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Addition formulas for polynomials built on classical combinatorial sequences

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Abstract

We prove addition formulas for some polynomials built on combinatorial sequences (Catalan numbers, Bell numbers, ...) or related to classical polynomials (Hermite, ...). These relations give also an explicit computation of several Hankel determinants. © 2000 Elsevier Science B.V. All rights reserved.

1. The Hankel matrix, of any order, built on Catalan numbers has determinant 1

It is well known that the *Catalan number* $c_n = \binom{2n}{n}/(n+1)$ counts the number of sequences of n integers 1 and n integers -1, with partial sums always positive.

Let us now introduce $c_{n,k} = [(2k+1)\binom{2n}{n+k}]/(n+k+1)$.

Here is the array of the first $c_{n,k}$

1									
1	1								
2	3	1							
5	9	5	1						
14	28	20	7	1					
42	90	75	35	9	1				
132	297	275	154	54	11	1			
429	1001	1001	637	273	77	13	1		
1430	3432	3640	2548	1260	440	104	15	1	
4862	11934	13260	9996	5508	2244	663	135	17	1.

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Because of the trivial relation $c_{n,k}=\binom{2n}{n+k}-\binom{2n}{n+k+1}$, we see that $c_{n,k}$ counts the number of sequences of $(n+k)$ integers 1 and $(n-k)$ integers -1, with partial sums always positive.

Now the coefficient of x^{i+j+1} in $(1-x)^2(1+x)^{2i+2j}$ is $-2c_{i+j}$.

On the other hand, the coefficient of x^a in $(1-x)(1+x)^{2i}$ is $\binom{2i}{a}(2i-2a+1)/(2i-a+1)$.

Identifying the term x^{i+j+1} in the trivial identity $(1-x)^2(1+x)^{2i+2j}=[(1-x)(1+x)^{2i}][(1-x)(1+x)^{2j}]$ and using the antisymmetry of the coefficients of these polynomials, we get easily [1]

$$c_{i+j}=\sum_{k=0}^{\min(i,j)} c_{i,k}c_{j,k}.$$

Another way of writing this formula is the following.

If A_n is the triangular matrix $(c_{i,k})_{i,k=0,\dots,n}$, if B_n is A_n^t, and if H_n is the Hankel matrix $(c_{i+j})_{i,j=0,\dots,n}$ then $A_nB_n=H_n$.

This relation implies $\det(H_n)=\det(A_n)\det(B_n)=1\cdot 1=1$.

2. The inverse of that matrix

Let $\alpha_{n,i,j}=(H_n^{-1})_{i,j}$.

On the one hand, we have $(1-x)(1+x)^{2n}=\sum_{k=0}^{n} c_{n,k}(x^{n-k}-x^{n+k+1})$.

Dividing by $x^{n+1/2}$ and replacing next x by $e^{2i\alpha}$, we obtain

$$\sum_{k=0}^{n} c_{n,k}\sin(2k+1)\alpha = 2^{2n}\sin\alpha\cos^{2n}\alpha.$$

On the other hand, by induction

$$\sin(2n+1)\alpha=\sum_{k=0}^{n}(-1)^{n+k}\binom{n+k}{n-k}2^{2k}\sin\alpha\cos^{2k}\alpha.$$

So, with the above notations, $(A_n^{-1})_{i,j}=(-1)^{i+j}\binom{i+j}{i-j}$ and, therefore,

$$\alpha_{n,i,j}=(-1)^{i+j}\sum_{k=\max(i,j)}^{n}\binom{k+i}{k-i}\binom{k+j}{k-j}.$$

