisfy the following observed pattern. For any c, assume $n \geqslant 4c-5$ (which we call *the stable range* for c).

(a) If $n \equiv 0, 1, 7 \pmod 8$, then $K_n \equiv -1 \pmod{2^c}$.

(b) If $n \equiv 3, 4, 5 \pmod 8$, then $K_n \equiv 1 \pmod{2^c}$.

(c) If $n \equiv 2 \pmod 4$, then $K_n \equiv 0 \pmod{2^c}$.

Call these congruences the *conjectured stable congruences*, and call

$$\text{the residues } 0, 1 \text{ and } -1 \pmod{2^c} \text{ the } \textit{stable values}. \tag{19}$$

If we denote the stable values by L_n, the stable congruences can be restated as $K_n \equiv L_n \pmod{2^{[(n+5)/4]}}$. Observe that L_n is clearly periodic with period 8, and in fact $L_{n+4} = -L_n$, i.e., L_n is *antiperiodic*.

Using the Jacobi symbol (a/b), since it is well known [8] that $(2/n) = (-1)^{(n^2-1)/8}$ for n odd, we can rewrite congruences (a) and (b) as $K_n \equiv (2/n) \pmod{2^c}$ for n odd, and $K_n \equiv -(-1)^{n/4} \pmod{2^c}$ if $4 \mid n$.

Although we have not been able to prove the conjectured stable congruences for all c, we can prove them for all c up to fairly large numbers. See Corollary 3 and the comment following it.

EXAMPLES. We give the values of K_n and their best congruences of stable type for $0 \leqslant n \leqslant 8$: $K_0 = 1 \equiv -1 \pmod 2$, $K_1 = 1 \equiv -1 \pmod 2$, $K_2 = 2/3 \equiv 0 \pmod 2$, $K_3 = 1 \equiv 1 \pmod{2^\infty}$, $K_4 = 37/45 \equiv 1 \pmod 8$, $K_5 = 13/9 \equiv 1 \pmod 4$, $K_6 = 1252/945 \equiv 0 \pmod 4$, $K_7 = 337/135 \equiv -1 \pmod 8$, and $K_8 = 33881/14175 \equiv -1 \pmod 8$.

COROLLARY 3. *The stable congruences hold for* $c \leqslant 7$.

Proof. $c = 0$ is trivial, and $c = 1$ restates Corollary 1. We use the recurrences (16) and (18) to proceed stepwise to $c = 2$ and $c = 3$. We leave $c = 4$, which is quite similar, as an exercise for the reader.

The idea is to first use (18) to get the stable values of $K_m \pmod{2^c}$ for m odd, by induction on m and c. Then use (16) to get the stable values of $K_m \pmod{2^c}$ for m even. Note that if K_n has the stable value $\pmod{2^{c-1}}$ and r is even, then rK_n has its proper value $\pmod{2^c}$.

For $c = 2$, we get the pair of congruences for $n \geqslant 7$ (by Lemma 2):

$$n \text{ odd implies } K_n \equiv -(K_{n-1} + 2K_{n-2} + K_{n-3} + 2K_{n-7}) \pmod 4;$$

$$n \text{ even implies that } 2K_n \equiv -(K_{n-1} + K_{n-3} + 2K_{n-7}) \pmod 4.$$

We observe by simple substitution, that the stable congruence values satisfy these congruences. (There are eight equations for the residues of n mod 8.) Then show by direct computation that K_n has the correct mod 4 value for $n = 3, 4, 5$, and observe that the above congruence recurrences and initial conditions uniquely determine the solution. (Starting with $n=8$, get the stable values for m odd, then get the stable values for m even.)

For $c=3$, we get the pair of congruences for $n \geqslant 7$

n odd: $K_n \equiv -(K_{n-1}+6K_{n-2}+K_{n-3}+4K_{n-4}+4K_{n-5}+2K_{n-7})$ (mod 8),

n even: $2K_n \equiv -(K_{n-1}+4K_{n-2}+K_{n-3}+4K_{n-5}+2K_{n-7})$ (mod 8).

Once again, case by case, each of the stable congruence values satisfies these congruences, as do the computed values of K_n for $n=7$ and 8, so the stable values give the unique solution. (Start with (18) and $n=10$.)

