COMBINATORIAL SUMS AND SERIES INVOLVING INVERSES OF BINOMIAL COEFFICIENTS

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(Submitted May 1998-Final Revision January 1999)

0. INTRODUCTION

In this note we deal with several combinatorial sums and series involving inverses of binomial coefficients. Some of them have already been considered by other authors (see, e.g., [3], [4]), but it should be noted that our approach is different. It is based on Euler's well-known Beta function defined by

$$B(m,n)=\int_0^1 t^{m-1}(1-t)^{n-1}dt$$

for all positive integers m and n. Since

$$B(m,n)=\frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}=\frac{(m-1)!(n-1)!}{(m+n-1)!},$$

we get

$$\binom{n}{k}^{-1}=(n+1)\int_0^1 t^k(1-t)^{n-k}dt \tag{1}$$

for all nonnegative integers n and k with $n\ge k$.

1. SUMS INVOLVING INVERSES OF BINOMIAL COEFFICIENTS

Theorem 1.1 ([4], Theorem 1): If n is a nonnegative integer, then

$$\sum_{k=0}^{n}\binom{n}{k}^{-1}=\frac{n+1}{2^{n+1}}\sum_{k=1}^{n+1}\frac{2^k}{k}.$$

Proof: Let S_n be the sum of inverses of binomial coefficients. From (1) we get

$$\begin{aligned}S_n&=\sum_{k=0}^{n}(n+1)\int_0^1 t^k(1-t)^{n-k}dt\\&=(n+1)\int_0^1\left\{(1-t)^n\sum_{k=0}^{n}\left(\frac{t}{1-t}\right)^k\right\}dt=(n+1)\int_0^1\frac{(1-t)^{n+1}-t^{n+1}}{1-2t}dt.\end{aligned}$$

Making the substitution $1-2t=x$, we obtain

$$\begin{aligned}S_n&=\frac{n+1}{2^{n+2}}\int_{-1}^{1}\frac{(1+x)^{n+1}-(1-x)^{n+1}}{x}dx=\frac{n+1}{2^{n+2}}\left\{\int_{-1}^{1}\frac{(1+x)^{n+1}-1}{x}dx+\int_{-1}^{1}\frac{1-(1-x)^{n+1}}{x}dx\right\}\\&=\frac{n+1}{2^{n+2}}\sum_{k=0}^{n}\left\{\int_{-1}^{1}(1+x)^k dx+\int_{-1}^{1}(1-x)^k dx\right\}=\frac{n+1}{2^{n+1}}\sum_{k=1}^{n+1}\frac{2^k}{k}.\end{aligned}$$

Theorem 1.2: If n is a positive integer, then

$$\sum_{k=0}^{2n}(-1)^k\binom{2n}{k}\binom{4n}{2k}^{-1}=\frac{4n+1}{2n+1}.$$

Proof: Formula (1) yields

$$\begin{aligned}\sum_{k=0}^{2n}(-1)^k\binom{2n}{k}\binom{4n}{2k}^{-1}&=\sum_{k=0}^{2n}(-1)^k\binom{2n}{k}(4n+1)\int_0^1 t^{2k}(1-t)^{4n-2k}dt\\&=(4n+1)\int_0^1\left\{(1-t)^{4n}\sum_{k=0}^{2n}\binom{2n}{k}\left(\frac{-t^2}{(1-t)^2}\right)^k\right\}dt\\&=(4n+1)\int_0^1(1-t)^{4n}\left(1-\frac{t^2}{(1-t)^2}\right)^{2n}dt\\&=(4n+1)\int_0^1(1-2t)^{2n}dt=\frac{4n+1}{2n+1}.\end{aligned}$$

