A class of formal Operators for Combinatorial identities and its Application

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Abstract—In this paper, we present some formulas of symbolic operator summation, which involving Generalization well-know number sequences or polynomial sequences, and mean while we obtain some identities about the sequences by employing M-R's substitution rule.

Keywords—Generating functions; Operators sequence group; $Riordan$ arrays; R.G Operator group; Combinatorial identities.

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I. Introduction

COMBINATORIAL identities play an important role in many areas of mathematics, including combinatorial analysis, graph theory, number theory, statistics and probability. So some methods should be attention that can help establish the Computation of Combinatorial Sums(see[6]), the symbolic calculus with operators $\triangle$(difference operator), E(displacement operator), D(derivative operator)(see[1],[5]), is one. Since all the symbolic expressed as power series in $\triangle(DorE)$(see[5]) over the real or complex number fields, it is clear that the theoretical basis of the calculus may be found within the general theory of the formal power series.

In this paper, we shall show that a variety of operators formulas and identities containing famous combinatorics number sequences by using a symbolic method with operators $\triangle$, E, D. The key idea is a suitable application of a certain symbolic substitution rule (see [4]) to the generation functions for those number sequences, so that a number of symbolic expressions could be obtained, which then can be used as stepping-stones to yielding particular formulas or combinatorial identities, as in [5].

II. The Definition of the formal R.G Operators

First, let $\triangle$, E, D is respectively difference, displacement and derivative operator. There is a operator T, and if $TE^{\alpha} = E^{\alpha}T$ and Tt is a non-zero constant, then T is called a $delat$ operator, as in [3]. Future, we will use the Q to represent the $delta$ operators.

Definition 2.1 A formal R.G Operator is a Operator sequence group that be composed by three formal power summation operator, that is $(G(Q), F(Q), H(Q))$, where $G(Q) = \sum_{k\geq 0} g_k Q^k, F(Q) = \sum_{k\geq 1} f_k Q^k, H(Q) = \sum_{k\geq 0} h_k Q^k$, then for any $\hbar(x) \in C^{\infty}$ at $x = a$, have

$$\begin{aligned}(G(Q), F(Q), H(Q))\hbar(a) &= \sum_{n\geq 0}\left(\sum_{k=0}^{n} d_{n,k}h_k\right)Q^n\hbar(a)\\ &= G(Q)H(F(Q))\hbar(a)\,,\end{aligned}$$

where, $d_{n,k} = [Q^n]G(Q)H(F(Q))$.
Now, let Ω is a set that composed by all R.G operators. Next we will definition a Computation $(\otimes)$, let $(G_1(Q), F_1(Q), H(Q))$, $(G_2(Q), F_2(Q), H(Q)) \in \Omega$ then

$$\begin{aligned}&(G_1(Q), F_1(Q), H(Q)) \otimes (G_2(Q), F_2(Q), H(Q))\\ &= (G_1(Q)G_2(F_1(Q)), F_2(F_1(Q)), H(Q))\ ,\end{aligned}$$

Still is a R.G Operators, therefore Ω is closed with mathematical operation $(\otimes)$. So we have the following theorem.

Theorem2.1(Ω is a formal R.G Operators Group) Suppose $G(Q)$, $F(Q)$ is respectively reversible formal power summation operator and $delta$ formal power summation operator,then Ω is a Group with mathematical operation $(\otimes)$, and $(1, Q, H(Q))$ is unit element of group, $(G(Q), F(Q), H(Q))$ is inverse element for $\big(1/G(\overline{F}(Q)), \overline{F}(Q), H(Q)\big)$, here $\overline{F}(Q)$ is composite inversion for $F(Q)$, be satisfied with $F(\overline{F}(Q)) = \overline{F}(F(Q)) = Q$.

Proof: First Ω is closed for the mathematical operation $(\otimes)$, and let $(G_1(Q), F_1(Q), H(Q))$, $(G_2(Q), F_2(Q), H(Q))$, $(G_3(Q), F_3(Q), H(Q)) \in \Omega$, Since

$$\begin{aligned}&[(G_1(Q), F_1(Q), H(Q)) \otimes (G_2(Q), F_2(Q), H(Q))]\\ &\otimes (G_3(Q), F_3(Q), H(Q))\\ &= (G_1(Q)\cdot G_2(F_1(Q)), F_2(F_1(Q)), H(Q)) \otimes (G_3(Q), F_3(Q), H(Q))\\ &= (G_1(Q)\cdot G_2(F_1(Q))\cdot G_3(F_2(F_1(Q))), F_3(F_2(F_1(Q))), H(Q))\ ,\end{aligned}$$

