A combinatorial approach to a general two-term recurrence

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ABSTRACT

We provide combinatorial proofs of explicit formulas for some sequences satisfying particular cases of the general recurrence $\left|{n \atop k}\right| = (\alpha(n-1) + \beta k + \gamma)\left|{n-1 \atop k}\right| + (\alpha'(n-1) + \beta'k + \gamma')\left|{n-1 \atop k-1}\right| + [n = k = 0]$, which have been previously shown using other methods. Many interesting combinatorial sequences are special cases of this recurrence, such as binomial coefficients, both kinds of Stirling numbers, Lah numbers, and two types of Eulerian numbers. Among the cases we consider are $\alpha' = 0$, $\alpha = -\beta$, and $\beta = \beta' = 0$. We also provide combinatorial proofs of some prior identities satisfied by $\left|{n \atop k}\right|$ when $\alpha' = 0$ and when $\beta = \beta' = 0$ as well as deduce some new ones in the former case. In addition, we introduce a polynomial generalization of $\left|{n \atop k}\right|$ when $\alpha' = 0$ which has among its special cases q-analogues of both kinds of Stirling numbers. Finally, we supply combinatorial proofs of two formulas relating binomial coefficients and the two kinds of Stirling numbers which were previously obtained by equating three different expressions for the solution of the aforementioned recurrence in the case when $\alpha' = \beta' = 0$ and all other weights are unity.

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1. Introduction

Graham, Knuth, and Patashnik propose the following open problem in their text *Concrete Mathematics* [6, p. 319, Problem 6.94]:

Develop a general theory of the solutions to the two-parameter recurrence

$$\left|{n \atop k}\right| = (\alpha n + \beta k + \gamma)\left|{n-1 \atop k}\right| + (\alpha'n + \beta'k + \gamma')\left|{n-1 \atop k-1}\right| + [n = k = 0], \quad \text{for } n, k \geq 0, \tag{1}$$

assuming that $\left|{n \atop k}\right| = 0$ when $n < 0$ or $k < 0$. What special values $(\alpha, \beta, \gamma, \alpha', \beta', \gamma')$ yield "fundamental solutions" in terms of which the general solution can be expressed?

Many combinatorial sequences of interest satisfy recurrences that are special cases of (1), which include binomial coefficients (see A007318 in [15]), both kinds of Stirling numbers (A008275, A008277), Lah numbers (A008297), two types of Eulerian numbers (A008292, A008517), and two types of associated Stirling numbers (A008306, A008299). Furthermore, several of the generalizations of the Stirling numbers that have been studied also satisfy recurrences of the form (1); see, for example, [7–9].

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Perhaps the most general result so far concerning solutions to (1) is due to Neuwirth [12], who shows that if $\alpha' = 0$, then

$$\left| {n \atop k} \right| = \prod_{i=1}^{k} (\beta' i + \gamma') \sum_{i=0}^{n} \sum_{j=0}^{n} \left[{n \atop i} \right] \binom{i}{j} \left\{ {j \atop k} \right\} \alpha^{n-i} \beta^{j-k} (\gamma + \alpha)^{i-j}, \quad n, k \geq 0, \tag{2}$$

where $\left[{m \atop r} \right]$ and $\left\{ {m \atop r} \right\}$ denote the Stirling numbers of the first and second kind, respectively. In deriving Eq. (2), Neuwirth uses infinite, triangular matrices whose entries are the $\left| {n \atop k} \right|$ values for recurrences of type (1) (which he terms *Galton arrays*). These matrices allow one to represent solutions to (1) in terms of simpler recurrences of the same type. Regev and Roichman [13] also obtain (2) with the additional assumption $\beta' = 0$, and they relate special cases of their solution to certain statistics on colored permutations. See also the papers by Mijajlović and Marković [10] and by Cakić [2] where solutions to (1) are given in the case $\beta = \gamma' = 1$, $\gamma = \alpha' = \beta' = 0$ (also, see [3]). Using a partial finite difference approach, Spivey [16, Theorem 6] provides another derivation of (2) and also obtains solutions to (1) for the cases (i) $\alpha = -\beta$, (ii) $\beta = \beta' = 0$, and (iii) $\frac{\alpha}{\beta} = \frac{\alpha'}{\beta'} + 1$. In all of these cases, as with Neuwirth's result (2), nothing more than binomial coefficients, the two kinds of Stirling numbers, and generalized factorials are required to express the solutions and thus are the "fundamental solutions" requested by Graham, Knuth, and Patashnik in these cases.

In this paper, we will consider some combinatorial aspects of recurrence (1). First, we will provide a combinatorial solution to (1) when $\alpha' = 0$ by describing a structure whose cardinality satisfies (1) in this case and then showing that the cardinality is also given by (2). We then use this structure to provide combinatorial proofs of some previous identities satisfied by $\left| {n \atop k} \right|$ when $\alpha' = 0$ as well as establish some new ones. Modifying slightly our proof of (2) yields a solution to (1) when $\alpha = -\beta$. We also supply combinatorial proofs of two formulas relating binomial coefficients and the two kinds of Stirling numbers which were previously obtained in [16] by equating three different expressions for the solution of (1) in the case when $\alpha' = \beta' = 0$ and all other weights are unity.

Next, we provide a combinatorial solution to (1) when $\beta = \beta' = 0$. Our expression for the solution in this case differs from the one obtained in [16] using algebraic methods, and in fact it can be shown, bijectively, that the two expressions are equivalent. We also provide in this case a combinatorial proof for an explicit formula of the row sum $\sum_{k=0}^{n} \left| {n \atop k} \right|$ and extend it to the case when $\beta + \beta' = 0$, which was obtained in [16, Corollary 19] and [12]. Finally, our proof of (2) above may be extended further to ascertain an explicit formula for the solution of a q-version of recurrence (1) when $\alpha' = 0$ and to deduce some identities satisfied by it.