Theorem 1.3 ([5]): If n is a positive integer, then

$$\sum_{k=0}^{2n}(-1)^k\binom{4n}{2k}\binom{2n}{k}^{-1}=-\frac{1}{2n-1}.$$

Proof: Let S_n be the sum to evaluate. From (1) we get

$$\begin{aligned}S_n&=\sum_{k=0}^{2n}(-1)^k\binom{4n}{2k}(2n+1)\int_0^1 t^k(1-t)^{2n-k}dt\\&=(2n+1)\int_0^1\left\{\sum_{k=0}^{2n}\binom{4n}{2k}(-1)^k t^k(1-t)^{2n-k}\right\}dt\\&=\frac{2n+1}{2}\int_0^1\left\{\left(\sqrt{1-t}+i\sqrt{t}\right)^{4n}+\left(\sqrt{1-t}-i\sqrt{t}\right)^{4n}\right\}dt.\end{aligned}$$

Since

$$\sqrt{1-t}\pm i\sqrt{t}=\cos\left(\arctan\sqrt{\frac{t}{1-t}}\right)\pm i\sin\left(\arctan\sqrt{\frac{t}{1-t}}\right),$$

it follows that

$$S_n=(2n+1)\int_0^1\cos\left(4n\arctan\sqrt{\frac{t}{1-t}}\right)dt.$$

Making the substitution $\arctan\sqrt{\frac{t}{1-t}}=x$, we obtain

$$\begin{aligned}S_n&=(2n+1)\int_0^{\pi/2}\cos(4nx)\sin(2x)dx\\&=\frac{2n+1}{2}\int_0^{\pi/2}\{\sin(4n+2)x-\sin(4n-2)x\}dx=-\frac{1}{2n-1}.\end{aligned}$$

Theorem 1.4 ([2]): If m, n, and p are nonnegative integers with $p\le n$, then

$$\sum_{k=0}^{m}\binom{m}{k}\binom{n+m}{p+k}^{-1}=\frac{n+m+1}{n+1}\binom{n}{p}^{-1}.$$

Proof: Formula (1) yields

$$\sum_{k=0}^{m}\binom{m}{k}\binom{n+m}{p+k}^{-1}=\sum_{k=0}^{m}\binom{m}{k}(n+m+1)\int_0^1 t^{p+k}(1-t)^{n+m-p-k}dt$$
$$=(n+m+1)\int_0^1\left\{t^p(1-t)^{n+m-p}\sum_{k=0}^{m}\binom{m}{k}\left(\frac{t}{1-t}\right)^k\right\}dt$$
$$=(n+m+1)\int_0^1 t^p(1-t)^{n+m-p}\left(1+\frac{t}{1-t}\right)^m dt$$
$$=(n+m+1)\int_0^1 t^p(1-t)^{n-p}dt=\frac{n+m+1}{n+1}\binom{n}{p}^{-1}.$$

Remark: In the special case $p=n$, from the above theorem we get

$$\sum_{k=0}^{m}\binom{m}{k}\binom{n+m}{n+k}^{-1}=\frac{n+m+1}{n+1}.$$

Theorem 1.5: If m and n are nonnegative integers, then

$$\sum_{k=0}^{n}(-1)^k\binom{m+n}{m+k}^{-1}=\frac{m+n+1}{m+n+2}\left(\binom{m+n+1}{m}^{-1}+(-1)^n\right).$$

Proof: We have

$$\sum_{k=0}^{n}(-1)^k\binom{m+n}{m+k}^{-1}=\sum_{k=0}^{n}(-1)^k(m+n+1)\int_0^1 t^{m+k}(1-t)^{n-k}dt$$
$$=(m+n+1)\int_0^1\left\{t^m(1-t)^n\sum_{k=0}^{n}\left(\frac{-t}{1-t}\right)^k\right\}dt$$
$$=(m+n+1)\left(\int_0^1 t^m(1-t)^{n+1}dt+(-1)^n\int_0^1 t^{m+n+1}dt\right)$$
$$=\frac{m+n+1}{m+n+2}\left(\binom{m+n+1}{m}^{-1}+(-1)^n\right).$$

Remark: In the special case $m=n$ we get

$$\sum_{k=0}^{n}(-1)^k\binom{2n}{n+k}^{-1}=\frac{2n+1}{2n+2}\left(\binom{2n+1}{n}^{-1}+(-1)^n\right),$$

while in the special case $m=0$ we obtain

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}^{-1}=\frac{n+1}{n+2}(1+(-1)^n).$$