As well as

$$\begin{aligned}&(G_1(Q), F_1(Q), H(Q)) \otimes [(G_2(Q), F_2(Q), H(Q))\\ &\otimes (G_3(Q), F_3(Q), H(Q))]\\ &= (G_1(Q), F_1(Q), H(Q)) \otimes (G_2(Q)\cdot G_3(F_2(Q)), F_3(F_2(Q)), H(Q))\\ &= (G_1(Q)\cdot G_2(F_1(Q))\cdot G_3(F_2(F_1(Q))), F_3(F_2(F_1(Q))), H(Q))\ .\end{aligned}$$

So the mathematical operation $(\otimes)$ of Ω is contented to the associative laws. Second, as $(1, Q, H(Q))$ is unit element of

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Ω, and for any $(G(Q), F(Q), H(Q)) \in \Omega$, we always have its inverse element $\left(1/G(\overline{F}(Q)), \overline{F}(Q)\right) \in \Omega$. In fact, for

$$\begin{aligned}&(G(Q), F(Q), H(Q)) \otimes (1, Q, H(Q))\\&= (G(Q), F(Q), H(Q))\\&= (1, Q, H(Q)) \otimes (G(Q), F(Q), H(Q))\\&= (1 \cdot G(Q), F(Q), H(Q)) \ ,\end{aligned}$$

and

$$\begin{aligned}&(G(Q), F(Q), H(Q)) \otimes \left(\frac{1}{G(\overline{F}(Q))}, \overline{F}(Q), H(Q)\right)\\&= \left(G(Q) \cdot \frac{1}{G(\overline{F}(F(Q)))}, \overline{F}(F(Q), H(Q))\right)\\&= (1, Q, H(Q))\\&= \left(\frac{1}{G(\overline{F}(Q))}, \overline{F}(Q), H(Q)\right) \otimes (G(Q), F(Q), H(Q)) \ ,\end{aligned}$$

So we can obtain that Ω is a Group, we call it for R.G Operators Group. ■

III. The Generalization of the formal R.G Operator

In this section, we will present Generalization of the formal R.G operator based on generalized *Riordan* array theory(see[2],[3]).

Definition 3.1. Let $G(Q) = \sum_{k\geq 0} \frac{g_k}{N_k} Q^k$, $F(Q) = \sum_{k\geq 1} \frac{f_k}{N_k} Q^k$, $H(Q) = \sum_{k\geq 0} \frac{h_k}{N_k} Q^k$ then, we note $(G(Q), F(Q), H(Q))_{N_k}$ for Generalization R.G operator, where N_k is a sequence with k. Then for any $\hbar(x) \in C^\infty$ at $x = a$, have

$$\begin{aligned}(G(Q), F(Q), H(Q))_{N_k} \hbar(a) &= G(Q) \cdot H(F(Q))\hbar(a)\\&= \sum_{n\geq 0}\left(\sum_{k\geq 0}^{n} d_{n,k} h_k\right) Q^n \hbar(a)\end{aligned}$$

Where, $d_{n,k} = [\frac{Q^n}{N_n}] G(Q) \left(F(Q)\right)^k$. Now, we note Ω_{N_k} is a set of Generalization R.G Operator, which composed by all $(G(Q), F(Q), H(Q))_{N_k}$.

Theorem 3.1(Ω_{N_k} is Generalization of the formal R.G Operators Group) Let $G(Q)_{N_k}$, $F(Q)_{N_k}$ is respectively reversible formal power summation operator and *delta* formal power summation operator, then for the mathematical operation($\otimes$), Ω_{N_k} also is a group. Especially, $(1, Q, H(Q)_{Nk})$ is unit element of Ω_{N_k}, $(G(Q)_{Nk}, F(Q)_{Nk}, H(Q)_{Nk})$ is inverse element for $\left(1/G(\overline{F}(Q)_{Nk})_{Nk}, \overline{F}(Q)_{Nk}, H(Q)_{Nk}\right)$. here, $\overline{F}(Q)_{Nk}$ is composite inversion for $F(Q)_{Nk}$, be satisfied with $F(\overline{F}(Q)_{Nk})_{Nk} = \overline{F}(F(Q)_{Nk})_{Nk} = Q$.