We will use the following notational conventions. Empty sums assume the value 0 and empty products the value 1, with $0^0 = 1$. If m and n are positive integers, then $[m, n] = \{m, m+1, \ldots, n\}$ if $m \leq n$, with $[m, n] = \varnothing$ if $m > n$. We will denote the special case $[1, n]$ by $[n]$ if $n \geq 1$, with $[0] = \varnothing$. Let $\mathcal{S}_{n,m}$ be the set of permutations of $[n]$ having m cycles and $\mathcal{S}_n$ be the set of all permutations of $[n]$. Recall that the cardinality of $\mathcal{S}_{n,m}$ is the (signless) Stirling number of the first kind $\left[{n \atop m} \right]$; see, e.g., [18, p. 18]. If one expresses $\sigma \in \mathcal{S}_{n,m}$ as $\sigma = (C_1)(C_2) \cdots (C_m)$, where the smallest letter is first within each cycle C_i and where $\min(C_1) < \min(C_2) < \cdots < \min(C_m)$, then σ is said to be in *standard cycle form*. For example, $\sigma = (134)(257)(69)(8) \in \mathcal{S}_{9,4}$ is in standard cycle form. In what follows, we will compare cycles of some permutation by comparing the sizes of the smallest elements contained within; that is, if (C) and (D) are distinct cycles, then we will say that (C) is smaller than (D) if and only if $\min(C) < \min(D)$.

A *partition* of a finite set is a collection of non-empty, pairwise disjoint subsets, called *blocks*, whose union is the set. The set of all partitions of $[n]$ having exactly m blocks will be denoted by $\mathcal{P}_{n,m}$ whose cardinality is given by the Stirling number of the second kind $\left\{ {n \atop m} \right\}$; see, e.g., [18, p. 33]. If one expresses $\pi \in \mathcal{P}_{n,m}$ as $\pi = B_1/B_2/\cdots/B_m$, where $\min(B_1) < \min(B_2) < \cdots < \min(B_m)$, then π is said to be in *standard form*. For example, $\pi = 1, 5, 7/2, 3/4, 6, 8 \in \mathcal{P}_{8,3}$ is in standard form. An *ordered partition* is one in which the blocks themselves are arranged in some order. Note that there are $m! \left\{ {n \atop m} \right\}$ ordered partitions of $[n]$ having exactly m blocks, which are synonymous with the surjective functions from $[n]$ to $[m]$.

2. A combinatorial approach

In this section, we provide combinatorial solutions of recurrence (1) in the cases when $\alpha' = 0$ and when $\beta = \beta' = 0$. In the former case, we also deduce some further identities satisfied by $\left| {n \atop k} \right|$.

2.1. A general two-term recurrence

We first provide a combinatorial proof of the following result, which occurs in [12,16].

Theorem 2.1. *Let* $\left| {n \atop k} \right|$ *denote the array of numbers defined by the recurrence*

$$\left| {n \atop k} \right| = (\alpha(n-1) + \beta k + \gamma) \left| {n-1 \atop k} \right| + (\beta' k + \gamma') \left| {n-1 \atop k-1} \right| + [n = k = 0], \quad n, k \geq 0. \tag{3}$$

Then $\left|{n \atop k}\right|$ is given explicitly by

$$\left|{n \atop k}\right| = \prod_{i=1}^{k}(\beta'i + \gamma') \sum_{i=0}^{n} \sum_{j=0}^{n} \left[{n \atop i}\right]\binom{i}{j}\left\{{j \atop k}\right\}\alpha^{n-i}\beta^{j-k}\gamma^{i-j}. \tag{4}$$

Proof. We first consider the case when $\gamma' = 0$. Note that (4) may then be written as

$$\left|{n \atop k}\right| = \sum_{i=0}^{n} \sum_{j=0}^{n} \left[{n \atop i}\right]\binom{i}{j}\left\{{j \atop k}\right\}k!\alpha^{n-i}\beta^{j-k}\gamma^{i-j}(\beta')^{k}. \tag{5}$$

Given $1 \le k \le j \le i \le n$, let $\mathcal{A}_{n,k,i,j}$ denote the set of all ordered pairs $\tau = (\rho, \mu)$, where $\rho \in \mathcal{S}_{n,i}$ is in standard cycle form and μ is an ordered partition of size j having k blocks whose elements are themselves cycles of ρ. Define the weight of each $\tau \in \mathcal{A}_{n,k,i,j}$, which we will denote by $w = w(\tau)$, by

$$w = \alpha^{n-i}\beta^{j-k}\gamma^{i-j}(\beta')^{k}.$$

Note that α marks the number of elements of $[n]$ which do not start cycles of ρ, γ marks the number of cycles of ρ which are not chosen to go within the blocks of μ, β marks the number of cycles of ρ belonging to the blocks of μ which are not the smallest within their respective blocks (where cycles are compared to one another as described above), and β' marks the number of blocks of μ.

Let $A_{n,k,i,j} = |\mathcal{A}_{n,k,i,j}|w$. Define the set $\mathcal{A}_{n,k}$ by $\mathcal{A}_{n,k} = \cup_{i,j}\mathcal{A}_{n,k,i,j}$ and define the array $A_{n,k}$ by

$$A_{n,k} = \sum_{\tau \in \mathcal{A}_{n,k}} w(\tau) = \sum_{i,j} A_{n,k,i,j}.$$

We first argue that $A_{n,k}$ satisfies the recurrence (3) when $\gamma' = 0$. To do so, first note that the total weight of all members of $\mathcal{A}_{n,k}$ in which n does not belong to a cycle by itself is $\alpha(n-1)A_{n-1,k}$ since n may directly follow a member of $[n-1]$ within any cycle. On the other hand, there are three options concerning the placement of the 1-cycle (n) within $(\rho, \mu) \in \mathcal{A}_{n,k}$: (i) it may not be chosen to occupy a block of μ, (ii) it occupies a block of μ with at least one other cycle, or (iii) it occupies its own block of μ. The total weights of the members of $\mathcal{A}_{n,k}$ corresponding to the three options are seen to be $\gamma A_{n-1,k}$, $\beta k A_{n-1,k}$, and $\beta'k A_{n-1,k-1}$, respectively. Combining all of the cases gives (3).