Consequently (see [3], p. 343),

$$\sum_{k=0}^{2n}(-1)^k\binom{2n}{k}^{-1}=\frac{2n+1}{n+1}.$$

Theorem 1.6: If n is a positive integer, then

$$\frac{1}{3n+1}\sum_{k=0}^{n}\binom{3n}{3k}^{-1}-\frac{1}{3n+2}\sum_{k=0}^{n}\binom{3n+1}{3k+1}^{-1}+\frac{1}{3n+3}\sum_{k=0}^{n}\binom{3n+2}{3k+2}^{-1}=\frac{1}{3n+3}\sum_{k=0}^{3n+2}\binom{3n+2}{k}^{-1}.$$

Proof: We have

$$\begin{aligned}&\frac{1}{3n+1}\sum_{k=0}^{n}\binom{3n}{3k}^{-1}-\frac{1}{3n+2}\sum_{k=0}^{n}\binom{3n+1}{3k+1}^{-1}+\frac{1}{3n+3}\sum_{k=0}^{n}\binom{3n+2}{3k+2}^{-1}\\&=\sum_{k=0}^{n}\int_0^1\{t^{3k}(1-t)^{3n-3k}-t^{3k+1}(1-t)^{3n-3k}+t^{3k+2}(1-t)^{3n-3k}\}\,dt\\&=\int_0^1\left\{(1-t)^{3n}(1-t+t^2)\sum_{k=0}^{n}\left(\frac{t^3}{(1-t)^3}\right)^k\right\}dt=\int_0^1\frac{(1-t)^{3n+3}-t^{3n+3}}{1-2t}dt.\end{aligned}$$

On the other hand,

$$\begin{aligned}\frac{1}{3n+3}\sum_{k=0}^{3n+2}\binom{3n+2}{k}^{-1}&=\sum_{k=0}^{3n+2}\int_0^1 t^k(1-t)^{3n+2-k}\,dt\\&=\int_0^1\left\{(1-t)^{3n+2}\sum_{k=0}^{3n+2}\left(\frac{t}{1-t}\right)^k\right\}dt=\int_0^1\frac{(1-t)^{3n+3}-t^{3n+3}}{1-2t}\,dt,\end{aligned}$$

completing the proof.

2. SERIES INVOLVING INVERSES OF BINOMIAL COEFFICIENTS

Theorem 2.1: If m and n are positive integers with $m>n$, then

$$\sum_{k=0}^{\infty}\binom{mk}{nk}^{-1}=\int_0^1\frac{1+(m-1)t^n(1-t)^{m-n}}{(1-t^n(1-t)^{m-n})^2}dt.$$

Proof: From (1) we get

$$\begin{aligned}\sum_{k=0}^{\infty}\binom{mk}{nk}^{-1}&=\sum_{k=0}^{\infty}(mk+1)\int_0^1 t^{nk}(1-t)^{(m-n)k}\,dt\\&=m\sum_{k=1}^{\infty}\int_0^1 k(t^n(1-t)^{m-n})^k\,dt+\sum_{k=0}^{\infty}\int_0^1(t^n(1-t)^{m-n})^k\,dt.\end{aligned}$$

Let $f:[0,1]\to\mathbf{R}$ be the function defined by $f(t)=t^n(1-t)^{m-n}$. It is immediately seen that f attains its maximum at the point $t_0=n/m$. Since $f(t_0)<1$, it follows that

$$\sum_{k=1}^{\infty}k(t^n(1-t)^{m-n})^k=\frac{t^n(1-t)^{m-n}}{(1-t^n(1-t)^{m-n})^2}$$

and

$$\sum_{k=1}^{\infty}(t^n(1-t)^{m-n})^k = \frac{1}{1-t^n(1-t)^{m-n}}$$

uniformly on [0, 1]. Therefore, we obtain

$$\sum_{k=0}^{\infty}\binom{mk}{nk}^{-1} = m\int_0^1 \frac{t^n(1-t)^{m-n}}{(1-t^n(1-t)^{m-n})^2}dt + \int_0^1 \frac{dt}{1-t^n(1-t)^{m-n}},$$

completing the proof.