Proof: First, let $(G_1(Q)_{Nk}, F_1(Q)_{Nk}, H(Q)_{Nk})$, $(G_2(Q)_{Nk}, F_2(Q)_{Nk}, H(Q)_{Nk})$, $(G_3(Q)_{Nk}, F_3(Q)_{Nk}, H(Q)_{Nk}) \in \Omega_{Nk}$,then since

$$\begin{aligned}&(G_1(Q)_{Nk}, F_1(Q)_{Nk}, H(Q)_{Nk}) \otimes\\&(G_2(Q)_{Nk}, F_2(Q)_{Nk}, H(Q)_{Nk})\\&= (G_1(Q)_{Nk} \cdot G_2(F_1(Q)_{Nk})_{Nk}, F_2(F_1(Q)_{Nk})_{Nk}, H(Q)_{Nk}) \ ,\end{aligned}$$

Where,$(G_1(Q)_{Nk} \cdot G_2(F_1(Q)_{Nk})_{Nk}, F_2(F_1(Q)_{Nk})_{Nk}, H(Q)_{Nk}) \in \Omega_{Nk}$, therefore Ω_{Nk} is closed with mathematical operation ($\otimes$). And also because

$$\begin{aligned}&[(G_1(Q)_{Nk}, F_1(Q)_{Nk}, H(Q)_{Nk}) \otimes (G_2(Q)_{Nk}, F_2(Q)_{Nk}, H(Q)_{Nk})]\\&\otimes (G_3(Q)_{Nk}, F_3(Q)_{Nk}, H(Q)_{Nk})\\&= (G_1(Q)_{Nk} \cdot G_2(F_1(Q)_{Nk})_{Nk}, F_2(F_1(Q)_{Nk})_{Nk}, H(Q)_{Nk})\\&\otimes (G_3(Q)_{Nk}, F_3(Q)_{Nk}, H(Q)_{Nk})\\&= (G_1(Q)_{Nk} \cdot G_2(F_1(Q)_{Nk})_{Nk} \cdot G_3(F_2(F_1(Q)_{Nk})_{Nk})_{Nk},\\&F_3(F_2(F_1(Q)_{Nk})_{Nk})_{Nk}, H(Q)_{Nk}) \ ,\end{aligned}$$

As well as

$$\begin{aligned}&(G_1(Q)_{Nk}, F_1(Q)_{Nk}, H(Q)_{Nk}) \otimes [(G_2(Q)_{Nk}, F_2(Q)_{Nk}, H(Q)_{Nk})\\&\otimes (G_3(Q)_{Nk}, F_3(Q)_{Nk}, H(Q)_{Nk})]\\&= (G_1(Q)_{Nk}, F_1(Q)_{Nk}, H(Q)_{Nk})\\&\otimes (G_2(Q)_{Nk} \cdot G_3(F_2(Q)_{Nk})_{Nk}, F_3(F_2(Q)_{Nk})_{Nk}, H(Q)_{Nk})\\&= (G_1(Q)_{Nk} \cdot G_2(F_1(Q)_{Nk})_{Nk} \cdot G_3(F_2(F_1(Q)_{Nk})_{Nk})_{Nk},\\&F_3(F_2(F_1(Q)_{Nk})_{Nk})_{Nk}, H(Q)_{Nk}) \ .\end{aligned}$$

So the mathematical operation ($\otimes$) in set of Ω_{Nk} is contented to associative laws. Second, as $(1, Q, H(Q)_{Nk})$ is unit element of Ω_{Nk}, and for any $(G(Q)_{Nk}, F(Q)_{Nk}, H(Q)_{Nk}) \in \Omega_{Nk}$, we always have its inverse element $\left(1/G(\overline{F}(Q)_{Nk})_{Nk}, \overline{F}(Q)_{Nk}\right) \in \Omega_{Nk}$. In fact, as

$$\begin{aligned}&(G(Q)_{Nk}, F(Q)_{Nk}, H(Q)_{Nk}) \otimes (1, Q, H(Q)_{Nk})\\&= (G(Q)_{Nk}, F(Q)_{Nk}, H(Q)_{Nk})\\&= (1, Q, H(Q)_{Nk}) \otimes (G(Q)_{Nk}, F(Q)_{Nk}, H(Q)_{Nk})\\&= (1 \cdot G(Q)_{Nk}, F(Q)_{Nk}, H(Q)_{Nk}) \ ,\end{aligned}$$

and

$$\begin{aligned}&(G(Q)_{Nk}, F(Q)_{Nk}, H(Q)_{Nk}) \otimes \left(\frac{1}{G(\overline{F}(Q)_{Nk})_{Nk}}, \overline{F}(Q)_{Nk}, H(Q)_{Nk}\right)\\&= \left(G(Q)_{Nk} \cdot \frac{1}{G(\overline{F}(F(Q)_{Nk})_{Nk})_{Nk}}, \overline{F}(F(Q)_{Nk})_{Nk}, H(Q)_{Nk}\right)\\&= (1, Q, H(Q)_{Nk})\\&= \left(\frac{1}{G(\overline{F}(Q)_{Nk})_{Nk}}, \overline{F}(Q)_{Nk}, H(Q)_{Nk}\right)\\&\otimes (G(Q)_{Nk}, F(Q)_{Nk}, H(Q)_{Nk}) \ ,\end{aligned}$$

