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European Journal of Combinatorics 24 (2003) 709–718

European Journal of Combinatorics

www.elsevier.com/locate/ejc

Combinatorial identities in dual sequences

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Received 24 November 2002; received in revised form 23 April 2003; accepted 24 April 2003

Abstract

In this paper we derive a general combinatorial identity in terms of polynomials with dual sequences of coefficients. Moreover, combinatorial identities involving Bernoulli and Euler polynomials are deduced. Also, various other known identities are obtained as particular cases.

MSC (2000): Primary 05A19; Secondary 11B65, 11B68, 11B73

1. Introduction

Let $\{a_n\}_{n\in\mathbb{N}}$ be a sequence of complex numbers where $\mathbb{N} = \{0, 1, 2, \ldots\}$. We call the sequence $\{a_n^*\}_{n\in\mathbb{N}}$ given by

$$a_n^* = \sum_{i=0}^{n} \binom{n}{i} (-1)^i a_i \tag{1.1}$$

the *dual sequence* of $\{a_n\}_{n\in\mathbb{N}}$. It is well known that $a_n^{**} = a_n$ for all $n \in \mathbb{N}$ (see, e.g. [2, pp. 192–193]). Those self-dual sequences are of particular interest and were investigated in [5]. The Bernoulli numbers $B_0, B_1, \ldots$ are given by $B_0 = 1$ and $\sum_{i=0}^{n} \binom{n+1}{i} B_i = 0\,(n = 1, 2, 3, \ldots)$; since $B_1 = -1/2$ and $B_{2k+1} = 0$ for $k = 1, 2, \ldots$ the sequence $\{(-1)^n B_n\}_{n\in\mathbb{N}}$ is self-dual as observed in [5]. Like the definition of Bernoulli polynomials (see, e.g. [6]), we introduce

$$A_n(x) = \sum_{i=0}^{n} \binom{n}{i} (-1)^i a_i x^{n-i} \quad \text{and} \quad A_n^*(x) = \sum_{i=0}^{n} \binom{n}{i} (-1)^i a_i^* x^{n-i}. \tag{1.2}$$

Obviously $A_n(0) = (-1)^n a_n$, $A_n(1) = a_n^*$ and

$$A'_{n+1}(x) = \sum_{i=0}^{n} \binom{n+1}{i} (-1)^i a_i (n+1-i) x^{n-i} = (n+1) A_n(x).$$

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doi:10.1016/S0195-6698(03)00062-3

In 1995 Kaneko [3] found the following new recursion formula for Bernoulli numbers:

$$\sum_{j=0}^{k}\binom{k+1}{j}(k+j+1)B_{k+j}=0 \qquad \text{for } k=1,2,\ldots.$$

By means of the Volkenborn integral, Momiyama [4] got the following symmetric extension: if $k,l\in\mathbb{N}$ and $k+l>0$, then

$$(-1)^k\sum_{j=0}^{k}\binom{k+1}{j}(l+j+1)B_{l+j}+(-1)^l\sum_{j=0}^{l}\binom{l+1}{j}(k+j+1)B_{k+j}=0.$$

Recently Wu et al. [7] proved further that for $k,l\in\mathbb{N}$ we have

$$\begin{aligned}&(-1)^k\sum_{j=0}^{k}\binom{k+1}{j}(l+j+1)B_{l+j}(t)+(-1)^l\sum_{j=0}^{l}\binom{l+1}{j}(k+j+1)B_{k+j}(-t)\\&\quad=(-1)^k(k+l+2)(k+l+1)t^{k+l}.\end{aligned}\tag{1.3}$$

Motivated by the above work, we obtain the following general theorem.

Theorem 1.1. *Let $k,l\in\mathbb{N}$ and $x+y+z=1$. Then*

$$\begin{aligned}&(-1)^k\sum_{j=0}^{k}\binom{k}{j}x^{k-j}\frac{A_{l+j+1}(y)}{l+j+1}+(-1)^l\sum_{j=0}^{l}\binom{l}{j}x^{l-j}\frac{A^*_{k+j+1}(z)}{k+j+1}\\&\quad=\frac{a_0(-x)^{k+l+1}}{(k+l+1)\binom{k+l}{k}}.\end{aligned}\tag{1.4}$$

Also,

$$(-1)^k\sum_{j=0}^{k}\binom{k}{j}x^{k-j}A_{l+j}(y)=(-1)^l\sum_{j=0}^{l}\binom{l}{j}x^{l-j}A^*_{k+j}(z)\tag{1.5}$$

and

$$\begin{aligned}&(-1)^k\sum_{j=0}^{k}\binom{k+1}{j}x^{k-j+1}(l+j+1)A_{l+j}(y)\\&\qquad+(-1)^l\sum_{j=0}^{l}\binom{l+1}{j}x^{l-j+1}(k+j+1)A^*_{k+j}(z)\\&\quad=(k+l+2)((-1)^{k+1}A_{k+l+1}(y)+(-1)^{l+1}A^*_{k+l+1}(z)).\end{aligned}\tag{1.6}$$

Remark 1.1. (1.5) in the case $l=0$ yields that

$$\begin{aligned}\sum_{j=0}^{k}\binom{k}{j}x^{k-j}A_j(y)&=(-1)^kA^*_k(1-x-y)=\sum_{j=0}^{k}\binom{k}{j}0^{k-j}A_j(x+y)\\&=A_k(x+y).\end{aligned}$$