We have also

$$\sum_{k=0}^{j}(-1)^{k+i}\binom{k+i}{k-i}c_{j,k}=\delta_{i,j}.$$

3. Some consequences

$$\sum_{k=0}^{n} c_{n,k}=\binom{2n}{n},$$

$$\sum_{k=0}^{n}(-1)^k c_{n,k}=0,$$

$$\sum_{k=0}^{n} c_{n,k}^2 = c_{2n},$$

$$\sum_{k=0}^{n} (2k+1)c_{n,k} = 2^{2n},$$

$$\sum_{k=0}^{n} (-1)^k(2k+1)c_{n,k} = -2c_{n-1},$$

$$\sum_{k=0}^{n} c_{n,k}c_{n,n-k} = \frac{2(2n^2+2n+1)(4n)!}{n!(3n+2)!}.$$

(Hint: use the Leibnitz rule on the Taylor coefficient $c_{n,k} = 1/(n-k)!((\mathrm{d}^{n-k}/\mathrm{d}x^{n-k})((1-x)(1+x)^{2n}))_{x=0}$.)

$$\forall k, \quad \sum_{n=k}^{\infty} \frac{c_{n,k}}{2^{2n}} = 2.$$

(Hint: first $c_{n,k}$ is also a Fourier coefficient: $c_{n,k} = (2^{2n+1})/\pi \int_0^{\pi} \cos^{2n}\alpha \sin(2k+1)\alpha \sin\alpha \, \mathrm{d}\alpha$, and next $\frac{\pi}{2}\sum_{n=0}^{\infty} c_{n,k}/2^{2n} = \int_0^{\pi} \left(\sum_{n=0}^{\infty} \cos^{2n}\alpha\right) \sin(2k+1)\alpha \sin\alpha \, \mathrm{d}\alpha = \int_0^{\pi} \frac{\sin(2k+1)\alpha}{\sin\alpha} \, \mathrm{d}\alpha = \pi$.)

4. Catalan polynomials

We define now the mth *Catalan polynomial* by

$$P_m(x) = \sum_{k=0}^{m} c_{m,k}x^k.$$

We also define

$$H_n(x) = \begin{vmatrix} P_0(x) & P_1(x) & P_2(x) & \cdots & P_n(x) \\ P_1(x) & P_2(x) & P_3(x) & \cdots & P_{n+1}(x) \\ P_2(x) & P_3(x) & P_4(x) & \cdots & P_{n+2}(x) \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ P_n(x) & P_{n+1}(x) & P_{n+2}(x) & \cdots & P_{2n}(x) \end{vmatrix}.$$

Note that $(1+x)$ divides every element of this determinant, except $P_0(x)$. Its value is therefore, as simple as possible

$$H_n(x) = (1+x)^n.$$

The proof is quite easy.

Let us put $a_{n,r}(x) = \sum_{k=r}^{n} c_{n,k}x^{k-r}$ (in particular, $a_{n,0}(x) = P_n(x)$ and $a_{n,n}(x) = 1$).

This polynomial is the coefficient of u^{n-r} in $(1-u)(1+u)^{2n}/(1-ux)$, i.e. in $\sum_{k=0}^{n} c_{n,k}(u^{n-k} - u^{n+k+1})\sum_{t=0}^{\infty} u^t x^t$.

So, we have the addition formula

$$a_{m+n,0}(x) = a_{m,0}(x)a_{n,0}(x) + (x+1)\sum_{k=1}^{\min(m,n)} a_{m,k}(x)a_{n,k}(x).$$

In other words, with the corresponding above notations, the matrix $\mathscr{H}$, i.e.

$$\begin{pmatrix} P_0(x) & P_1(x) & P_2(x) & P_3(x) & \cdots & P_n(x) \\ P_1(x) & P_2(x) & P_3(x) & P_4(x) & \cdots & P_{n+1}(x) \\ P_2(x) & P_3(x) & P_4(x) & P_5(x) & \cdots & P_{n+2}(x) \\ P_3(x) & P_4(x) & P_5(x) & P_6(x) & \cdots & P_{n+3}(x) \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ P_n(x) & P_{n+1}(x) & P_{n+2}(x) & P_{n+3}(x) & \cdots & P_{2n}(x) \end{pmatrix},$$

is the product of $\mathscr{A}$, i.e.

