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Discrete Mathematics 239 (2001) 33–51

DISCRETE MATHEMATICS

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Recursively defined combinatorial functions: extending Galton's board

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Received 9 April 1998; revised 23 March 2000; accepted 30 May 2000

Abstract

Many functions in combinatorics follow simple recursive relations of the type $F(n,k) = a_{n-1,k}F(n-1,k)+b_{n-1,k-1}F(n-1,k-1)$. Treating such functions as (infinite) triangular matrices and calling $a_{n,k}$ and $b_{n,k}$ generators of F, our paper will study the following question: Given two triangular arrays and their generators, how can we give explicit formulas for the generators of the product matrix? Our results can be applied to factor infinite matrices with specific types of generators (e.g. $a_{n,k} = a'_n + a''_k$) into matrices with 'simpler' types of generators. These factorization results then can be used to give construction methods for inverse matrices (yielding conditions for self-inverse matrices), and results for convolutions of recursively defined functions. Slightly extending the basic techniques, we will even be able to deal with certain cases of nontriangular infinite matrices. As a side-effect, many seemingly separate results about recursive combinatorial functions will be shown to be special cases of the general framework developed here.

Keywords: Combinatorial identities; Combinatorial functions; Triangular matrices; Inverse relations; Stirling numbers; Binomial coefficients

1. Introduction

In this paper we will study real-valued functions $F(n,k)$ of two nonnegative integer arguments defined by the following recursive relations:

$$F(n,k) = a_{n-1,k}F(n-1,k) + b_{n-1,k-1}F(n-1,k-1) \quad \text{for } n\geqslant 1,\ k\geqslant 1. \tag{1}$$

We will denote these relations by

$$F = \mathscr{G}(a_{n,k}, b_{n,k}).$$

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PII: S0012-365X(00)00373-3

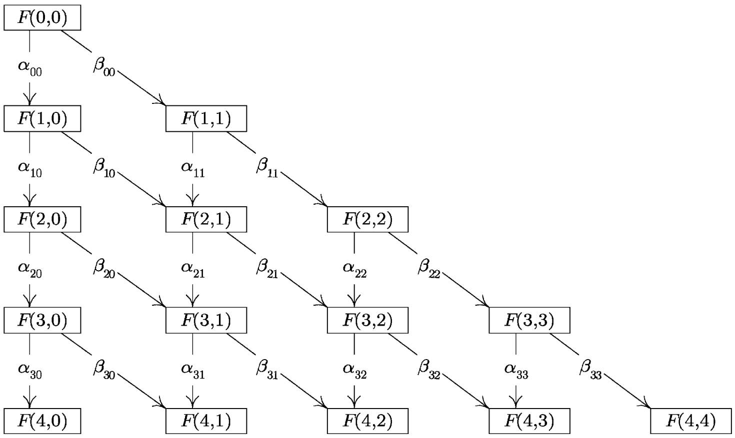

Fig. 1. Triangular Galton array.

We will always have $F(0,0)=1$, and we will consider different 'boundary conditions' for $F(n,0)$ with $n\geqslant 1$ and $F(0,k)$ with $k\geqslant 1$, the simplest case being

$$\begin{aligned} &F(0,k)=0 \quad \text{for } k\geqslant 1,\\ &F(n,0)=a_{n-1,0}F(n-1,0) \quad \text{for } n\geqslant 1. \end{aligned} \tag{2}$$

In this case, we trivially have $F(n,k)=0$ for $k\geqslant n$ and the values of $a_{n,k}$ and $b_{n,k}$ for $k\geqslant n$ do not have any influence on $F(n,k)$.

Relations of this kind can be visualized as in Fig. 1.

The illustration in Fig. 1 brings to mind Galton's board (a physical device for producing the binomial distribution). In this context, we shall refer to the pair $(a_{n,k},b_{n,k})$ as the *Galton scheme* generating the *Galton array* $F(n,k)$. Galton arrays of the triangular type just described will be called *pure Galton arrays*.

The best-known example of (pure) Galton arrays are, of course, the binomial coefficients $F(n,k)=\binom{n}{k}$, satisfying the well-known recursion

$$F(n,k)=F(n-1,k)+F(n-1,k-1).$$

This recursion is also satisfied by the binomials with noninteger upper index, $F_\alpha(n,k)=\binom{\alpha+n}{k}$. This motivates the study of Galton arrays with $F(0,k)=y_k\neq 0$ for some $k\geqslant 1$, yielding nontriangular arrays.

Many results in combinatorics deal with properties of Galton arrays. Here are some examples of arrays which can be written as Galton arrays (we also show the notation established in [3]):

$B=\mathscr{G}(1,1)$	$B(n,k)=\binom{n}{k}$	binomial coefficients,
$s=\mathscr{G}(-n,1)$		Stirling numbers (first kind) sign corrected,
$s'=\mathscr{G}(n,1)$	$s'(n,k)=\left[{n \atop k}\right]$	Stirling numbers (first kind),

$S = \mathscr{G}(k,1) \quad S(n,k) = \left\{ {n \atop k} \right\}$ Stirling numbers (second kind),

$L = \mathscr{G}(-n-k,-1)$ Lah numbers,

$E = \mathscr{G}(1+k,n-k) \quad E(n,k) = \left\langle {n \atop k} \right\rangle$ Euler numbers,

$E_2 = \mathscr{G}(1+k,2n-k) \quad E_2(n,k) = \left\langle\!\left\langle {n \atop k} \right\rangle\!\right\rangle$ second-order Euler numbers,

$G_q = \mathscr{G}(q^k,1)$ Gaussian coefficients,

$S_q = \mathscr{G}\left(\sum_{l=0}^{k} q^l, 1\right)$ q-Stirling numbers.

It is worth noting that $a_{n,k}$ and $b_{n,k}$ are linear functions of n and k in all these cases except for the Gaussian coefficients and the q-Stirling numbers. In all cases except the Euler numbers of both kinds and the Lah numbers the $a_{n,k}$ and $b_{n,k}$ depend either on n or on k, but not on both variables. In these cases, we even have $b_{n,k} = 1$ for all n and k.

Our main result (Theorem 10) deals with matrix products of Galton arrays, i.e. expressions of the form

$$H(n,k) = \sum_{i=0}^{n} F(n,i)G(i,k). \qquad (3)$$

We will write $H = F * G$ to denote this set of equations. Theorem 10 states that $\mathscr{G}(a_n,b_n) * \mathscr{G}(c_k,d_k) = \mathscr{G}(a_n + b_n c_k, b_n d_k)$ for two Galton arrays with (respectively) row dependent and column-dependent generators (a_{nk} is called row dependent if the value of a_{nk} depends on n, but is independent of k, and column dependent if the value of a_{nk} depends on k, but is independent of n).

Results for matrix products of Galton arrays can be derived not only for cases in which both F and G are triangular, but also for the case in which F is triangular and G is nontriangular, e.g. the generalized binomials.

A very important special case is $H(n,k) = \delta_{n,k}$ ($\delta_{n,k}$, as usual, is the Kronecker symbol) because then H is the identity matrix. When two matrices F and G give matrix product H, they are inverse matrices. Results about sums of recursively defined arrays and about inverse relations have been studied extensively in [1,2,5–10,12]. Some of the results in these papers are more general (but under somewhat different conditions about recursion depth) than the inversion results we will derive in this paper.

Matrix multiplication of recursively defined arrays is studied in [4,13], and (somewhat hidden) in [11]. Our approach in a certain sense unifies many results obtained separately in these books and papers. Furthermore, many of the results proved separately in [3, Chapters 5 and 6] can be seen as special cases of Theorem 10.