So to solve the recurrence (3) when $\gamma' = 0$, it remains to determine an explicit expression for $A_{n,k}$. To do so, note that $|\mathcal{A}_{n,k,i,j}| = k!\left[{n \atop i}\right]\binom{i}{j}\left\{{j \atop k}\right\}$ since there are $\left[{n \atop i}\right]$ choices concerning the cycles of ρ in $(\rho, \mu) \in \mathcal{A}_{n,k,i,j}$ and $\binom{i}{j}\left\{{j \atop k}\right\}k!$ ways in which to select cycles of ρ and arrange them in an ordered partition having k blocks. Summing over all i and j and using the weights as defined, we see that $A_{n,k}$ is given by (5), which implies the result when $\gamma' = 0$.

Let us now assume $\gamma \neq 0$. If $(\rho, \mu) \in \mathcal{A}_{n,k}$, then we will compare two blocks of μ by comparing the sizes of the smallest elements of $[n]$ lying within each block. Given $\tau = (\rho, \mu) \in \mathcal{A}_{n,k}$, let $\lambda = (\rho, \omega)$ be obtained from τ as follows: leave ρ unchanged and let ω be obtained from μ by circling a subset of the blocks of μ such that if a block B of μ is circled in ω, then any block of μ smaller than B comes to the left of B in the ordering of blocks. Let $\mathcal{B}_{n,k}$ denote the set of configurations λ that so arise and, similarly, define $\mathcal{B}_{n,k,i,j}$. Define the weight of $\lambda \in \mathcal{B}_{n,k,i,j}$, which we will denote by $v(\lambda)$, by

$$v(\lambda) = w(\tau)\left(\frac{\gamma'}{\beta'}\right)^{\ell} = \alpha^{n-i}\beta^{j-k}\gamma^{i-j}(\beta')^{k-\ell}(\gamma')^{\ell},$$

where λ is obtained from τ as described and ℓ denotes the number of blocks of λ that are circled. Note that β' now marks the number of blocks which are not circled in λ, with γ' marking the number that are.

If

$$B_{n,k} = \sum_{\lambda \in \mathcal{B}_{n,k}} v(\lambda),$$

then reasoning as before shows that $B_{n,k}$ satisfies recurrence (3) and is given by the explicit formula (4). Note that in showing (3), there is now a fourth option concerning placement of the cycle (n), that is, it may go within a circled block by itself (which would then automatically be last among the blocks in the ordering). The total v-weight of all the members of $\mathcal{B}_{n,k}$ in which this occurs is seen to be $\gamma'B_{n-1,k-1}$, which completes the proof. □

Remark 2.2. The prior proof shows further that if the constants β' and γ' in recurrence (3) are replaced, more generally, by sequences β_i' and γ_i', then the result continues to hold with the product $\prod_{i=1}^{k}(i\beta' + \gamma')$ replaced by $\prod_{i=1}^{k}(i\beta_i' + \gamma_i')$.

Given $n \ge k \ge 0$, let $\left\|{n \atop k}\right\| = \sum_{j=k}^{n}\left|{n \atop j}\right|\binom{j}{k}$ denote what is known as the *upper binomial transform*. A three-term recurrence for $\left\|{n \atop k}\right\|$ is given in [16, Theorem 10] for $\left|{n \atop k}\right|$ in its full generality. Here, we provide a combinatorial proof of the recurrence in the case when $\alpha' = \beta' = 0$ using the interpretation for $\left|{n \atop k}\right|$ above.

Theorem 2.3. *Suppose* $\left|{n \atop k}\right|$ *is given by*

$$\left|{n \atop k}\right| = (\alpha(n-1) + \beta k + \gamma)\left|{n-1 \atop k}\right| + \gamma'\left|{n-1 \atop k-1}\right| + [n = k = 0].$$

Let $\left\|{n \atop k}\right\|$ *denote the sum* $\sum_{j=k}^{n}\left|{n \atop j}\right|\binom{j}{k}$. *Then*

$$\left\|{n \atop k}\right\| = \beta(k+1)\left\|{n-1 \atop k+1}\right\| + (\alpha(n-1) + \beta k + \gamma + \gamma')\left\|{n-1 \atop k}\right\| + \gamma'\left\|{n-1 \atop k-1}\right\| + [n = k = 0]. \tag{6}$$

Proof. Let $\widetilde{\mathcal{B}}_{n,j}$ denote the set $\mathcal{B}_{n,j}$ defined in the proof of Theorem 2.1 above corresponding to the case $\beta' = 0$; i.e., the assumption concerning the ordering of the blocks of ω in $\lambda = (\rho, \omega)$ is dropped. Given $0 \le k \le j \le n$, let $\mathcal{C}_{n,j,k}$ denote the set of configurations obtained from $\widetilde{\mathcal{B}}_{n,j}$ by marking k of the blocks of ω in $\lambda = (\rho, \omega) \in \widetilde{\mathcal{B}}_{n,j}$. Let $\mathcal{C}_{n,k} = \cup_{j=k}^{n}\mathcal{C}_{n,j,k}$. Then $\left|{n \atop k}\right|$ gives the total weight of all the members of $\mathcal{C}_{n,k}$ (where the weight of a member of $\mathcal{C}_{n,k}$ is understood to be the v-weight of the corresponding member of $\widetilde{\mathcal{B}}_{n,j}$ from which it was obtained).