Thus Ω_{Nk} also is a Group, and we call it for Generalization R.G Operator Group. ■

IV. Some Theorem of the formal R.G operator and its Application

This section we will introduce some theorem about the formal R.G operator, and then we will gives some Corollary based on this.

Theorem 4.1 Suppose $(G(Q), F(Q), H(Q)) \in \Omega$, and here $H(Q) = Q^k$ that is

$(G(Q),F(Q),H(Q))$=$\left(G(Q),F(Q),Q^k\right)$, then for any $\hbar(x)\in C^\infty$ at $x=a$ we have,

$$\begin{aligned}\left(G(Q),F(Q),Q^k\right)\hbar(a)&=G(Q)\cdot(F(Q))^k\hbar(a)\\&=\sum_{n\geq0}\left(\sum_{k=0}^{n}d_{n,k}h_k\right)Q^n\hbar(a)\\&=\sum_{n\geq0}d_{n,k}Q^n\hbar(a)\end{aligned}$$

where, $d_{n,k}=[Q^n]G(Q)\cdot(F(Q))^k$.

Proof: For any $(G(Q),F(Q),H(Q))\in\Omega$, where $G(Q)$, $F(Q)$, Q is respectively reversible formal power summation operator, $delta$ formal power summation operator and $delta$ operator, we always have

$$\begin{aligned}\left(G(Q),F(Q),(Q)^k\right)&=G(Q)\cdot(F(Q))^k\\&=\sum_{n\geq0}d_{n,k}Q^n,\end{aligned}$$

So for any $\hbar(x)\in C^\infty$ evaluate at $x=a$ we have,

$$\begin{aligned}\left(G(Q),F(Q),(Q)^k\right)\hbar(a)&=G(Q)\cdot(F(Q))^k\hbar(a)\\&=\sum_{n\geq0}\left(\sum_{k=0}^{n}d_{n,k}h_k\right)Q^n\hbar(a)\\&=\sum_{n\geq0}d_{n,k}Q^n\hbar(a)\,.\end{aligned}$$

The proof is complete. ■

Corollary 4.1.1 Let $n,k(n\geq k)$ be a nonnegative integer, and $\hbar(x)\in C^\infty$, then

$$\frac{(\triangle)^k}{k!}\hbar(a)=\sum_{n\geq0}\left\{\begin{array}{c}n\\k\end{array}\right\}\frac{D^n}{n!}\hbar(a)$$

where, $\left\{\begin{array}{c}n\\k\end{array}\right\}$ is the second Stirling numbers, see [3].

Proof: Let $G(Q)=1$, $F(Q)=e^Q-1$, $H(Q)=Q^k$ so $\left(1,e^Q-1,Q^k\right)\in\Omega$, then for any $\hbar(x)\in C^\infty$ evaluate at $x=a$ have

$$\begin{aligned}\left(1,e^Q-1,Q^k\right)\hbar(a)&=\left(e^Q-1\right)^k\hbar(a)\\&=\sum_{n\geq0}\left\{\begin{array}{c}n\\k\end{array}\right\}\frac{k!}{n!}Q^k\hbar(a)\end{aligned}$$

if we let $D=Q$ then we have

$$\left(e^D-1\right)^k\hbar(a)=\sum_{n\geq0}\left\{\begin{array}{c}n\\k\end{array}\right\}\frac{k!}{n!}D^k\hbar(a)$$

and

$$\frac{(\triangle)^k}{k!}\hbar(a)=\sum_{n\geq0}\left\{\begin{array}{c}n\\k\end{array}\right\}\frac{D^n}{n!}\hbar(a)$$