We will also be able to derive results for expressions of the form

$$H(n,k) = \sum_{i} F(n,i+l)G(i,k)$$

and

$$H(n,k)=\sum_i F(n,i)G(i+l,k).$$

Technically speaking, this can be achieved by defining

$$F'(n,k)=F(n+l,k+l)$$

and

$$G'(n,k)=G(n+l,k).$$

Even if F and G are pure Galton arrays, F' and G' are not since $F(n,0)$ and $G'(0,k)$ do not satisfy Eqs. (2). To be able to cover results of this kind by Eq. (3), we will need to extend our definition of Galton arrays by allowing the modified conditions

$$F(n,0)=a_{n-1,0}F(n-1,0)+x_n$$

and

$$F(0,k)=y_k$$

So the most general Galton array we will study has the visual representation given in Fig. 2.

It is clear that the coefficients $a_{n,k}$ and $b_{n,k}$ together with x_n and y_k completely determine the function $F(n,k)$. The converse, however, is not true, as we can see by the following argument:

It is easily checked that in addition to the defining recursion the 'standard' binomials for $n\geqslant 1$ and $1\leqslant k\leqslant n$ also satisfy the recursion

$$F(n,k)=\left(1+\frac{\beta k}{\alpha+\beta(n-1)}\right)F(n-1,k)$$
$$+\left(1-\frac{\beta(n-k)}{\alpha+\beta(n-1)}\right)F(n-1,k-1),$$

if $\alpha+\beta n\neq 0$ for all $n\geqslant 0$.

Therefore, F does not uniquely determine its generators $a_{n,k}$ and $b_{n,k}$. We will, however, derive results showing that generators satisfying some additional conditions are unique.

2. Basic results about Galton arrays

Definition 1. F is called a *Galton array* with *row inflow* x_n and *column inflow* y_k and *generators* $(a_{n,k},b_{n,k})$ if $F(n,k)$ is defined for all $n\geqslant 0$ and $k\geqslant 0$ and

$$\begin{aligned}
&F(0,0)=1 && \\
&F(0,k)=y_k && \text{for } k\geqslant 1,\\
&F(n,0)=a_{n-1,0}F(n-1,0)+x_n && \text{for } n\geqslant 1,\\
&F(n,k)=b_{n-1,k-1}F(n-1,k-1)+a_{n-1,k}F(n-1,k) && \text{for } n\geqslant 1 \text{ and } k\geqslant 1.
\end{aligned}$$

This will be expressed by $F=\mathcal{G}(x_n|a_{n,k},b_{n,k}|y_k)$.

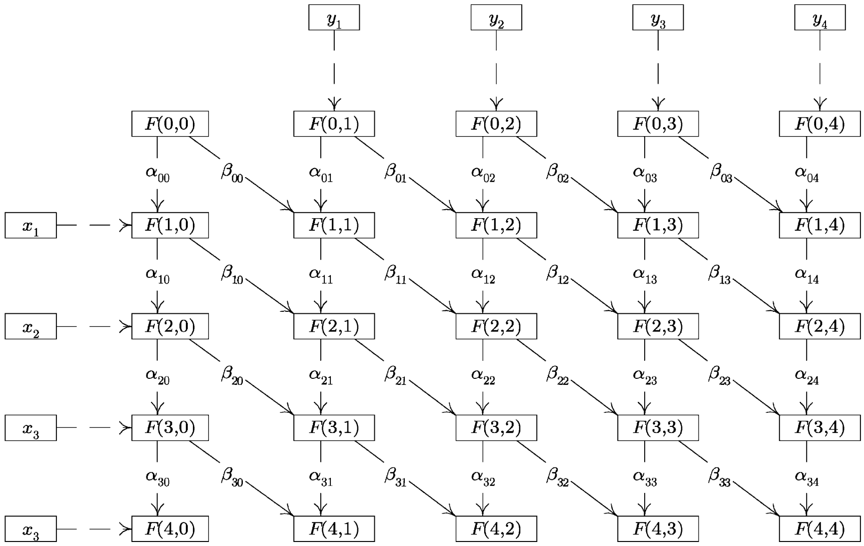

Fig. 2. General Galton array with inflows.

We will deal with the following special cases:

$x_n = 0,\ y_k = 0$	$F = \mathcal{G}(a_{n,k}, b_{n,k})$	pure Galton array,
$y_k = 0$	$F = \mathcal{G}(x_n \vert a_{n,k}, b_{n,k})$	Galton array with row inflow,
$x_n = 0$	$F = \mathcal{G}(a_{n,k}, b_{n,k} \vert y_k)$	Galton array with column inflow.

The condition $F(0,0) = 1$ is not really a restriction because multiplying a Galton array by a constant c retains the recursion equations for $F(n,k)$ with $n \geqslant 1$ and $k \geqslant 1$, and the x_n and y_k multiplied by c retain the 'boundary' equations.

Remark 2. It is clear that for a Galton array $F = \mathcal{G}(x_n|a_{n,k}, b_{n,k}|y_k)$ the array F' defined by

$$F'(n,k) = \frac{F(n+N, k+K)}{F(N,K)}$$

is also a Galton array with

$$F' = \mathcal{G}\left(b_{n+N-1,K-1}\frac{F(n+N-1,K-1)}{F(N,K)}|a_{n+N,k+K}, b_{n+N,k+K}|\frac{F(N,k+K)}{F(N,K)}\right).$$

As we have seen, in many cases we have $b_{n,n} = 1$ for all n, and in these cases division by $F(N,K)$ is unnecessary to retain the Galton array property $F'(0,0) = 1$.

In the special case of a pure Galton array $F = \mathcal{G}(a_{n,k}, b_{n,k})$ with $a_{00} = 0$ and $b_{00} = 1$, the 'cut off' Galton array F' defined by $F'(n,k) = F(n+1, k+1)$ is also a pure Galton array with $F' = \mathcal{G}(a_{n+1,k+1}, b_{n+1,k+1})$ (see Fig. 2).

So 'cut off' Galton arrays still are Galton arrays, but except under special circumstances the property of being a pure Galton array gets usually lost for the 'cut off' array. Riordan arrays, as studied in [13], retain their properties under cutting, so to give our results the same degree of generality, we have to allow for inflows in our Galton arrays.

It is also clear that the binomials are a very special case of a Galton array. The generators ($a_{n,k}=1, b_{n,k}=1$) are shift invariant (i.e. $a_{n+i,k+j}=a_{n,k}$ for arbitrary 'shifts' by i and j, and similarly for $a_{n,k}$), and the x_n and y_n needed after cutting are just the values of the array from the row and the column preceding the cut out part.

Let us now study the question of uniqueness of a pair of generators $a_{n,k}$ and $b_{n,k}$.

Theorem 3. *Let $F=\mathscr{G}(a_{n,k},b_{n,k})$ and $F'=\mathscr{G}(a'_{n,k},b'_{n,k})$. Let furthermore $F(n,k)\neq 0$ for all $0\leqslant k\leqslant n$. If additionally for all $k\leqslant n$ one of the following four conditions is satisfied*:

(i) $a_{n,k}=a_{n,0}$ *and* $a'_{n,k}=a'_{n,0}$,
(ii) $a_{n,k}=a_{n-k,0}$ *and* $a'_{n,k}=a'_{n-k,0}$,
(iii) $b_{n,k}=b_{n,n}$ *and* $b'_{n,k}=b'_{n,n}$,
(iv) $b_{n,k}=b_{k,k}$ *and* $b'_{n,k}=b'_{k,k}$,

then $a_{n,k}=a'_{n,k}$ and $b_{n,k}=b'_{n,k}$ for all $k\leqslant n$.