The case $c=4$ is similar, but a bit more complicated, since we now have $N=15$, and there are a couple of additional terms in the congruences (with $n=11, 12, 13$ basing the induction). In fact, $N=15$ is sufficient for $c \leqslant 7$, but more terms must be taken into account, the congruences are mod higher powers, and initial values are bigger. The computations appear routine. ∎

The following proposition enables us to extend Corollary 3 to bigger values of c, namely using MAPLE to verify the congruences up to $n=460$, the conclusion holds at least for $c \leqslant 115$.

Proposition 1. *Suppose that* $K_n \equiv L_n \pmod{2^{[(n+5)/4]}}$ *for* $n \leqslant 4c$. *Then the stable congruences* (19) *hold* mod 2^d *for all* $d \leqslant c$.

Proof. Start with (18) with c replaced by $c-1$, and take even $n \geqslant 4c-2$. Now replace n by $n+4$, and use Lemma 2 and antiperiodicity of L_n to prove the stable congruence for $n+3$ by induction on c and n: Small c and $n=4c-2$ and $4c$ provide the base for the induction. The key is observing that if $j=4u+r$ with $r \leqslant 3$ and $u \geqslant 1$, then $(a_j/2^{[j/2]})\, K_{n+4-j} \equiv -(a_j/2^{[j/2]})\, K_{n-j} \pmod{2^c}$. It follows easily that $K_{n+3} \equiv -K_{n-1} \pmod{2^c}$, establishing the stable value for K_{n+3}. Once the stable values have been established for odd n, we can similarly use (16) to establish the even $n \pmod{2^c}$, completing the proof. ∎

Remarks. We have observed the following empirical values of v, which go beyond the conjectured stable congruences:

$$v(K_{8u+1}+1)=v(K_{8u-1}+1)=v(K_{8u}+1)=2u+1, \qquad v(K_{8u-4}-1)=3u,$$
$$v(K_{8u-3}-1)=2u, \qquad v(K_{8u-5}-1)=2u+1 \qquad \text{if} \quad u>1,$$

and

$$v(K_{4u-2})=u \qquad (\textit{which is equivalent to } v(C_{4u-2})=3u). \tag{20}$$

These empirical values imply that $v(K_n - L_n) \geqslant [(n+5)/4]$, with equality unless $n \equiv 3, 4 \pmod 8$.

The last formula in (20) is particularly interesting, since it is the conjectural answer to the missing case in the remarks preceding Corollary 1. It says that if $n \equiv 2 \pmod 4$, then $v(C_n) = 3(n+2)/4$.

We have no proofs for any of these formulas. Neither do we know why the residue class of n mod 8 is apparently so important, or why there should be a connection with the Jacobi symbol.

There are corresponding *conjectured stable congruences* for the b_n, which are $k_n \equiv 1$ if $n \equiv 1, 2, 3 \pmod 8$; $k_n \equiv -1$ if $n \equiv 5, 6, 7 \pmod 8$; and $k_n \equiv 0$ if $4 \mid n$. We can only derive special cases from corresponding K_n congruences. The situation for values of v here is slightly more complicated, but we have found observable patterns analogous to the values given by (20). Specifically, it appears from substantial MAPLE computation that $v(k_{8u-3}+1) = v(k_{8u-5}-1) = v(k_{8u-6}-1) = 2u$; that $v(k_{8u+1}-1) = v(k_{8u-4}) = 2u+1$; and that $v(k_{8u-1}+1) = v(k_{8u}) = v(k_{8u-2}+1) - u$. The formula for $v(k_{8u-4})$ gives $v(c_{8u-4}) = 6u$, or $v(c_n) = 3(n+4)/4$ if $n \equiv 4 \pmod 8$, which is a missing value in Corollary 2.

Our empirical formula for $v(k_{8u})$, hence also for $v(c_{8u})$, the remaining missing value, is quite complicated so is omitted. The values for $1 \leqslant u \leqslant 6$ of $v(k_{8u})$ are 5, 8, 9, 13, 13, and 16, and the values for $v(c_{8u})$ are 10, 17, 22, 30, 34, and 41, respectively.

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