Remark: As special cases of Theorem 2.1 we get

$$\sum_{k=0}^{\infty}\binom{2k}{k}^{-1} = \frac{4}{3}+\frac{2\pi\sqrt{3}}{27}, \tag{2}$$

$$\sum_{k=0}^{\infty}\binom{4k}{2k}^{-1} = \frac{16}{15}+\frac{\pi\sqrt{3}}{27}-\frac{2\sqrt{5}}{25}\ln\frac{1+\sqrt{5}}{2}. \tag{3}$$

Indeed, according to Theorem 2.1, we have

$$\sum_{k=0}^{\infty}\binom{2k}{k}^{-1} = \int_0^1 \frac{1+t-t^2}{(1-t+t^2)^2}dt = 2\int_0^1 \frac{dt}{(1-t+t^2)^2} - \int_0^1 \frac{dt}{1-t+t^2} \tag{4}$$

and

$$\sum_{k=0}^{\infty}\binom{4k}{2k}^{-1} = \int_0^1 \frac{1+3t^2(1-t)^2}{(1-t^2(1-t)^2)^2}dt,$$

respectively. Since

$$\frac{1+3x^2}{(1-x^2)^2} = \frac{1+x}{2(1-x)^2}+\frac{1-x}{2(1+x)^2},$$

we obtain

$$\begin{aligned}\sum_{k=0}^{\infty}\binom{4k}{2k}^{-1} &= \int_0^1 \frac{1+t-t^2}{2(1-t+t^2)^2}dt + \int_0^1 \frac{1-t+t^2}{2(1+t-t^2)^2}dt \\ &= \int_0^1 \frac{dt}{(1-t+t^2)^2} - \frac{1}{2}\int_0^1 \frac{dt}{1-t+t^2} + \int_0^1 \frac{dt}{(1+t-t^2)^2} - \frac{1}{2}\int_0^1 \frac{dt}{1+t-t^2}.\end{aligned} \tag{5}$$

Taking into account that

$$\int_0^1 \frac{dt}{1-t+t^2} = \frac{2\pi\sqrt{3}}{9} \quad \text{and} \quad \int_0^1 \frac{dt}{1+t-t^2} = \frac{4\sqrt{5}}{5}\ln\frac{1+\sqrt{5}}{2},$$

and that (see, e.g., [1])

$$\int \frac{dt}{(a+bt+ct^2)^2} = \frac{b+2ct}{(4ac-b^2)(a+bt+ct^2)} + \frac{2c}{4ac-b^2}\int \frac{dt}{a+bt+ct^2},$$

from (4) and (5) one can easily obtain (2) and (3).

Theorem 2.2 ([4], Theorem 2): If $n \geq 2$ is an integer, then

$$\sum_{k=0}^{\infty}\binom{n+k}{k}^{-1} = \frac{n}{n-1}.$$

Proof: For each positive integer p, we have

$$\begin{aligned} s_p &:= \sum_{k=0}^{p}\binom{n+k}{k}^{-1} = \sum_{k=0}^{p}(n+k+1)\int_0^1 t^k(1-t)^n dt \\ &= \int_0^1\left\{(n+1)(1-t)^n\sum_{k=0}^{p}t^k + (1-t)^n\sum_{k=0}^{p}kt^k\right\}dt \\ &= (n+1)\int_0^1(1-t)^n dt - (n+1)\int_0^1 t^{p+1}(1-t)^{n-1}dt + \int_0^1 t(1-t)^{n-2}dt \\ &\qquad - (p+1)\int_0^1 t^{p+1}(1-t)^{n-2}dt + p\int_0^1 t^{p+2}(1-t)^{n-2}dt. \end{aligned}$$

Formula (1) yields

$$s_p = \frac{n}{n-1} - (n-2)!\frac{(np+p+1)(p+1)!}{(p+n+1)!} - (n+1)(n-1)!\frac{(p+1)!}{(p+n+1)!}.$$

Taking into account that $n \geq 2$, we conclude that $s_p \to \frac{n}{n-1}$ when $p \to \infty$.

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AMS Classification Number: 11B65

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