$$\begin{pmatrix} a_{0,0}(x) & 0 & 0 & 0 & \cdots & 0 \\ a_{1,0}(x) & (x+1)a_{1,1}(x) & 0 & 0 & \cdots & 0 \\ a_{2,0}(x) & (x+1)a_{2,1}(x) & (x+1)a_{2,2}(x) & 0 & \cdots & 0 \\ a_{3,0}(x) & (x+1)a_{3,1}(x) & (x+1)a_{3,2}(x) & (x+1)a_{3,3}(x) & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{n,0}(x) & (x+1)a_{n,1}(x) & (x+1)a_{n,1}(x) & (x+1)a_{n,3}(x) & \cdots & (x+1)a_{n,n}(x) \end{pmatrix},$$

by $\mathscr{B}$, i.e. by

$$\begin{pmatrix} a_{0,0}(x) & a_{1,0}(x) & a_{2,0}(x) & a_{3,0}(x) & \cdots & a_{n,0}(x) \\ 0 & a_{1,1}(x) & a_{2,1}(x) & a_{3,1}(x) & \cdots & a_{n,1}(x) \\ 0 & 0 & a_{2,2}(x) & a_{3,2}(x) & \cdots & a_{n,2}(x) \\ 0 & 0 & 0 & a_{3,3}(x) & \cdots & a_{n,3}(x) \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & \cdots & a_{n,n}(x) \end{pmatrix}.$$

The special case $x=0$ is the numerical theorem above, since $a_{m,0}(0)=P_m(0)=c_m$. $P_m(1)=\binom{2m}{m}$ gives also another amusing little theorem.

The case $x=-1$ is quite uninteresting: $\forall m\geqslant 1$, $P_m(-1)=0$.

The inversion formula becomes of course

$$\sum_{k=0}^{j}(-1)^{k+i}\left(\binom{k+i-1}{k-i-1}x+\binom{k+i}{k-i}\right)a_{j,k}(x)=\delta_{i,j}.$$

5. A related problem

Consider now the polynomial $Q_n(x)=\sum_{k=0}^{n}\binom{n}{k}^2x^k$.

We have

$$\begin{vmatrix} Q_0(x) & Q_1(x) & Q_2(x) & \cdots & Q_n(x) \\ Q_1(x) & Q_2(x) & Q_3(x) & \cdots & Q_{n+1}(x) \\ Q_2(x) & Q_3(x) & Q_4(x) & \cdots & Q_{n+2}(x) \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ Q_n(x) & Q_{n+1}(x) & Q_{n+2}(x) & \cdots & Q_{2n}(x) \end{vmatrix} = 2^n x^{n(n+1)/2}.$$

I have already given [2] an analytic proof of that identity, but the following one is almost obvious.

Indeed, let us define by induction the $r_{m,n}(x)$ by

$$r_{m,0}(x) = Q_m(x)$$

and

$$\forall n \geqslant 1, \quad r_{m,n}(x) = r_{m-1,n-1}(x) + (x+1)r_{m-1,n}(x) + xr_{m-1,n+1}(x)$$

(with $r_{m,n}(x) = 0$ if $n > m$).

Here are the first $r_{m,n}(x)$

1				
$x+1$	1			
x^2+4x+1	$2x+2$	1		
x^3+9x^2+9x+1	$3x^2+9x+3$	$3x+3$	1	
$x^4+16x^3+36x^2+16x+1$	$4x^3+24x^2+24x+4$	$6x^2+16x+6$	$4x+4$	$1.$

By induction, we get an addition formula analogous to the previous one for Catalan polynomials

$$r_{m+n,0}(x) = r_{m,0}(x)r_{n,0}(x) + 2\sum_{k=1}^{\min(m,n)} x^k r_{m,k}(x)r_{n,k}(x).$$

Writing it as a product of matrices proves the theorem at once.