Proof. We note that for $k<n$ we have

$$F(n+1,k+1)=a_{n,k+1}F(n,k+1)+b_{n,k}F(n,k),$$

therefore we have

$$b_{n,k}=\frac{F(n+1,k+1)-a_{n,k+1}F(n,k+1)}{F(n,k)}.$$

For the case $a_{n,k}=a_{n,0}$ for all $k\leqslant n$ we have

$$b_{n,k}=\frac{F(n+1,k+1)-a_{n,k+1}F(n,k+1)}{F(n,k)}$$
$$=\frac{F(n+1,k+1)-a_{n,0}F(n,k+1)}{F(n,k)}.$$

For the case $a_{n,k}=a_{n-k,0}$ for all $k\leqslant n$ we have

$$b_{n,k}=\frac{F(n+1,k+1)-a_{n,k+1}F(n,k+1)}{F(n,k)}$$
$$=\frac{F(n+1,k+1)-a_{n-k,0}F(n,k+1)}{F(n,k)}.$$

In both cases, the $a_{n,0}$ have to satisfy $a_{n,0}=F(n+1,0)/F(n,0)$ because of (2) and therefore both the $a_{n,k}$ and the $b_{n,k}$ are completely determined. For $n=k$, of course, we use $F(n,k+1)=0$ in these equations.

For the remaining cases we note that for $k \geqslant 1$ we have

$$F(n+1,k) = a_{n,k}F(n,k) + b_{n,k-1}F(n,k-1)$$

and therefore

$$a_{n,k} = \frac{F(n+1,k) - b_{n,k-1}F(n,k-1)}{F(n,k)}.$$

For $b_{n,k} = b_{n,n}$ we have $a_{n,0} = F(n+1,0)/F(n,0)$ and for all $1 \leqslant k \leqslant n$ we have

$$a_{n,k} = \frac{F(n+1,k) - b_{n,k-1}F(n,k-1)}{F(n,k)} = \frac{F(n+1,k) - b_{n,n}F(n,k-1)}{F(n,k)}.$$

For $b_{n,k} = b_{k,k}$ we have $a_{n,0} = F(n+1,0)/F(n,0)$ and for all $1 \leqslant k \leqslant n$ we have

$$a_{n,k} = \frac{F(n+1,k) - b_{n,k-1}F(n,k-1)}{F(n,k)}$$
$$= \frac{F(n+1,k) - b_{k-1,k-1}F(n,k-1)}{F(n,k)}. \qquad \square$$

We note that $a_{n,k} > 0$ and $b_{n,k} > 0$ for all $0 \leqslant k \leqslant n$ imply $F(n,k) > 0$ for all $0 \leqslant k \leqslant n$.

From this theorem we see that for a pure Galton array F there is only one possible Galton scheme with the property that the vertical generators $a_{n,k}$ are constant in every row generating F. So, if we have $F = \mathscr{G}(a_n, b_{n,k})$ and $F' = \mathscr{G}(a'_n, b'_{n,k})$ and we know either $a_n \neq a'_n$ or $b_{n,k} \neq b'_{n,k}$, and furthermore $F(n,k) \neq 0$ and $F'(n,k) \neq 0$ for all $0 \leqslant k \leqslant n$, then F and F' are different. Similarly, different sets of generators with the other properties stated in Theorem 3 produce different Galton arrays.

As a consequence, any representation $B = \mathscr{G}(a_{n,k}, b_{n,k})$ of the binomial coefficients with the additional constraint that either $a_{n,k}$ or $b_{n,k}$ depends either on n or on k only, necessarily satisfies $a_{n,k} = 1$ and $b_{n,k} = 1$.

More generally, Galton arrays with $b_{n,k} = 1$ and $F(n,k) \neq 0$ for $0 \leqslant k \leqslant n$ uniquely determine the corresponding generator $a_{n,k}$.

We already noted that $F(n,k) = \delta_{n,k}$ is a very important special case. More generally, diagonal matrices will play an important role in some later results. Therefore we will now derive a result for diagonal and 'band' matrices, and also for 'vertical strip' matrices.

Theorem 4. *Let $F = \mathscr{G}(a_{n,k}, b_{n,k})$. If for $K \geqslant 0$ we have $b_{n,K} = 0$ for all $n \geqslant K$ then $F(n,k) = 0$ for all $n \geqslant 0$ and $k > K$. If for $N \geqslant 0$ we have $a_{N+k,k} = 0$ for all $k \geqslant 0$, then $F(n,k) = 0$ for $n > N + k$. In particular, if $a_{k,k} = 0$ for all $k \geqslant 0$, then $F(n,k) = 0$ for all $0 \leqslant k < n$. If furthermore $b_{k,k} = 1$, then $F(n,k) = \delta_{n,k}$ and F is the (infinite) identity matrix.*

An end of proof symbol directly following a theorem like the previous theorem indicates that the proof is essentially self evident and therefore omitted.

As a consequence of this theorem, $\mathscr{G}(a_n - a_k, 1)$ is the identity matrix for any sequence a_n.

We can reformulate this result in the following way: If all diagonal generators starting in one column are 0, then all values of F to the right of this column are 0. If on the other hand all the vertical generators starting in one lower diagonal are 0, then all values of F below that diagonal are 0. Furthermore, if all the vertical generators starting in the main diagonal are 0, then F is an infinite diagonal matrix. This result is in contrast to the uniqueness result in Theorem 3, but there is no contradiction. Theorem 3 relies on the fact that $F(n,k) \neq 0$ for all $0 \leqslant k \leqslant n$, and Theorem 4 is a consequence of $F(n,k) = 0$ for some n and k.

Here is another non-uniqueness result.

Theorem 5. *Let $F = \mathscr{G}(a_{n,k}, b_{n,k})$ and $F = \mathscr{G}(a'_{n,k}, b'_{n,k})$. Then for any real number λ*

$$F = \mathscr{G}(\lambda a_{n,k} + (1-\lambda)a'_{n,k}, \lambda a_{n,k} + (1-\lambda)b'_{n,k}).$$

3. Matrix multiplication results about Galton arrays

For the special case of Galton arrays with vertical generator $a_{n,k} = 0$ (i.e. $F = \mathscr{G}(0, b_{n,k})$), we have $F(n,n) = \prod_{i=0}^{n-1} b_{ii}$ and $F(n,k) = 0$ for $n \neq k$. So the values of F are completely independent of $b_{n,k}$ for $n \neq k$ and F is a diagonal array. By informally defining $b_n = b_{n,n}$ we can write $F = \mathscr{G}(0, b_n)$. Therefore, pure diagonal Galton arrays also can be represented as $F = \mathscr{G}(0, b_{n,k} = \xi_n)$ or equivalently as $F = \mathscr{G}(0, b_{n,k} = \xi_k)$ with $F(n,n) = \prod_{i=0}^{n-1} \xi_i$. In particular, $\mathscr{I} = \mathscr{G}(0, b_{n,k} = 1)$ is the (infinite) identity matrix.

Furthermore, we recall that multiplying any matrix F from the left with a diagonal matrix D, yielding $D * F$, is the same as multiplying each row of F with diagonal element of D taken from the same row. Similarly, multiplying any matrix F from the right with a diagonal matrix D, yielding $F * D$, is the same as multiplying each column of F with diagonal element of D taken from the same column.

Using these observations, we are ready to formulate the next theorem.