The proof is complete. ■

Example . Let $\hbar(x)=x^n(n\geq1)$in Corollary(4.1.1), and evaluate at $x=0$ then

$$\begin{aligned}\sum_{n\geq0}\left\{\begin{array}{c}n\\k\end{array}\right\}=\frac{1}{k!}(\triangle)^k\hbar(0)&=\frac{1}{k!}\sum_{m\geq0}^{k}\binom{k}{m}(-1)^{k-m}\hbar(m)\\&=\sum_{m\geq0}^{k}\frac{(-1)^{k-m}m^n}{(k-m)!m!}\,.\end{aligned}$$

and by the same way, if we let $\hbar(x)=\alpha^x(\alpha>0,\alpha\neq1)$, the same time since $(\triangle)^k\alpha^a=\alpha^a(\alpha-1)^k$ and $D^n\alpha^a=\alpha^a(\ln\alpha)^n$, so we have the following result,

$$\frac{(\alpha-1)^k}{k!}=\sum_{n\geq0}\left\{\begin{array}{c}n\\k\end{array}\right\}\frac{(\ln\alpha)^n}{n!}\,,$$

when $\alpha=2$ we have

$$\frac{1}{k!}=\sum_{n\geq0}\left\{\begin{array}{c}n\\k\end{array}\right\}\frac{(\ln2)^n}{n!}\,.$$

So we obtain two beautiful results.

Theorem 4.2 Let $(G(Q),F(Q),H(Q))\in\Omega$, and Q is a $delta$ operator, then for any $\hbar(x)\in C^\infty$, if $G(Q)=\frac{Q^m}{(1-Q)^{m+1}}$, $F(Q)=\frac{Q^{(\beta-\alpha)}}{(1-Q)^\beta}$ evaluate at $x=a$, we have

$$\frac{Q^m}{(1-Q)^{m+1}}\cdot H\left(\frac{Q^{(\beta-\alpha)}}{(1-Q)^\beta}\right)\hbar(a)=\sum_{n\geq0}\left(\sum_{k=0}^{n}\binom{n+\alpha k}{m+\beta k}h_k\right)\cdot Q^n\hbar(a)\,,$$

where $\alpha>\beta$, $\alpha-\beta$ is the integers, and n is the variables, m is the parameter, and if $G(Q)=(1+Q)^n$, $F(Q)=\frac{Q^{(-\beta)}}{(1+Q)^{(-\alpha)}}$ then we have

$$(1+Q)^n\cdot H\left(\frac{Q^{(-\beta)}}{(1+Q)^{(-\alpha)}}\right)\hbar(a)=\sum_{m\geq0}\left(\sum_{k=0}^{m}\binom{n+\alpha k}{m+\beta k}h_k\right)\cdot Q^m\hbar(a)\,.$$

where b is the integer($b<0$), n is the parameter, m is the variables.

Proof: Use the same method as Theorem(4.1), we can obtain these results easily, not be repeated here. ■

Corollary 4.2.1 Let n,k $(n\geq k\geq0)$ be a nonnegative integer, and then we have

$$H((-\triangle)^{b-a}\cdot E^{-b})\hbar(a-m-1)=\sum_{n\geq0}\left(\sum_{k=0}^{n}\binom{n+ak}{m+bk}h_k\right)\cdot(-\triangle)^{n-m}\hbar(a)\,,$$

and

$$H((\triangle)^{-b}\cdot E^{a})\hbar(a+n)=\sum_{n\geq0}\left(\sum_{k=0}^{m}\binom{n+ak}{m+bk}h_k\right)\triangle^n\hbar(a)\,.$$

Proof: Let $Q = -\triangle$ in the first identities of Theorem(4.1), then we have

$$\frac{(-\triangle)^m}{(1+\triangle)^{m+1}} \cdot H(\frac{(-\triangle)^{(b-a)}}{(1+\triangle)^b})\hbar(a) = \sum_{n\geq 0}\left(\sum_{k=0}^{n}\binom{n+\alpha k}{m+\beta k}h_k\right)(-\triangle)^n\hbar(a),$$

or

$$H((-\triangle)^{b-a}\cdot E^{-b})\hbar(a-m-1) = \sum_{n\geq 0}\left(\sum_{k=0}^{n}\binom{n+\alpha k}{m+\beta k}h_k\right)(-\triangle)^{n-m}\hbar(a)\,.$$

and then if we let $Q = \triangle$ in the second identities of Theorem(4.1), then we have

$$(1+\triangle)^n \cdot H(\triangle^{(-b)}\cdot E^a)\hbar(a) = H((\triangle)^{-b}\cdot E^a)\hbar(a+n) = \sum_{n\geq 0}\left(\sum_{k=0}^{m}\binom{n+ak}{m+bk}h_k\right)\triangle^n\hbar(a)\,.$$

So we got the result.