To show that the right-hand side of (6) also gives this total weight, let us denote an arbitrary member of $\mathcal{C}_{n,k}$ by $\lambda' = (\rho, \omega')$, where ρ is a permutation of $[n]$ and ω' is a partition of some subset of the cycles of ρ having at least k blocks in which exactly k of the blocks of ω' are circled. Note first that the total weight of all the members of $\mathcal{C}_{n,k}$ in which n occurs in a cycle of length at least two is $\alpha(n-1)\left\|{n-1 \atop k}\right\|$. The total weight of those members in which the 1-cycle (n) occurs in a block of ω' by itself is $\gamma'\left\|{n-1 \atop k-1}\right\| + \gamma'\left\|{n-1 \atop k}\right\|$, depending on whether or not the block is marked. If (n) occurs in ρ, but not in any block of ω', then there are $\gamma\left\|{n-1 \atop k}\right\|$ possibilities. Finally if (n) occurs in a block of ω' with at least one other cycle, then there are $\beta k\left\|{n-1 \atop k}\right\|$ or $\beta(k+1)\left\|{n-1 \atop k+1}\right\|$ possibilities, depending on whether or not the block is marked. To see the latter case, arrange the elements of $[n-1]$ according to a configuration in $\mathcal{C}_{n-1,k+1}$ and select one of the marked blocks. Then erase the mark on this block and insert the cycle (n) into it. This covers all possible cases concerning the position of the element n within a member of $\mathcal{C}_{n,k}$ and completes the proof of (6). $\quad\square$

The expression in (4) can be reduced when any one of the parameters is zero and the others are unity. For example, the following formula, see [16, Identity 7], was shown algebraically by finding three different expressions for the solution of (3) in the case when $\beta' = 0$ and all other weights are unity. Here, we provide a bijective proof. In what follows, if $w = w_1 w_2 \cdots w_n$ is a word in some alphabet, then let $stan(w)$ denote the word obtained by replacing all occurrences of the i-th smallest letter of w with i for each i.

Proposition 2.4. *If* $n, k \ge 0$, *then*

$$\binom{n}{k}^2 (n-k)! = \sum_{j=0}^{n}\left[{n+1 \atop j+1}\right]\left\{{j \atop k}\right\} = \sum_{j=0}^{n}\left[{n \atop j}\right]\left\{{j+1 \atop k+1}\right\}. \tag{7}$$

Proof. Multiplying both sides by $k!$, we see that the first identity may be rewritten in the more suggestive form

$$\binom{n}{k}n! = \sum_{j=0}^{n}\left[{n+1 \atop j+1}\right]\left\{{j \atop k}\right\}k!. \tag{8}$$

Given $0 \le k \le n$, let $\sigma = \sigma_1\sigma_2\cdots\sigma_n \in \mathcal{S}_n$ and let $S = \{i_1 < i_2 < \cdots < i_k\}$ be a subset of $[n]$ of size k. We divide up σ according to S as follows: let $x_1 = \sigma_1\cdots\sigma_{i_1}$, $x_j = \sigma_{i_{j-1}+1}\cdots\sigma_{i_j}$ if $2 \le j \le k$, and $x_{k+1} = \sigma_{i_k+1}\cdots\sigma_n$ (note that x_{k+1} is empty if $i_k = n$). If $1 \le j \le k$, then let y_j be the permutation of the letters in x_j obtained by writing $stan(x_j)$ in standard cycle form and putting back the letters of x_j. For example, if $x_1 = 628395$, then $stan(x_1) = 415263 = (142)(356)$ and $y_1 = (263)(589)$.

Note that the sub-permutations $y_1, y_2, \ldots, y_k$, expressed as cycles, taken together with the cycle obtained by writing $n+1$ followed by any letters in x_{k+1} (and enclosing the word that results in parentheses), constitute a permutation δ of $[n+1]$ having at least $k+1$ cycles. If $1 \le j \le k$, we now let the cycles of y_j comprise the j-th block of an ordered partition π having k blocks. It is then seen that all ordered pairs (δ, π) arise, where $\delta \in \mathcal{S}_{n+1}$ has at least $k+1$ cycles and π is an ordered partition of the cycles of δ not including the one containing the element $n+1$. Such ordered pairs are clearly enumerated by the right-hand side of (8), upon conditioning on the number of cycles of δ. Since the operation described above is reversible, starting with an ordered pair (δ, π), identity (8) follows.

Note that the second identity may be expressed as

$$(k+1)\binom{n}{k}n! = \sum_{j=0}^{n}\left[{n \atop j}\right]\left\{{j+1 \atop k+1}\right\}(k+1)!. \tag{9}$$

To show (9), let σ, S, the x_j, and the y_j be as before. In addition, let y_{k+1} be defined like the other y_j if $i_k < n$, with $y_{k+1} = \varnothing$ if $i_k = n$. If $i_k < n$, then select any one of the y_j. If $i_k = n$, then select some number r in $[k+1]$ and let $z_1, z_2, \ldots, z_{k+1}$ be obtained from $y_1, y_2, \ldots, y_k$ by letting $z_j = y_j$ if $1 \le j < r$, $z_r = \varnothing$, and $z_j = y_{j-1}$ if $r < j \le k+1$.

Let τ be the permutation of $[n]$ obtained by putting together the cycles of all of the y_j. If $i_k < n$, then let the cycles of y_j comprise the j-th block in an ordered partition υ, where we insert a special element inside the block containing the cycles of the selected y_j. If $i_k = n$, then we let the cycles of z_i non-empty comprise the i-th block of υ and, for the index r where $z_r = \varnothing$, we let the r-th block of υ contain the special object by itself. Combining the two cases, we obtain all ordered pairs (τ, υ) in which τ is some member of $\mathscr{S}_n$ having at least k cycles and υ is an ordered partition of the cycles of τ, taken together with a special object, having $k+1$ blocks. The right-hand side of (9) counts all such ordered pairs (τ, υ) according to the number of cycles j of τ, which completes the proof. $\square$

2.2. Some further identities

In this section, we prove some further relations involving the numbers $\left|{n \atop k}\right|$ that satisfy (3) which do not seem to have been previously noted, using the combinatorial interpretation described above. We will use the same notation as in the proof of Theorem 2.1 above in those that follow. Our first relation generalizes several known recurrences.