Theorem 6. *Let $F = \mathscr{G}(x_n | a_{n,k}, b_{n,k} | y_k)$ and ξ_n and η_k be arbitrary sequences. Then for $\mu_n = \prod_{i=0}^{n-1} \xi_i$ and $v_k = \prod_{i=0}^{k-1} \eta_i$ we have*

$$\mathscr{G}(0, \xi_n) * \mathscr{G}(x_n | a_{n,k}, b_{n,k} | y_k) * \mathscr{G}(0, \eta_k) = \mathscr{G}(\mu_n x_n | \xi_n a_{n,k}, \xi_n \eta_k b_{n,k} | v_k y_k).$$

Proof. We have

$$\begin{aligned}
\mu_n F(n,0) &= \mu_n(a_{n-1,k} F(n-1,0) + x_n) \\
&= \xi_{n-1} a_{n-1,0} \mu_{n-1} F(n-1,0) + \mu_n x_n, \\
\eta_k F(0,k) &= \eta_k y_k,
\end{aligned}$$

$$\begin{aligned}
\mu_n v_k F(n,k) &= \mu_n v_k (a_{n-1,k} F(n-1,k) + b_{n-1,k-1} F(n-1,k-1)) \\
&= \xi_{n-1} a_{n-1,k} \mu_{n-1} v_k F(n-1,k) \\
&\quad + \xi_{n-1} \eta_{k-1} b_{n-1,k-1} \mu_{n-1} v_{k-1} F(n-1,k-1). \qquad \square
\end{aligned}$$

Remark 7. This theorem allows us to multiply any Galton array with diagonal matrices from left and right. Since $\mathscr{G}(0,1)$ is the identity matrix, two special cases of the theorem can easily be stated by omitting either the left or the right diagonal matrix.

Remark 8. Some special cases of this theorem are helpful for 'rescaling' and 'correcting signs'. For

$$F = \mathscr{G}(x_n|a_{n,k}, b_{n,k}|y_k) \quad \text{and} \quad F'(n,k) = \frac{\xi^n}{\eta^k} F(n,k),$$

we have

$$F' = \mathscr{G}(0,\xi) * \mathscr{G}(x_n|a_{n,k}, b_{n,k}|y_k) * \mathscr{G}(0,\eta) = \mathscr{G}(\xi^n x_n|\xi a_{n,k}, \xi\eta b_{n,k}|\eta^k y_k).$$

For $\xi = \eta = -1$ this is

$$F'(n,k) = (-1)^{n+k} F(n,k)$$

and

$$F' = \mathscr{G}((-1)^n x_n| - a_{n,k}, b_{n,k}|(-1)^k y_k).$$

We have seen that some operations on F which can be viewed as matrix multiplication can be translated into transformations of the generators $a_{n,k}$ and $b_{n,k}$. Now we will extend this kind of operation to deal with nondiagonal matrix operations applied to Galton arrays. We will not, however, be able to deal with the most general case. Instead, we will multiply two matrices, where the generators for the left matrix will only depend on the row index and the generators for the right factor will only depend on the column index. Additionally, we will allow inflows.

In the proof we will use a matrix representation for the recursion equation, and we will use some special matrices that we will define now. (Some of the techniques we are developing here were used in [4,10] also uses similar matrix proof techniques.)

$\mathscr{S}$ defined by $\mathscr{S}(n,k) = \delta_{n+1,k}$ has ones in the diagonal above the main diagonal and zeros everywhere else, $\mathscr{T}$ defined by $\mathscr{T}(n,k) = \delta_{n,k+1}$ has ones in the diagonal below the main diagonal and zeros everywhere else, $\mathscr{I}$ defined by $\mathscr{I}(n,k) = \delta_{n,k}$ is the (infinite) identity matrix, and $\mathscr{O}$ defined by $\mathscr{O}_{n,k}(i,j) = \delta_{n,i}\delta_{k,j}$ has $\mathscr{O}_{n,k}(n,k) = 1$ and zero everywhere else. $\mathscr{O}$ will be shorthand for $\mathscr{O}_{0,0}$. Matrix multiplication with any of these matrices as one factor and an arbitrary infinite (possibly nontriangular) matrix is well defined since all sums only have finitely many terms.

The matrices satisfy the following equations:

$$\begin{aligned} &\mathscr{S} * \mathscr{T} = \mathscr{I}, \quad \mathscr{T} * \mathscr{S} = \mathscr{I} - \mathscr{O}, \\ &\mathscr{T} * \mathscr{O} = \mathscr{O}_{1,0}, \quad \mathscr{O} * \mathscr{T} = 0, \quad \mathscr{S} * \mathscr{O} = 0, \quad \mathscr{O} * \mathscr{S} = \mathscr{O}_{0,1}, \\ &\mathscr{O} * \mathscr{O} = \mathscr{O}. \end{aligned} \tag{4}$$

If $\mathscr{C}_1$ and $\mathscr{C}_2$ are any matrices with $\mathscr{C}_i(0,0) = 0$, $\mathscr{C}_i(n,k) = 0$ for $i = 1,2$ and $k \geqslant 1$ and $\mathscr{R}$ is a matrix with $\mathscr{R}(n,k) = 0$ for $n \geqslant 1$, then we also have

$$\mathscr{C}_i * \mathscr{T} = 0, \quad \mathscr{S} * \mathscr{R} = 0, \quad \mathscr{C}_i * \mathscr{O} = \mathscr{C}_i, \quad \mathscr{O} * \mathscr{R} = \mathscr{R}, \quad \mathscr{C}_1 * \mathscr{C}_2 = 0. \tag{5}$$

Furthermore, for a sequence a_n, $n\geqslant 0$, $\mathscr{D}(a_n)$ will denote the infinite diagonal matrix D with $D(n,k)=a_n\delta_{n,k}$. For x_n, $n\geqslant 1$, $\mathscr{C}(x_n)$ will denote matrix C with $C(n,k)=x_n\delta_{k,0}$ (the x_n's are put into the first column of C). For y_k, $k\geqslant 1$, $\mathscr{R}(y_n)$ will denote matrix R with $R(n,k)=y_k\delta_{n,0}$ (the y_k's are put into the first row of R).

Using these matrices, we can rewrite our recursive equations from Definition 1 for some special cases.

- $F=\mathscr{G}(x_n|a_n,b_n|y_k)$ with generators $a_{n,k}$ and $b_{n,k}$ depending only on the row index is equivalent to

$$F=\mathscr{O}+\mathscr{C}(x_n)+\mathscr{R}(y_k)+\mathscr{T}*\mathscr{D}(a_n)*F+\mathscr{T}*\mathscr{D}(b_n)*F*\mathscr{S}.$$

- $F=\mathscr{G}(x_n|a_k,b_k|y_k)$ with generators $a_{n,k}$ and $b_{n,k}$ depending only on the column index is equivalent to

$$F=\mathscr{O}+\mathscr{C}(x_n)+\mathscr{R}(y_k)+\mathscr{T}*F*\mathscr{D}(a_k)+\mathscr{T}*F*\mathscr{D}(b_k)*\mathscr{S}.$$

With these tools we can prove matrix multiplication results for Galton arrays $F*G$. It is clear that we cannot handle the case in which F has column inflow and G has row inflow, because then both arrays are nontriangular and therefore matrix multiplication is not well defined. If F does not have column inflow, then F is triangular and $F*G$ is well defined.

Therefore, as our most general case we will study the case in which the left array has only row inflow. We will see that for this general case we can derive recursion equations, but these equations will not describe a Galton array. We will need additional conditions to make sure that the matrix product also is a Galton array.