Remark. The case $x = 1$ gives again the theorem concerning the Hankel determinant built on the central binomial coefficients, since $Q_m(1) = \binom{2m}{m}$.

6. Bell polynomials

Let us call $S_{m,k}$ the Stirling number of the second kind, counting the equivalences with k classes over a set of n elements.

The Bell polynomial $B_m(x)$ is defined by

$$B_0(x) = 1,$$

$$\forall m \geqslant 1, \quad B_m(x) = \sum_{k=1}^{m} S_{m,k}x^k$$

($B_m(1)$ is the mth Bell number, number of all equivalences over a set of n elements).

The classical relation $S_{m+1,k} = k\,S_{m,k} + S_{m,k-1}$ gives $B_{m+1}(x) = x(B_m(x) + B'_m(x))$ and leads, by induction, to the addition formula

$$B_{m+n}(x) = \sum_{k=0}^{\min(m,n)} \frac{x^k}{k!}\left(\frac{\mathrm{d}^k}{\mathrm{d}x^k}B_m(x)\right)\left(\frac{\mathrm{d}^k}{\mathrm{d}x^k}B_n(x)\right).$$

Of course, $(\mathrm{d}^m/\mathrm{d}x^m)B_m(x) = m!$

So, again with the same technique, we find (see also [3])

$$\det(B_{i+j}(x))_{i,j=0,\ldots,n} = \left(\prod_{k=0}^{n} k!\right) x^{n(n+1)/2}.$$

7. Hermite polynomials

The generating series for these polynomials is

$$e^{(x^2)}e^{-(t-x)^2}=\sum_{n=0}^{\infty}\frac{H_n(x)}{n!}t^n.$$

Let us introduce the polynomials $s_{m,n}(x)$ by

$$s_{0,0}(x)=1,$$
$$\forall n,\quad s_{m,n}(x)=s_{m-1,n-1}(x)+2xs_{m-1,n}(x)-2(n+1)s_{m-1,n+1}(x)$$

(with, of course $s_{m,n}(x)=0$ if $n\notin\{0,\dots,m\}$).

Here are of the first $s_{m,n}(x)$:

1					
$2x$	1				
$4x^2-2$	$4x$	1			
$8x^3-12x$	$12x^2-6$	$6x$	1		
$16x^4-48x^2+12$	$32x^3-48x$	$24x^2-12$	$8x$	1	
$32x^5-160x^3+120x$	$80x^4-240x^2+60$	$80x^3-120x$	$40x^2-20$	$10x$	1

By induction, we get

$$s_{m,0}(x)=H_m(x)$$

and the addition formula

$$s_{m+n,0}(x)=\sum_{k=0}^{\min(m,n)}(-2)^k k!s_{m,k}(x)s_{n,k}(x).$$

So, the usual technique gives (see also [4]), for the corresponding Hankel determinant, the constant value

$$\det(H_{i+j}(x))_{i,j=0,\dots,n}=\left(\prod_{k=0}^{n}k!\right)(-2)^{n(n+1)/2}.$$

8. Derangement polynomials

Our last example is the derangement polynomial

$$d_n(x)=\sum_{k=0}^{n}(-1)^k\frac{n!}{k!}x^{n-k}.$$

Note that, indeed, $d_n(1)$ is the number of derangements, i.e. of permutations without any fixed point, over a set of n elements.

Here, we introduce the polynomials $t_{n,k}(x)=(1/k!^2)(\mathrm{d}^k/\mathrm{d}x^k)d_n(x)$.

Of course, $t_{n,0}(x)=d_n(x)$ and $t_{n,n}(x)=1$.

The induction gives

$$t_{m+n,0}(x) = \sum_{k=0}^{\min(m,n)} k!^2 x^{2k} t_{m,k}(x) t_{n,k}(x)$$

and, always with the same technique (see also [5]),

$$\det(d_{i+j}(x))_{i,j=0,\dots,n} = \left(\prod_{k=0}^{n} k!^2\right) x^{n(n+1)}.$$

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