■

Theorem 4.3 Let $n, k (n \geq k)$ be a nonnegative integer, then we have

$$H(-D)\hbar(a) = \sum_{n\geq 0}\left(\sum_{k\geq 0}^{n}\frac{k!}{n!}\left[\begin{array}{c} n \\ k \end{array}\right]h_k\right)\cdot(-\triangle)^n\hbar(a)$$

where, D, $\triangle$ respectaly is different and difference operator, and $\left[\begin{array}{c} n \\ k \end{array}\right]$ is the first Stirling numbers, as in [3].

Proof: Use the same method as Theorem(4.2), let $G(Q) = 1$, $F(Q) = -\ln(1-Q)$, $H(Q) = \sum_{k\geq 0} h_k Q^k$, then for any $\hbar(x) \in C^\infty$ evaluate $x = a$ we have

$$H(\ln\frac{1}{1-Q})\hbar(a) = \sum_{n\geq 0}\left(\sum_{k=0}^{n}\frac{k!}{n!}\left[\begin{array}{c} n \\ k \end{array}\right]h_k\right)Q^n\hbar(a)\,,$$

let $Q = -\triangle$ in this formula, then we can obtain

$$H(\ln\frac{1}{1+\triangle})\hbar(a) = H(-D)\hbar(a) = \sum_{n\geq 0}\left(\sum_{k=0}^{n}\frac{k!}{n!}\left[\begin{array}{c} n \\ k \end{array}\right]h_k\right)(-\triangle)^n\hbar(a)$$

The proof is complete.

■

Corollary 4.3.1 Let $n, k (n \geq k)$ be a nonnegative integer, for any $\hbar(x) \in C^\infty$ then we have

$$\hbar(a+1) = \sum_{n\geq 0}\left(\sum_{k=0}^{n}\frac{k!}{n!}\left[\begin{array}{c} n \\ k \end{array}\right]\right)\frac{(\triangle)^n}{n!}\hbar(a)$$

Proof: Let $H(Q) = e^{-Q}$ in the Theorem(4.3), meanwhile $h_k = \frac{(-1)^n}{n!}$ then we have

$$e^D(\hbar(a)) = \sum_{n\geq 0}\left(\sum_{k=0}^{n}\frac{k!}{n!}\left[\begin{array}{c} n \\ k \end{array}\right]\frac{(-1)^n}{n!}\right)(-1)^n(-\triangle)^n\hbar(a)\,,$$

Since $e^D(\hbar(a)) = E(\hbar(a))$, so we have

$$\hbar(a+1) = \sum_{n\geq 0}\left(\sum_{k=0}^{n}\frac{k!}{n!}\left[\begin{array}{c} n \\ k \end{array}\right]\right)\frac{(\triangle)^n}{n!}\hbar(a)\,,$$

The proof is complete.

■

Example. Let $\hbar(x) = \alpha^x (\alpha > 0, \alpha \neq 1)$ in the Corollary(4.3.1), since $\triangle^n(\alpha^x) = (\alpha-1)^n\alpha^x$, so we have

$$\alpha^{a+1} = \sum_{n\geq 0}\left(\sum_{k=0}^{n}\frac{k!}{n!}\left[\begin{array}{c} n \\ k \end{array}\right]\right)\frac{(\alpha-1)^n}{n!}\alpha^a\,,$$

if $a = 0$ then we have

$$\alpha = \sum_{n\geq 0}\left(\sum_{k=0}^{n}\frac{k!}{n!}\left[\begin{array}{c} n \\ k \end{array}\right]\right)\frac{(\alpha-1)^n}{n!}\,,$$

and if $\alpha = 1$ then

$$1 = \sum_{n\geq 0}\left(\sum_{k=0}^{n}\frac{k!}{n!}\left[\begin{array}{c} n \\ k \end{array}\right]\right)\frac{0^n}{n!}\,,$$

and if $\alpha = 2$ then we have

$$2 = \sum_{n\geq 0}\left(\sum_{k=0}^{n}\frac{k!}{n!}\left[\begin{array}{c} n \\ k \end{array}\right]\right)\frac{1}{n!}\,.$$

This is a beautiful result.

Theorem 4.4 Let $G(Q) = 1, F(Q) = e^Q - 1$, and $(G(Q), F(Q), H(Q)) \in R.G$, $Q = D$ is a *delta* operator, then for any $\hbar(x) \in C^\infty$ evaluate at $x = a$ we have

$$H(\triangle)\cdot\hbar(a) = \sum_{n\geq 0}\left(\sum_{k=0}^{n}\frac{k!}{n!}\left\{\begin{array}{c} n \\ k \end{array}\right\}h_k\right)D^n\cdot\hbar(a)\,.$$

where, D, $\triangle$ respectaly is different and difference operators, and $\left\{\begin{array}{c} n \\ k \end{array}\right\}$ is the second Stirling numbers.