Lemma 9 (Matrix multiplication recursion lemma). *Let* $F=\mathscr{G}(x_n|a_n,b_n)$ *and* $G=\mathscr{G}(y_n|c_k,d_k|z_k)$. *Then* $H=F*G$ *satisfies the following equations for* $n\geqslant 1$ *and* $k\geqslant 1$:

$$H(0,k)=z_k,$$

$$H(n,0)=(a_{n-1}+b_{n-1}c_0)H(n-1,0)+x_n+b_{n-1}\sum_{i=1}^{n}F(n-1,i-1)y_i,$$

$$H(n,k)=(a_{n-1}+b_{n-1}c_k)H(n-1,k)+b_{n-1}d_{k-1}H(n-1,k-1)+x_nz_k.$$

Proof. Using the representation we introduced just before this lemma we have

$$F=\mathscr{O}+\mathscr{C}(x_n)+\mathscr{T}*\mathscr{D}(a_n)*F+\mathscr{T}*\mathscr{D}(b_n)*F*\mathscr{S},$$

$$G=\mathscr{O}+\mathscr{C}(y_n)+\mathscr{R}(z_k)+\mathscr{T}*G*\mathscr{D}(c_k)+\mathscr{T}*G*\mathscr{D}(d_k)*\mathscr{S},$$

and therefore, using the canceling equations (4) and (5), we have

$$\begin{aligned}F*G&=(\mathscr{O}+\mathscr{C}(x_n)+\mathscr{T}*\mathscr{D}(a_n)*F+\mathscr{T}*\mathscr{D}(b_n)*F*\mathscr{S})*G\\&=\mathscr{O}*G+\mathscr{C}(x_n)*G+\mathscr{T}*\mathscr{D}(a_n)*F*G+\mathscr{T}*\mathscr{D}(b_n)*F*\mathscr{S}*G\\&=\mathscr{T}*\mathscr{D}(a_n)*F*G\end{aligned}$$

$$
\begin{aligned}
&+\mathcal{O}*(\mathcal{O}+\mathcal{C}(y_n)+\mathcal{R}(z_k)+\mathcal{T}*G*\mathcal{D}(c_k)+\mathcal{T}*G*\mathcal{D}(d_k)*\mathcal{S})\\
&+\mathcal{C}(x_n)*(\mathcal{O}+\mathcal{C}(y_n)+\mathcal{R}(z_k)+\mathcal{T}*G*\mathcal{D}(c_k)+\mathcal{T}*G*\mathcal{D}(d_k)*\mathcal{S})\\
&+\mathcal{T}*\mathcal{D}(b_n)*F*\mathcal{S}*(\mathcal{O}+\mathcal{C}(y_n)+\mathcal{R}(z_k)\\
&+\mathcal{T}*G*\mathcal{D}(c_k)+\mathcal{T}*G*\mathcal{D}(d_k)*\mathcal{S})\\
=&\,\mathcal{T}*\mathcal{D}(a_n)*F*G+\mathcal{O}+\mathcal{R}(z_k)+\mathcal{C}(x_n)\\
&+\mathcal{T}*\mathcal{D}(b_n)*F*\mathcal{S}*\mathcal{C}(y_n)+\mathcal{T}*\mathcal{D}(b_n)*F*G*\mathcal{D}(c_k)\\
&+\mathcal{T}*\mathcal{D}(b_n)*F*G*\mathcal{D}(d_k)*\mathcal{S}. \qquad \square
\end{aligned}
$$

The term $x_n z_k$ in the last equation is responsible for the fact that the matrix product is not necessarily a Galton array under the most general circumstances. If, however, either $x_n = 0$ or $y_k = 0$, then the matrix product is a Galton array, and we have found a representation of its generators directly from the generators of the matrix factors.

Theorem 10 (Matrix multiplication of Galton arrays). *For Galton arrays with generators depending on either the row only or the column only we have*

$$\mathcal{G}(x_n|a_n,b_n)*\mathcal{G}(y_n|c_k,d_k)=\mathcal{G}\left(x_n+b_{n-1}\sum_{i=1}^{n}F(n-1,i-1)y_i|a_n+b_nc_k,b_nd_k\right),$$

$$\mathcal{G}(a_n,b_n)*\mathcal{G}(y_n|c_k,d_k|z_k)=\mathcal{G}\left(b_{n-1}\sum_{i=1}^{n}F(n-1,i-1)y_i|a_n+b_nc_k,b_nd_k|z_k\right).$$

For pure Galton arrays we have.

$$\mathcal{G}(a_n,b_n)*\mathcal{G}(c_k,d_k)=\mathcal{G}(a_n+b_nc_k,b_nd_k).$$

Now we will use the multiplication theorem to factor Galton arrays with certain structures for the generators into simpler ones. First, we will derive results for pure Galton arrays. Using these results, we will be able to derive results about inversion of pure Galton arrays, and using these we will be able to factor Galton arrays with inflows.

As we will see in the rest of the paper, Galton schemes with $(a_{n,k}=c'_n(a'_n+a''_k), b_{n,k}=c'_nb'_k)$ have useful algebraic properties. Arrays of this type will be called arrays with *almost additive vertical generators*. Arrays with the additional conditions $c'_n=1$ and $b_k=1$ will be called arrays *with additive vertical generators*.

Theorem 12 (Factorization of pure Galton arrays).

$$\mathcal{G}(c_n(a'_n+a''_k),c_nb_k)=\mathcal{G}(0,c_n)*\mathcal{G}(a'_n,1)*\mathcal{G}(a''_k,1)*\mathcal{G}(0,b_k).$$

Remark 12. Setting $c'_n=1$ or $b'_k=1$ gives the following special cases:

$$\mathcal{G}(a'_n+a''_k,b_k)=\mathcal{G}(a'_n,1)*\mathcal{G}(a''_k,1)*\mathcal{G}(0,b_k),$$

$$\mathcal{G}(c_n(a'_n+a''_k),c_n)=\mathcal{G}(0,c_n)*\mathcal{G}(a'_n,1)*\mathcal{G}(a''_k,1).$$

This result can be used to explicitly construct the inverses of certain Galton arrays. For all the arrays discussed in the following we will have $b_{n,k} \neq 0$ for all $n \geqslant 0$, and $0 \leqslant k \leqslant n$ since otherwise $\mathscr{G}(a_{n,k}, b_{n,k})$ has columns containing only zeros and is therefore not invertible.

The theorem immediately implies $\mathscr{G}(-a_n, 1) * \mathscr{G}(a_k, 1) = \mathscr{G}(a_k - a_n, 1)$. According to Theorem 4 we see that $\mathscr{G}(a_k - a_n, 1)$ is the identity matrix. Therefore, we also have $\mathscr{G}(a_k, 1) * \mathscr{G}(-a_n, 1) = \mathscr{I}$, and, using the multiplication theorem and the factorization theorem

$$\mathscr{G}(a'_n + a''_k, 1) * \mathscr{G}(-a''_n - a'_k, 1) = \mathscr{G}(a'_n, 1) * \mathscr{G}(a''_k, 1) * \mathscr{G}(-a''_n, 1) * \mathscr{G}(-a'_k, 1) = \mathscr{I}.$$

We further note that $(\mathscr{G}(0, b_n))^{-1} = (\mathscr{G}(0, b_k))^{-1} = \mathscr{G}(0, 1/b_n) = \mathscr{G}(0, 1/b_k)$.