Proposition 2.5. *If $n, k \ge 0$, then*

$$\left|{n+1 \atop k+1}\right| = \sum_{i=0}^{n-k-1} \left|{n-i \atop k+1}\right| \binom{n}{i} i! \gamma \alpha^i + (\beta'(k+1) + \gamma') \sum_{i=0}^{n-k} \left|{n-i \atop k}\right| \binom{n}{i} \prod_{j=1}^{i}(\alpha j + \beta), \tag{10}$$

where $\left|{n \atop k}\right|$ is given by (3).

Proof. The first sum gives the total weight of all members of $\mathscr{B}_{n+1,k+1}$ in which the element 1 does not belong to a block (circled or not) of ω in $\lambda = (\rho, \omega) \in \mathscr{B}_{n+1,k+1}$. To see this, note that the weight of all such members of $\mathscr{B}_{n+1,k+1}$ in which there are exactly i members of $[2, n+1]$ occurring in the cycle containing 1 is $\gamma \left|{n-i \atop k+1}\right| \binom{n}{i} i! \alpha^i$. Summing over all possible i gives the first sum.

The second sum gives the total weight of all members of $\mathscr{B}_{n+1,k+1}$ in which 1 does lie within a block of ω again according to the number i of additional elements in the block that contains 1. There are then $\binom{n}{i}$ choices regarding these elements and $\left|{n-i \atop k}\right|$ ways to arrange the other members of $[2, n+1]$, which constitute a member of $\mathscr{B}_{n-i,k}$. Concerning the block containing the element 1, if it is uncircled, then there are $k+1$ ways in which to arrange it relative to the k other blocks of ω, whence the $\beta'(k+1)$ part of the factor preceding the sum. If the block is circled, then there is only one way to arrange it relative to the others, whence the γ' term. Finally, suppose that there are exactly j cycles occurring in the block of ω containing 1. Note that these cycles and the elements contained therein contribute a factor of $\alpha^{i+1-j}\beta^{j-1}$ towards the υ-weight as defined in the proof of Theorem 2.1 above, and there are $\left[{i+1 \atop j}\right]$ choices regarding these cycles. Summing over j, we see that there are

$$\sum_{j=1}^{i+1} \left[{i+1 \atop j}\right] \alpha^{i+1-j}\beta^{j-1} = \frac{\alpha^{i+1}}{\beta} \left(\frac{\beta}{\alpha}\right)^{\overline{i+1}} = \frac{\alpha^{i+1}}{\beta} \left(\frac{\beta}{\alpha}\right)\left(\frac{\beta}{\alpha}+1\right) \cdots \left(\frac{\beta}{\alpha}+i\right) = \prod_{j=1}^{i}(\beta + j\alpha)$$

possibilities for the cycles in all, which completes the proof. $\square$

Remark 2.6. Note that the product appearing in the second sum on the right-hand side of (10) could also have been justified directly by considering, independently, the positions of the i additional elements of $[2, n+1]$ in the block of ω containing 1, starting with the smallest. Each such element may either start its own cycle or follow one of the preceding elements or 1 within a prior cycle.

Example 2.7. Taking $\beta = \gamma' = 1$ in (10) and all other weights to be zero gives a well-known recurrence for the Stirling numbers of the second kind (see, e.g., [1, Identity 198]):

$$\left\{{n+1 \atop k+1}\right\} = \sum_{i=0}^{n-k} \left\{{n-i \atop k}\right\} \binom{n}{i}, \quad n, k \ge 0.$$

Proposition 2.8. *If $n, k \ge 0$, then*

$$\left|{n+k+1 \atop k}\right| = \sum_{i=0}^{k}(\alpha(n+i) + \beta i + \gamma)\left|{n+i \atop i}\right| \prod_{j=i+1}^{k}(\beta' j + \gamma'), \tag{11}$$

where $\left|{n \atop k}\right|$ is given by (3).

Proof. Condition on the largest index i, $0 \le i \le k$, such that the element $n + i + 1$ does not occur as a 1-cycle by itself in a block of ω within $\lambda = (\rho, \omega) \in \mathcal{B}_{n+k+1,k}$. Note that such an index exists for all $\lambda \in \mathcal{B}_{n+k+1,k}$, for otherwise there would be at least $k + 1$ blocks in ω. The elements of $[n + i]$ then constitute a member of $\mathcal{B}_{n+i,i}$, and there are $\alpha(n + i) + \beta i + \gamma$ choices for the element $n + i + 1$, which may directly follow a member of $[n + i]$ in a cycle or occur as its own cycle either within a block of ω or outside of all the blocks. Finally, the product $\prod_{j=i+1}^{k}(\beta'j + \gamma')$ reflects the choices concerning placement of the elements in $[n + i + 2, n + k + 1]$. $\square$

Example 2.9. Taking $\alpha = \gamma' = 1$ or $\beta = \gamma' = 1$, respectively, with all other weights zero, gives known recurrences for the Stirling numbers of first and second kind, respectively (see, for example, formulas (6.23) and (6.22) in [6]):

$$\begin{bmatrix} n + k + 1 \\ k \end{bmatrix} = \sum_{i=0}^{k}(n + i)\begin{bmatrix} n + i \\ i \end{bmatrix}, \quad n, k \ge 0,$$

and

$$\left\{\begin{matrix} n + k + 1 \\ k \end{matrix}\right\} = \sum_{i=0}^{k} i \left\{\begin{matrix} n + i \\ i \end{matrix}\right\}, \quad n, k \ge 0.$$

Our next relation for $\left|{n \atop k}\right|$ generalizes the well-known binomial recurrence $\binom{n+1}{k+1} = \sum_{i=k}^{n}\binom{i}{k}$.