Proof: Let n, k be a nonnegative integer, by the same ways as Theorem(4.3) have,

$$H(e^Q - 1)\hbar(a) = \sum_{n\geq 0}\left(\sum_{k=0}^{n}\frac{k!}{n!}\left\{\begin{array}{c} n \\ k \end{array}\right\}h_k\right)Q^n\hbar(a)\,,$$

and let $Q = D$ in the formula, then we have following result

$$\sum_{n\geq 0}\left(\sum_{k=0}^{n}\frac{k!}{n!}\left\{\begin{array}{c} n \\ k \end{array}\right\}h_k\right)D^n\cdot\hbar(a) = H(e^D-1)\cdot\hbar(a) = H(\triangle)\cdot\hbar(a)$$

So we got the result.

■

Corollary 4.4.1 Let n, k be a nonnegative integer, $\hbar(x) \in C^\infty$ evaluate at $x = a$ we have

$$\hbar(a-1) = \sum_{n\geq 0}\left(\sum_{k=0}^{n}\frac{k!}{n!}\left\{\begin{array}{c} n \\ k \end{array}\right\}(-1)^k\right)D^n\cdot\hbar(a)$$

Proof: let $H(Q)=\frac{1}{1+Q}$, in the Theorem(4.4) then

$$\sum_{n\geq 0}\left(\sum_{k=0}^{n}\frac{k!}{n!}\left\{\begin{array}{c}n\\k\end{array}\right\}(-1)k\right)D^n\cdot\hbar(a)$$
$$=H(\frac{1}{1+\triangle})\cdot\hbar(a)=\hbar(a-1).$$

The proof is complete. ∎

Theorem 4.5 Let $G(Q)=(\ln\frac{1}{1-Q})^k$, $F(Q)=e^Q-1$, $H(Q)=\ln\frac{1}{1+Q}$ and $(G(Q),F(Q),H(Q))\in R.G$, then for any $\hbar(x)\in C^\infty$ evaluate $x=a$ we have,

$$\sum_{n\geq 0}\left(\sum_{j=0}^{n}\left(\sum_{l=0}^{n}\frac{(j!)^2}{l!(n-l)!}\left[\begin{array}{c}n-l\\j\end{array}\right]\left\{\begin{array}{c}n\\j\end{array}\right\}\frac{(-1)^j}{j}\right)\right)$$
$$\cdot(-1)^{n+k}(\triangle)^n\hbar(a)$$
$$=\left(D^k\hbar(a+1)-D^k\hbar(a)\right)$$

Proof: Let n,j be a nonnegative integer, by the same methods as Thorem(4.4) we have

$$\sum_{n\geq 0}\left(\sum_{j=0}^{n}\left(\sum_{l=0}^{n}\frac{(j!)^2}{l!(n-l)!}\left[\begin{array}{c}n-l\\j\end{array}\right]\left\{\begin{array}{c}n\\j\end{array}\right\}\right)h_j\right)$$
$$\cdot Q^n\hbar(a)$$
$$=(\ln\frac{1}{1-Q})^k\cdot H(e^Q-1)\hbar(a)$$

Let $Q=-\triangle$, then

$$\sum_{n\geq 0}\left(\sum_{j=0}^{n}\left(\sum_{l=0}^{n}\frac{(j!)^2}{l!(n-l)!}\left[\begin{array}{c}n-l\\j\end{array}\right]\left\{\begin{array}{c}n\\j\end{array}\right\}h_j\right)\right)$$
$$\cdot(-\triangle)^n\hbar(a)$$
$$=(-D)^k\cdot H(e^{-\triangle}-1)\hbar(a)\,,$$

and $H(Q)=\ln\frac{1}{1+Q}$, so $H(e^{-\triangle}-1)=\triangle$, then we have

$$\sum_{n\geq 0}\left(\sum_{j=0}^{n}\left(\sum_{l=0}^{n}\frac{(j!)^2}{l!(n-l)!}\left[\begin{array}{c}n-l\\j\end{array}\right]\left\{\begin{array}{c}n\\j\end{array}\right\}\frac{(-1)^j}{j}\right)\right)$$
$$\cdot(-\triangle)^n\hbar(a)$$
$$=\triangle\cdot(-D)^k\hbar(a)\,,$$

or

$$\sum_{n\geq 0}\left(\sum_{j=0}^{n}\left(\sum_{l=0}^{n}\frac{(j!)^2}{l!(n-l)!}\left[\begin{array}{c}n-l\\j\end{array}\right]\left\{\begin{array}{c}n\\j\end{array}\right\}\frac{(-1)^j}{j}\right)\right)$$
$$\cdot(-1)^{n+k}(\triangle)^n\hbar(a)$$
$$=\left(D^k\hbar(a+1)-D^k\hbar(a)\right)$$