Theorem 13 (Inversion of pure Galton arrays).

$$(\mathscr{G}(b_n(a'_n + a''_k), b_n c_k))^{-1} = \mathscr{G}\left(-\frac{a''_n + a'_k}{c_n}, \frac{1}{c_n b_k}\right).$$

Proof. We have

$$\mathscr{G}(b_n(a'_n + a''_k), b_n c_k) = \mathscr{G}(0, b_n) * \mathscr{G}(a'_n + a''_k, 1) * \mathscr{G}(0, c_k).$$

Therefore we have

$$(\mathscr{G}(b_n(a'_n + a''_k), b_n c_k))^{-1} = \mathscr{G}\left(0, \frac{1}{c_n}\right) * \mathscr{G}(-a''_n - a'_k, 1) * \mathscr{G}\left(0, \frac{1}{b_k}\right)$$
$$= \mathscr{G}\left(\frac{-a''_n - a'_k}{c_n}, \frac{1}{c_n b_k}\right). \qquad \square$$

Remark 14. This theorem allows for many special cases:

$$(\mathscr{G}(a_k, 1))^{-1} = \mathscr{G}(-a_n, 1),$$
$$(\mathscr{G}(a_n, 1))^{-1} = \mathscr{G}(-a_k, 1),$$
$$(\mathscr{G}(a'_n + a''_k, 1))^{-1} = \mathscr{G}(-a''_n - a'_k, 1),$$
$$(\mathscr{G}(a_n, b_n))^{-1} = \mathscr{G}\left(-\frac{a_k}{b_k}, \frac{1}{b_k}\right),$$
$$(\mathscr{G}(a_k, c_k))^{-1} = \mathscr{G}\left(-\frac{a_n}{c_n}, \frac{1}{c_n}\right),$$
$$(\mathscr{G}(a'_n + a''_k, c_k))^{-1} = \mathscr{G}\left(-\frac{a''_n + a'_k}{c_n}, \frac{1}{c_n}\right),$$
$$(\mathscr{G}(b_n(a'_n + a''_k), b_n))^{-1} = \mathscr{G}\left(-a''_n - a'_k, \frac{1}{b_k}\right),$$
$$\left(\mathscr{G}\left(\frac{a'_n + a''_k}{b_n}, \frac{b_k}{b_n}\right)\right)^{-1} = \mathscr{G}\left(-\frac{a''_n + a'_k}{b_n}, \frac{b_k}{b_n}\right).$$

As already mentioned, [10] has shown $(\mathscr{G}(a_k, 1))^{-1} = \mathscr{G}(-a_n, 1)$. The same result also can be derived as a consequence of Theorem 5 in [5]. Even more is true, the full result of Theorem (13) can be derived as a consequence of the results in [5] or [10], but has not been explicitly stated in these papers. Moreover, it seems logical to derive inversion as direct consequence of matrix multiplication results.

An interesting consequence of Theorem 13 is

Corollary 15 (Self-inverse Galton arrays). *For every constant c the Galton arrays* $\mathscr{G}(c + a_n + a_k, -1)$ *and* $\mathscr{G}(b_n(c + a_n + a_k), -b_n/b_k)$ *are self-inverse.*

We note that among all the examples given in Section 1, only the Lah number satisfy this condition.

The last result in this section will show how in certain cases we can 'factor out' row or column inflows in Galton arrays with row dependent or column dependent generators.

Theorem 16 (Factorization of inflows).

$$\mathscr{G}(x_n|a_k, b_k) = \mathscr{G}(x_n|0, 1) * \mathscr{G}(a_k, b_k),$$
$$\mathscr{G}(a_n, b_n|y_k) = \mathscr{G}(a_n, b_n) * \mathscr{G}(0, 1|y_k),$$
$$\mathscr{G}(x_n|a_n, b_n) = \mathscr{G}(a_n, b_n) * \mathscr{G}(z_n|0, 1),$$

where $z_n = \sum_{i=1}^{n} G(n-1, i-1)x_i$ *and* $G = \mathscr{G}(-a_k/b_k, 1/b_k) = (\mathscr{G}(a_n, b_n))^{-1}$.

To express these equations in terms of vector operations we note that $F = \mathscr{G}(x_n|0, 1)$ is a lower triangular matrix satisfying $F(n, k) = F(n-k, 0) = x_{n-k}$ and is therefore 'constant along diagonals'. The result of multiplying such a matrix F with a column vector v, $F * v$, is a vector which is the convolution of v and the first column of F, i.e. the sequence x_n.

$F' = \mathscr{G}(0, 1|y_k)$ is an upper triangular matrix which also is constant along diagonals. The result of multiplying such a matrix F' with a row vector v', $v' * F'$, is a vector which is the convolution of v' and the first row of F', i.e. the sequence y_k.

Therefore the first two equations can be reformulated: every column dependent Galton array with row inflow can be obtained by convoluting the columns of the corresponding pure Galton array with the row-inflow vector. A similar statement is true for every row dependent Galton array with column inflows. The third equation does not directly translate into convolutions of rows or columns of the Galton array with the inflow vector. But if we consider the 'flipped' rows of $\mathscr{G}(a_n, b_n)$ (defined by $G(n, k) = F(n, n-k)$), then the rows of $\mathscr{G}(a_n, b_n) * \mathscr{G}(z_n|1, 0)$ are produced by flipping the rows of $\mathscr{G}(a_n, b_n)$, convoluting with z_n and flipping back.

An interesting consequence of this theorem is the following: Galton arrays with constant vertical and diagonal generators have the same set of generators if the first few rows or the first few columns are 'cut off'. Therefore, the 'cut off' arrays can be produced by convolution of the pure complete array with either the first row of the 'cut off' array or with a column that is easily calculated from the first column of the 'cut off' array. As we will see later, arrays with constant horizontal and vertical generators are just binomial coefficients multiplied by powers of constants (i.e. the terms occurring in the binomial theorem), and therefore the equivalence of shifting a Galton array either rowwise or columnwise with convolution of the rows of this array either rowwise or columnwise is a special property of the binomials or, more generally, the ‘binomial terms’.

Let us apply these multiplication and factorization results. It is easily seen that for the matrix $J(n,k)=\delta_{n,k}-\delta_{n,k+1}$ and its inverse J^{-1} we have $J^{-1}(n,k)=1$ for $0\leqslant k\leqslant n$ and $J^{-1}(n,k)=0$ for $k>n$. So J^{-1} is the lower triangular matrix with all entries in and below the main diagonal 1. For any sequence a_n with $a_0=0$ and $a_n\neq 0$ for $n\geqslant 1$ for $F=\mathscr{G}(a_n-a_{k+1},1)$ we have $F(n,k)=\delta_{n,k}-a_n\delta_{n,k+1}$. Therefore,

$$\mathscr{G}\left(0,\frac{1}{a_{n+1}}\right)*\mathscr{G}(a_n-a_{k+1},1)*\mathscr{G}(0,a_{k+1})=\mathscr{G}\left(\frac{a_n-a_{k+1}}{a_{n+1}},\frac{a_{k+1}}{a_{n+1}}\right)=J.$$

As a consequence, applying Theorem 13 we have

$$\mathscr{G}\left(0,\frac{1}{a_{n+1}}\right)*\mathscr{G}(a_{n+1}-a_k,1)*\mathscr{G}(0,a_{k+1})=\mathscr{G}\left(\frac{a_{n+1}-a_k}{a_{n+1}},\frac{a_{k+1}}{a_{n+1}}\right)=J^{-1},$$

which illustrates the nonuniqueness of Galton generators. Setting $a_n=n$ we get

$$\mathscr{G}\left(\frac{n+1-k}{n+1},\frac{k+1}{n+1}\right)=\mathscr{G}\left(1-\frac{k}{n+1},1-\frac{n-k}{n+1}\right)=J^{-1}.$$

4. Row sum, column sums, and diagonal sums of Galton arrays

Three special cases of the multiplication theorem,

$$\mathscr{G}(a_n,b_n)*\mathscr{G}(0,1|1)=\mathscr{G}(a_n,b_n|1),$$

$$\mathscr{G}(a_n,b_n)*\mathscr{G}(1|0,1)=\mathscr{G}\left(b_{n-1}\sum_{i=1}^{n}F(n-1,i-1)|a_n,b_n\right),$$

$$\mathscr{G}(1|0,1)*\mathscr{G}(a_k,b_k)\mathscr{G}(1|a_k,b_k),$$

deal with ‘running row sums’, ‘running row tail sums’, and ‘running column sums’ of special types of pure Galton arrays. Our equations show that these running sums are Galton arrays with the same generators as the original arrays, but instead of being pure Galton arrays they have row or column inflows.

We can derive a result for row sums of Galton arrays even in cases in which Theorem 10 is not directly applicable like in the examples just given.