Proposition 2.10. If $n, k \ge 0$, then

$$\left|\begin{matrix} n + 1 \\ k + 1 \end{matrix}\right| = (\beta'(k + 1) + \gamma')\sum_{i=k}^{n}\left|\begin{matrix} i \\ k \end{matrix}\right|\sum_{j=0}^{n-i}\binom{n - i}{j}\frac{(n - j)!}{i!}\alpha^{n-i-j}\prod_{\ell=0}^{j-1}[\alpha\ell + \beta(k + 1) + \gamma], \tag{12}$$

where $\left|{n \atop k}\right|$ is given by (3).

Proof. Consider the block of ω within $\lambda = (\rho, \omega) \in \mathcal{B}_{n+1,k+1}$ whose smallest element of $[n + 1]$ contained within is the largest among all of the blocks and let $i + 1$ denote this smallest element; note that $k \le i \le n$. The right-hand side of (12) then gives the total weight of all the members of $\mathcal{B}_{n+1,k+1}$ according to the value of i. Note that $\left|{i \atop k}\right|$ accounts for the ways in which to arrange the members of $[i]$ within λ and that $\beta'(k + 1) + \gamma'$ accounts for the placement of the element $i + 1$ as a 1-cycle within a new block. Finally, the inner sum accounts for the choices concerning the positions of the elements in $[i + 2, n + 1]$ once all the members of $[i + 1]$ have been placed. To see this, suppose exactly $n - i - j$ elements of $[i + 2, n + 1]$ occur in cycles containing at least one element of $[i + 1]$. There are $\binom{n-i}{n-i-j} = \binom{n-i}{j}$ ways to select these elements and

$$(i + 1)(i + 2)\cdots(i + n - i - j)\alpha^{n-i-j} = \frac{(n - j)!}{i!}\alpha^{n-i-j}$$

ways to position them relative to other members of $[n + 1]$. The remaining j elements of $[i + 2, n + 1]$, the set of which we will denote by R, appear in cycles of ρ that do not contain any elements of $[i + 1]$. Starting with the smallest member of R, one may decide, in an independent fashion, whether to start a new cycle within some block of ω using a member of R, start a new cycle not belonging to any of the blocks of ω, or add the element to a cycle already containing a member of R, whence the factor of $\prod_{\ell=0}^{j-1}[\beta(k + 1) + \gamma + \alpha\ell]$ appearing. $\square$

The next result generalizes known recurrences for Stirling numbers of both kinds.

Proposition 2.11. Suppose $n, k, m \ge 0$. If $\left|{n \atop k}\right|$ is given by (3) with $\beta' = 0$, then

$$\left|\begin{matrix} n \\ k + m \end{matrix}\right|\binom{k + m}{k} = \sum_{i=0}^{n}\sum_{j=0}^{n-i}\left|\begin{matrix} i \\ k \end{matrix}\right|^{*}\left|\begin{matrix} j \\ m \end{matrix}\right|^{*}\binom{n}{i}\binom{n - i}{j}\prod_{\ell=0}^{n-i-j-1}(\alpha\ell + \gamma), \tag{13}$$

where $\left|{n \atop k}\right|^{*}$ satisfies the same recurrence as $\left|{n \atop k}\right|$ but with $\gamma = 0$. If $\left|{n \atop k}\right|$ is given by (3) with $\gamma' = 0$, then

$$\left|\begin{matrix} n \\ k + m \end{matrix}\right| = \sum_{i=0}^{n}\sum_{j=0}^{n-i}\left|\begin{matrix} i \\ k \end{matrix}\right|^{*}\left|\begin{matrix} j \\ m \end{matrix}\right|^{*}\binom{n}{i}\binom{n - i}{j}\prod_{\ell=0}^{n-i-j-1}(\alpha\ell + \gamma), \tag{14}$$

where $\left|{n \atop k}\right|^{*}$ satisfies the same recurrence as $\left|{n \atop k}\right|$ but with $\gamma = 0$.

Proof. The left-hand side of (13) counts the members $\lambda = (\rho, \omega)$ in $\mathcal{B}_{n,k+m}$ in which k of the blocks of ω are distinguished. Alternatively, one can count these configurations by selecting i elements of $[n]$ to occupy the distinguished blocks and j

more elements of $[n]$ to occupy those that are not, and then arranging all of these elements in the blocks, which can be done in $\left|\begin{matrix} i \\ k \end{matrix}\right|^* \left|\begin{matrix} j \\ m \end{matrix}\right|^* \binom{n}{i} \binom{n-i}{j}$ ways. Then the remaining $n - i - j$ members of $[n]$ are arranged in a permutation as cycles which belong to none of the blocks of ω. They contribute weight $\prod_{l=0}^{n-i-j-1}(\gamma + \alpha \ell)$ since each remaining element may either start a new cycle or follow a smaller element within a prior cycle. Summing over all possible i and j gives the first relation. To show (14), use similar reasoning and let i now denote the number of elements of $[n]$ occupying the first k (block) positions of ω and j denote the number of elements occupying the last m positions. $\square$

Example 2.12. Taking $\alpha = \gamma' = 1$ in (13) and all other weights to be zero gives the following formula for $\left[\begin{matrix} n \\ k \end{matrix}\right]$ which occurs as (6.29) in [6]:

$$\left[\begin{matrix} n \\ k+m \end{matrix}\right] \binom{k+m}{k} = \sum_{i=0}^{n} \left[\begin{matrix} i \\ k \end{matrix}\right] \left[\begin{matrix} n-i \\ m \end{matrix}\right] \binom{n}{i}, \quad n, k, m \geq 0.$$

One may generalize the proof of the Bell number formula due to Spivey [17] to obtain a comparable formula for the row sum $\sum_{k=0}^{n} \left|\begin{matrix} n \\ k \end{matrix}\right|$ when $\beta' = 0$.