The proof is complete. ∎

Theorem 4.6 Let $G(Q)=(e^Q-1)^k$, $F(Q)=\ln\frac{1}{1-Q}$, $H(Q)=e^{(-Q)}$ and $(G(Q),F(Q),H(Q))\in R.G$, then for any $\hbar(x)\in C^\infty$ evaluate $x=a$ we have

$$\sum_{n\geq 0}\left(\sum_{j=0}^{n}\left(\sum_{l=0}^{n}\frac{(j!)^2}{l!(n-l)!}\left[\begin{array}{c}n-l\\j\end{array}\right]\left\{\begin{array}{c}n\\j\end{array}\right\}\frac{(-1)^j}{j!}\right)\right)$$
$$\cdot D^n\hbar(a)$$
$$=\sum_{l=0}^{k}\binom{k}{l}(-1)^{k-l}\left(\hbar(a+k)-\hbar^{'}(a+k)\right)$$

Proof: Let n,j,l be a nonnegative integer, by the same methods as Theorem(4.5) we have,

$$\sum_{n\geq 0}\left(\sum_{j=0}^{n}\left(\sum_{l=0}^{n}\frac{(j!)^2}{l!(n-l)!}\left[\begin{array}{c}n-l\\j\end{array}\right]\left\{\begin{array}{c}n\\j\end{array}\right\}h_j\right)\right)\cdot Q^n\hbar(a)$$
$$=(e^Q-1)^k\cdot H(\ln\frac{1}{1-Q})\hbar(a)\,,$$

then let $Q=D$ in it, we have

$$\sum_{n\geq 0}\left(\sum_{j=0}^{n}\left(\sum_{l=0}^{n}\frac{(j!)^2}{l!(n-l)!}\left[\begin{array}{c}n-l\\j\end{array}\right]\left\{\begin{array}{c}n\\j\end{array}\right\}h_j\right)\right)\cdot Q^n\hbar(a)$$
$$=\triangle^k\cdot H(\ln\frac{1}{1-D})\hbar(a)\,,$$

and since $H(Q)=e^{-Q}$, so $h_k=\frac{(-1)^k}{k!}$, then we have

$$\sum_{n\geq 0}\left(\sum_{j=0}^{n}\left(\sum_{l=0}^{n}\frac{(j!)^2}{l!(n-l)!}\left[\begin{array}{c}n-l\\j\end{array}\right]\left\{\begin{array}{c}n\\j\end{array}\right\}\frac{(-1)^j}{j!}\right)\right)\cdot D^n\hbar(a)$$
$$=\triangle^k\cdot(1-D)\hbar(a)$$

so we got the result,

$$\sum_{n\geq 0}\left(\sum_{j=0}^{n}\left(\sum_{l=0}^{n}\frac{(j!)^2}{l!(n-l)!}\left[\begin{array}{c}n-l\\j\end{array}\right]\left\{\begin{array}{c}n\\j\end{array}\right\}\frac{(-1)^j}{j!}\right)\right)\cdot D^n\hbar(a)$$
$$=\sum_{l=0}^{k}\binom{k}{l}(-1)^{k-l}\left(\hbar(a+k)-\hbar^{'}(a+k)\right)$$

The proof is complete. ∎

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References

[1] Ch. Jordan, Calculus of Finite Differences, Chelsea Publishing Co. New York, 1970, pp. 66-192.
[2] L. Comtet, Advanced Combinatorics, D. Reidel Publishing Co., Dordrecht, 1974, pp. 169-174.
[3] H. S. Wilf, Generatingfunctionology, Academic Press, New York, 1990, pp. 48-96.
[4] R. Mullin snd G.-C. ROTA, On the foundations of combinatorial theory: III. Theory of binomial enumeration, in: Graph Theory and its Applications, New York and London, 1965, pp. 156-210.
[5] T. X. He, L. C. Hsu, and P.J. S. Shiue, Symbolization of generating functions, an qpplication of Mullin-Rota's theory of binomial enumeration, Comp. Math. with Applications, 54 (2007), pp. 664-678.
[6] G.P.EGORYCHEV, Integral Representation and the Computation of Combinatorial Sums. American Mathematical Society Providence RhodeIsLand, 1984, pp. 68-90.