Theorem 17 (Row sums of triangular Galton arrays). *Let* $F=\mathscr{G}(x_n|a_{n,k},b_{n,k})$. *If* $a_{n,k}+b_{n,k}=q_n$ *for* $0\leqslant k\leqslant n$, *then the row sums* $S_n=\sum_{i=0}^{n}F(n,i)$ *satisfy the equation*

$$S_n=q_{n-1}S_{n-1}+x_n.$$

The proof, of course, is rather trivial (just plug in the definitions), but nevertheless there are two reasons for explicitly stating this fact as a theorem: So far, the Euler numbers of both kinds did not fit too well into our framework of Galton arrays. They are covered by the definition, but the results derived so far are not applicable. Theorem 17 applies for the Euler numbers, too. So Theorem 17 shows that writing down a recursively defined number array as Galton array is helpful in recognizing interesting properties of the array.

We can also generalize the running column sums to general Galton arrays. Just unfolding the recursion for $F = \mathscr{G}(x_n|a_{n,k}, a_{n,k}|y_n)$ yields the following equation (if necessary, set $y_0 = 0$):

$$F(n,k) = \sum_{i=0}^{n-1} b_{i,k-1} \left(\prod_{j=i+1}^{n-1} a_{jk} \right) F(i, k-1) + \left(\prod_{j=0}^{n-1} a_{jk} \right) y_k.$$

For $0 \leqslant k < n$ we also have

$$F(n,k) = \sum_{i=0}^{k} a_{n-k+i-1,i} \left(\prod_{j=i}^{k-1} b_{n-k+j,j} \right) F(n-k+i-1, i) + \left(\prod_{i=0}^{k-1} b_{n-k+i,i} \right) x_{n-k}.$$

We see that the first of these equations gives nice results for $a_{n,k} = 1$ and $y_k = 0$ and the second one for $b_{n,k} = 1$ and $x_n = 0$, and especially for $a_{n,k} = a_{n-k,0}$.

5. Galton arrays with linear generators (and some more examples)

As already noted in the introduction, quite a few well known arrays can easily be represented as Galton arrays. In addition to the ones mentioned there, we will need some more (relatively trivial) examples.

Using the notation $\alpha^{\underline{n}} = \prod_{i=0}^{n-1} (\alpha - i)$ and $\alpha^{\bar{n}} = \prod_{i=0}^{n-1} (\alpha + i)$ for the falling and rising factorial powers we have the following equations (representing sequences as diagonal matrices):

$$F = \mathscr{G}(0, \alpha) \Rightarrow F(n,k) = \delta_{n,k} \alpha^n,$$

$$F = \begin{cases} \mathscr{G}(0, \alpha + n) \\ \mathscr{G}(0, \alpha + k) \end{cases} \Rightarrow F(n,k) = \delta_{n,k} \alpha^{\bar{n}},$$

$$F = \begin{cases} \mathscr{G}(0, \alpha - n) \\ \mathscr{G}(0, \alpha - k) \end{cases} \Rightarrow F(n,k) = \delta_{n,k} \alpha^{\underline{n}},$$

$$F = \begin{cases} \mathscr{G}(0, \alpha + \beta n) \\ \mathscr{G}(0, \alpha + \beta k) \end{cases} \Rightarrow F(n,k) = \begin{cases} \delta_{n,k} (\frac{\alpha}{\beta})^{\bar{n}} \beta^n, \\ \delta_{n,k} (-\frac{\alpha}{\beta})^{\underline{n}} (-\beta)^n, \end{cases}$$

$$F = \begin{cases} \mathscr{G}(0, 1 + n) \\ \mathscr{G}(0, 1 + k) \end{cases} \Rightarrow F(n,k) = \delta_{n,k} n!.$$

For triangular matrices we have the following equations:

$$F = \mathscr{G}(1+n-k, 1+k) \Rightarrow F(n,k) = \begin{cases} n! & \text{for } 0 \leqslant k \leqslant n, \\ 0 & \text{for } k > n. \end{cases}$$

$$F = \mathscr{G}(1, 1+k) \Rightarrow F(n,k) = n^{\underline{k}},$$

$$F = \begin{cases} \mathscr{G}(1+n, 1+n) \\ \mathscr{G}(1+n+k, 1+k) \end{cases} \Rightarrow F(n,k) = n^{\underline{k}} n^{\underline{n-k}},$$

$$F = \begin{cases} \mathscr{G}(1, 1+k|\alpha^{\underline{k}}) \\ \mathscr{G}(1|0, 1+\alpha+n|\alpha^{\underline{k}}) \end{cases} \Rightarrow F(n,k) = (\alpha+n)^{\underline{k}},$$

$$F = \mathscr{G}(x,y) \Rightarrow F(n,k) = \binom{n}{k} x^{n-k} y^k.$$

All these equations can be checked easily.

An important nonuniqueness result for Galton arrays with linear generators is

$$\mathscr{G}(\alpha+\beta n, \alpha+\beta n) = \mathscr{G}(\alpha+\beta(n+k), \alpha+\beta k). \tag{6}$$

This can be seen by noting that for $F = \mathscr{G}(\alpha+\beta n, \alpha+\beta n)$ we have $F(n,k) = (\alpha/\beta)^{\bar{n}} \beta^n \binom{n}{k}$ and checking the recursion given by the right-hand side of this Eq. (6) for this function.

Before giving some examples using these representations we note that in general the matrix product of Galton arrays with linear generators need not to have linear generators, since quadratic terms may occur. Here are the cases in which the product is linear:

$$\mathscr{G}(\alpha+\bar{\alpha}n, \beta) * \mathscr{G}(\varphi, \psi+\hat{\psi}k) = \mathscr{G}((\alpha+\beta\varphi)+\bar{\alpha}n, \beta\psi+\beta\hat{\psi}k),$$

$$\mathscr{G}(\alpha+\bar{\alpha}n, \beta+\bar{\beta}n) * \mathscr{G}(\varphi, \psi) = \mathscr{G}((\alpha+\beta\varphi)+(\bar{\alpha}+\bar{\beta}\varphi)n, \beta\psi+\bar{\beta}\psi n),$$

$$\mathscr{G}(\alpha+\bar{\alpha}n, \beta) * \mathscr{G}(\varphi+\hat{\varphi}k, \psi) = \mathscr{G}((\alpha+\beta\varphi)+\bar{\alpha}n+\beta\hat{\varphi}k, \beta\psi),$$

$$\mathscr{G}(\alpha, \beta) * \mathscr{G}(\varphi+\hat{\varphi}k, \psi+\hat{\psi}k) = \mathscr{G}((\alpha+\beta\varphi)+\beta\hat{\varphi}k, \beta\psi+\beta\hat{\psi}k).$$

Now, let us rewrite and extend the factorization Theorem 11 for Galton arrays with linear generators.

Theorem 18 (Factorization of linear Galton arrays). *For Galton arrays with linear generators we have*

$$\mathscr{G}(\alpha+\bar{\alpha}n+\hat{\alpha}k, \beta+\hat{\beta}k) = \mathscr{G}(\bar{\alpha}n, 1) * \mathscr{G}(\alpha, 1) * \mathscr{G}(\hat{\alpha}k, 1) * \mathscr{G}(0, \beta+\hat{\beta}k).$$

If furthermore $\alpha \neq 0,\ \bar{\alpha} \neq 0,\ \hat{\alpha} \neq 0,\ \beta \neq 0,\ \hat{\beta} \neq 0$, *we have*

$$\mathscr{G}(\alpha+\bar{\alpha}n+\hat{\alpha}k, \beta+\hat{\beta}k) = \mathscr{G}(0, \bar{\alpha}) * \mathscr{G}(n, 1) * \mathscr{G}\left(0, \frac{\alpha}{\bar{\alpha}}\right) * \mathscr{G}(1,1)$$

$$* \mathscr{G}\left(0, \frac{\hat{\alpha}}{\alpha}\right) * \mathscr{G}(k, 1) * \mathscr{G}\left(0, \frac{\hat{\beta}}{\hat{\alpha}}\right) * \mathscr{G}\left(0, \frac{\beta}{\hat{\beta}} + k\right).$$

This theorem implies that all nontrivial pure Galton arrays with linear generators with the diagonal generator depending only on the column index can be obtained as a matrix product of Stirling numbers of both kinds, binomials, powers, and falling factorial powers.