Proposition 2.13. Let $P(n) = \sum_{k=0}^{n} \left|\begin{matrix} n \\ k \end{matrix}\right|$, where $\left|\begin{matrix} n \\ k \end{matrix}\right|$ is given by (3) with $\beta' = 0$. If $n, m \geq 0$, then

$$P(n+m) = \sum_{i=0}^{n} \sum_{j=0}^{m} \left|\begin{matrix} m \\ j \end{matrix}\right| \binom{n}{i} P(n-i) \prod_{\ell=m}^{m+i-1} (\alpha \ell + \beta j). \tag{15}$$

Proof. Let $\mathcal{B}_n = \cup_{k=0}^{n} \mathcal{B}_{n,k}$. Condition on the number of blocks j of ω in $\lambda = (\rho, \omega) \in \mathcal{B}_{n+m}$ containing at least one member of $[m]$ and on the number of elements i in $[m+1, m+n]$ either which lie in a cycle of ρ belonging to one of these blocks or which lie in a cycle of ρ which does not belong to any of the blocks of ω but contains at least one member of $[m]$. There are then $\left|\begin{matrix} m \\ j \end{matrix}\right|$ ways to arrange the members of $[m]$ and $\binom{n}{i}$ choices for the aforementioned elements of $[m+1, m+n]$, with $\prod_{\ell=m}^{m+i-1}(\alpha \ell + \beta j)$ ways to arrange these elements relative to those in $[m]$. Finally, there are $P(n-i)$ ways to configure the remaining members of $[m+1, m+n]$ in new blocks and cycles. $\square$

Remark 2.14. Requiring the configurations to contain a fixed number of blocks in the prior proof yields the following refinement of (15):

$$\left|\begin{matrix} n+m \\ k \end{matrix}\right| = \sum_{i=0}^{n} \sum_{j=0}^{m} \left|\begin{matrix} m \\ j \end{matrix}\right| \binom{n}{i} \left|\begin{matrix} n-i \\ k-j \end{matrix}\right| \prod_{\ell=m}^{m+i-1} (\alpha \ell + \beta j). \tag{16}$$

Taking $\beta = \gamma' = 1$ in (15) and all other weights to be zero gives the following refinement of Spivey's formula in [17]:

$$\left\{\begin{matrix} n+m \\ k \end{matrix}\right\} = \sum_{i=0}^{n} \sum_{j=0}^{m} \left\{\begin{matrix} m \\ j \end{matrix}\right\} \binom{n}{i} \left\{\begin{matrix} n-i \\ k-j \end{matrix}\right\} j^i. \tag{17}$$

Other choices of the parameters yield similar formulas for other arrays such as $\left[\begin{matrix} n \\ k \end{matrix}\right]$.

2.3. A second recurrence

Solutions to the recurrence (1) may be given in some cases when $\alpha' \neq 0$. For example, the solution to (1) when $\alpha = -\beta$ may be obtained from Theorem 2.1 by replacing k with $n - k$ in (3) and renaming the coefficients, as noted in [16, Theorem 8], and the combinatorial proof above can be modified slightly to show this case. An algebraic solution to recurrence (1) when $\alpha' \neq 0$ and $\beta = \beta' = 0$ is given in [16, Theorem 9]. Here, we provide a combinatorial solution in this case and obtain a different formula for $\left|\begin{matrix} n \\ k \end{matrix}\right|$, one which applies when $\alpha' = 0$ as well.

Theorem 2.15. If the array $\left|\begin{matrix} n \\ k \end{matrix}\right|$ is defined by

$$\left|\begin{matrix} n \\ k \end{matrix}\right| = (\alpha(n-1) + \gamma) \left|\begin{matrix} n-1 \\ k \end{matrix}\right| + (\alpha'(n-1) + \gamma') \left|\begin{matrix} n-1 \\ k-1 \end{matrix}\right| + [n = k = 0], \tag{18}$$

then

$$\left|\begin{matrix} n \\ k \end{matrix}\right| = \sum_{i=0}^{n} \sum_{j=0}^{n} \left[\begin{matrix} n \\ i \end{matrix}\right] \binom{i}{n-j} \binom{n-i}{j-k} \alpha^{j-k} (\gamma \alpha')^{n-j} (\alpha')^{k-i} (\gamma')^{i+j-n}. \tag{19}$$

Proof. Given $0 \leq i \leq k \leq n$ and $\max\{k, n - i\} \leq j \leq n$, let $\mathcal{D}_{n,k,i,j}$ denote the set of ordered pairs $\tau = (\rho, S)$, where ρ is a member of $\mathscr{S}_{n,i}$ and S is a subset of $[n]$ of size $n - k$ such that exactly $n - j$ members of S lie within $[i]$. Define the weight of $\tau \in \mathcal{D}_{n,k,i,j}$, which we will denote by $w = w(\tau)$, by

$$w = \alpha^{j-k}\gamma^{n-j}(\alpha')^{n-i-(j-k)}(\gamma')^{i+j-n},$$

and let $D_{n,k,i,j} = |\mathcal{D}_{n,k,i,j}|w$. Let $\mathcal{D}_{n,k} = \cup_{i,j} \mathcal{D}_{n,k,i,j}$ and let $D_{n,k} = \sum_{i,j} D_{n,k,i,j}$. Note that

$$|\mathcal{D}_{n,k,i,j}| = \begin{bmatrix} n \\ i \end{bmatrix} \binom{i}{n-j} \binom{n-i}{j-k},$$

with the weight of each member of $\mathcal{D}_{n,k,i,j}$ given by w above. Summing over all i and j then shows that $D_{n,k}$ is given by the right-hand side of (19) above.