Rewriting Theorem 13 we see that the inverse of a linearly generated Galton array also has linear generators if $b_{n,k}$ is constant. Then we have

$$(\mathscr{G}(\alpha + \bar{\alpha}n + \hat{\alpha}k, \beta))^{-1} = \mathscr{G}\left(-\frac{\alpha + \hat{\alpha}n + \bar{\alpha}k}{\beta}, \frac{1}{\beta}\right).$$

Studying row sum properties of Galton arrays with linear generators we see that $\mathscr{G}(\alpha + \bar{\alpha}n + \hat{\alpha}k, \beta) * \mathscr{G}(0, (\gamma - \hat{\alpha}k)/\beta) = \mathscr{G}(\alpha + \bar{\alpha}n + \hat{\alpha}k, \gamma - \hat{\alpha}k)$ satisfies the conditions of Theorem 17. The result can be rewritten in the following way: Let $F = \mathscr{G}(\alpha + \bar{\alpha}n + \hat{\alpha}k, \beta)$. If for arbitrary γ we define the sequence $v_n = (\hat{\alpha}/\beta)^n(\gamma/\hat{\alpha})^{\underline{n}}$, then the weighted row sums $S_n = \sum_{i=0}^{n} F(n,i)v_i$ satisfy the equation $S_{n+1} = (\alpha + \gamma + \bar{\alpha}n)S_n$. For $\alpha = \bar{\alpha} = 0$ and $\beta = 1$ this is the well known equation

$$\sum_{i=0}^{n} \left\{ {n \atop i} \right\} \gamma^{\underline{i}} = \gamma^n.$$

Finally, let us use these results to show that the machinery we have developed can be helpful in deriving formulas for recursive functions. The results by themselves are not new, they only serve to show that the (new) theorems we have proved can be applied to give a unified and easy to use mechanism for proving equations about a wide class of arrays studied in combinatorics. Applying our techniques

$$\mathscr{G}(1,1) * \mathscr{G}\left(1, 1 \middle| \binom{\alpha}{k}\right) = \mathscr{G}(2,1) * \mathscr{G}\left(0, 1 \middle| \binom{\alpha}{k}\right)$$

is equivalent to

$$\sum_{i=0}^{n} \binom{n}{i} \binom{\alpha + i}{k} = \sum_{i=0}^{\min(n,k)} \binom{n}{i} \binom{\alpha}{k - i} 2^{n-i}.$$

Next, we rewrite Remark 2 for the case of linear generators.

Let

$$F = \mathscr{G}(\bar{\alpha}n + \hat{\alpha}k, 1 + \bar{\beta}n + \hat{\beta}k)$$

and

$$F' = \mathscr{G}((\bar{\alpha} + \hat{\alpha}) + \bar{\alpha}n + \hat{\alpha}k, (1 + \bar{\beta} + \hat{\beta}) + \bar{\beta}n + \hat{\beta}k).$$

Then

$$F'(n,k) = F(n+1, k+1).$$

This is applicable for the Stirling numbers of both kinds and the Lah numbers. For the rest of the paper we will use Knuth's notation for the Stirling numbers explained in section (1).

Rewriting Galton array equations in standard notation

$$\mathscr{G}(n,1)*\mathscr{G}(1,1)=\mathscr{G}(1+n,1) \quad \text{becomes} \quad \sum_{i=0}^{n}\begin{bmatrix} n \\ i \end{bmatrix}\binom{i}{k}=\begin{bmatrix} n+1 \\ k+1 \end{bmatrix},$$

$$\mathscr{G}(1,1)*\mathscr{G}(k,1)=\mathscr{G}(1+k,1) \quad \text{becomes} \quad \sum_{i=0}^{n}\binom{n}{i}\begin{Bmatrix} i \\ k \end{Bmatrix}=\begin{Bmatrix} n+1 \\ k+1 \end{Bmatrix}.$$

We also already know from Eq. (6) that $\mathscr{G}(2+n,2+n)=\mathscr{G}(2+n+k,2+k)$, therefore

$$\begin{aligned}\mathscr{G}(2+n+k,1)&=\mathscr{G}(2+n+k,2+k)*(\mathscr{G}(0,2+k))^{-1}\\ &=\mathscr{G}(2+n,2+n)*(\mathscr{G}(0,2+k))^{-1}\\ &=\mathscr{G}(0,2+n)*\mathscr{G}(1,1)*(\mathscr{G}(0,2+k))^{-1}\end{aligned}$$

and this implies $L(n+1,k+1)(-1)^n=((n+1)!/(k+1)!)\binom{n}{k}$.

Another consequence of Eq. (6) is

$$\mathscr{G}(1,1)=(\mathscr{G}(0,1+k))^{-1}*\mathscr{G}(n,1)*\mathscr{G}(1,1)*\mathscr{G}(k,1)*\mathscr{G}(0,1+k),$$

connecting the binomials, Stirling numbers, and factorials which also can be written as

$$\mathscr{G}(-k,1)*\mathscr{G}(0,1+n)*\mathscr{G}(1,1)=\mathscr{G}(1,1)*\mathscr{G}(k,1)*\mathscr{G}(0,1+k),$$

$$\begin{aligned}\mathscr{G}(0,1+n)*\mathscr{G}(1,1)*(\mathscr{G}(0,1+k))^{-1}&=\mathscr{G}(n,1)*\mathscr{G}(1,1)*\mathscr{G}(k,1)\\ &=\mathscr{G}(1+n,1)*\mathscr{G}(k,1)\\ &=\mathscr{G}(n,1)*\mathscr{G}(1+k,1).\end{aligned}$$

According to the shift properties of Galton arrays with linear generators, $\mathscr{G}(1+n,1)$ and $\mathscr{G}(1+k,1)$ are the 'cut off' or 'shifted' Stirling numbers. Therefore the last equation can be rewritten as

$$\sum_{i=0}^{n}\begin{bmatrix} n \\ i \end{bmatrix}\begin{Bmatrix} i+1 \\ k+1 \end{Bmatrix}=\sum_{i=0}^{n}\begin{bmatrix} n+1 \\ i+1 \end{bmatrix}\begin{Bmatrix} i \\ k \end{Bmatrix}=\frac{n!}{k!}\binom{n}{k}.$$

We also have

$$\mathscr{G}(1+n,1)*\mathscr{G}(-k,1)=\mathscr{G}(1+n-k,1)=\mathscr{G}(1+n-k,1+k)*(\mathscr{G}(0,1+k))^{-1}.$$

This implies

$$\sum_{i=0}^{n}\begin{bmatrix} n+1 \\ i+1 \end{bmatrix}\begin{Bmatrix} i \\ k \end{Bmatrix}(-1)^{i+k}=\frac{n!}{k!}=(n-k)!\binom{n}{k}.$$

Acknowledgements

The author wishes to thank Renzo Sprugnoli, who was extremely supportive and helpful in many ways. Stephen Milne, Christian Krattenthaler and the unknown referees supplied extremely fruitful comments and suggestions, and Richard Stanley offered helpful hints. Special thanks go to Paul Horwitz for serving as a midwife for the ideas underlying this paper.

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