To complete the proof, we show that $D_{n,k}$ satisfies recurrence (18). In what follows, it will be convenient to represent the subset S defined above as a sequence of 1's and 0's, where a 1 occurs in the ℓ-th position if $\ell \in S$ and a 0 occurs if not. We will consider four cases concerning members of $\mathcal{D}_{n,k}$. First note that the total weight of all $\tau = (\rho, S) \in \mathcal{D}_{n,k}$ for which the 1-cycle (n) occurs in ρ and the i-th position of S is a 0, where i denotes the number of cycles of ρ, is given by $\gamma' D_{n-1,k-1}$. To see this, suppose $\tau' = (\rho', S') \in \mathcal{D}_{n-1,k-1}$ and that we obtain τ from τ' by adding the cycle (n) to ρ' and inserting a 0 just after the $(i - 1)$-st letter of S', where the number of cycles of ρ' is $i - 1$. Observe that the parameters n, k, i, and j all increase by 1 in the transition from τ' to τ. For n increasing implies k does since $|S'| = |S| = n - k$, and then either n or k increasing implies j does as well. Thus, we have $w(\tau) = \gamma' w(\tau')$ and since all such members $\tau \in \mathcal{D}_{n,k}$ arise as described from $\tau' \in \mathcal{D}_{n-1,k-1}$, it follows that the total weight of all these members of $\mathcal{D}_{n,k}$ is $\gamma' D_{n-1,k-1}$. By similar reasoning, the total weight of all the members $\tau = (\rho, S)$ of $\mathcal{D}_{n,k}$ for which (n) occurs in ρ and the i-th position of S is a 1, where i denotes the number of cycles of ρ, is given by $\gamma D_{n-1,k}$. Note that in this case, n and i both increase by one in the transition, with k and j remaining unchanged.

The total weight of all the members $\tau = (\rho, S) \in \mathcal{D}_{n,k}$ in which n occurs in a cycle of ρ with at least one member of $[n - 1]$ with S ending in a 1 is $\alpha(n - 1)\mathcal{D}_{n-1,k}$. Note that all such members of $\mathcal{D}_{n,k}$ may be obtained from members $\tau' = (\rho', S') \in \mathcal{D}_{n-1,k}$ by inserting the element n just after any member of $[n - 1]$ within a cycle of ρ' and then adding a 1 to the end of S'. In this transition from τ' to τ, note that the parameters n and j both increase, while k and i remain the same, whence $w(\tau) = \alpha w(\tau')$ (as n increasing and i constant implies j must increase since $S \cap [i] = S' \cap [i]$, with k remaining unchanged since $|S| = |S'| + 1$). By similar reasoning, the total weight of all the members $\mathcal{D}_{n,k}$ in which n occurs in a cycle of ρ with at least one member of $[n - 1]$ with S ending in a 0 is $\alpha'(n - 1)\mathcal{D}_{n-1,k-1}$. Combining the four cases above implies that $D_{n,k}$ satisfies recurrence (18), which completes the proof. $\square$

Example 2.16. If $\alpha = \alpha'$ and $\gamma = \gamma'$ in (18), then we have, by (19) and Vandermonde's identity,

$$\begin{vmatrix} n \\ k \end{vmatrix} = \sum_{i=0}^{n} \begin{bmatrix} n \\ i \end{bmatrix} \alpha^{n-i}\gamma^i \sum_{j=k}^{n} \binom{i}{n-j} \binom{n-i}{j-k}$$

$$= \binom{n}{k} \alpha^n \sum_{i=0}^{n} \begin{bmatrix} n \\ i \end{bmatrix} \left(\frac{\gamma}{\alpha}\right)^i = \binom{n}{k} \prod_{i=0}^{n-1} (\gamma + i\alpha),$$

which can also be seen combinatorially by modifying the proof of Theorem 2.15.

Remark 2.17. The solutions to (3) when $\beta = \beta' = 0$ and to (18) when $\alpha' = 0$ are seen to coincide, the corresponding explicit formulas (4) and (19) in the respective cases both reducing to

$$\begin{vmatrix} n \\ k \end{vmatrix} = (\gamma')^k \sum_{i=0}^{n} \begin{bmatrix} n \\ i \end{bmatrix} \binom{i}{k} \alpha^{n-i}\gamma^{i-k}.$$

Remark 2.18. If $\gamma = 0$ in (19), then we have

$$\begin{vmatrix} n \\ k \end{vmatrix} = \alpha^{n-k} \sum_{i=0}^{k} \begin{bmatrix} n \\ i \end{bmatrix} \binom{n-i}{k-i} (\alpha')^{k-i}(\gamma')^i.$$

In this case, a simpler combinatorial interpretation for $\begin{vmatrix} n \\ k \end{vmatrix}$ may be given as follows. Let $I_{n,k}$ denote the set of permutations σ belonging to $\mathscr{S}_{n,i}$ for some $i \in [k]$ in which exactly $k - i$ of the $n - i$ elements of $[n]$ not starting cycles of σ are marked. Then $\begin{vmatrix} n \\ k \end{vmatrix}$ gives the total weight of all members of $I_{n,k}$, where σ as described has weight $\alpha^{n-k}(\alpha')^{k-i}(\gamma')^i$, and it is seen to satisfy recurrence (18) when $\gamma = 0$ using this interpretation.

The following result for row sums of $\begin{vmatrix} n \\ k \end{vmatrix}$ occurs in [12] and [16]. Here, we provide a combinatorial proof by modifying the interpretation above for $\begin{vmatrix} n \\ k \end{vmatrix}$.

Theorem 2.19. *Suppose* $\left|{n \atop k}\right|$ *is given by*

$$\left|{n \atop k}\right| = (\alpha(n-1) + \beta k + \gamma) \left|{n-1 \atop k}\right| + (\alpha'(n-1) + \beta'(k-1) + \gamma') \left|{n-1 \atop k-1}\right| + [n = k = 0], \tag{20}$$

where $\beta + \beta' = 0$. *Then*

$$\sum_{k=0}^{n} \left|{n \atop k}\right| = \prod_{i=0}^{n-1} ((\alpha + \alpha')i + \gamma + \gamma'). \tag{21}$$

Proof. Let us first assume $\beta = \beta' = 0$. Let $\mathcal{D}_n = \cup_{k=0}^{n} \mathcal{D}_{n,k}$, where $\mathcal{D}_{n,k}$ is as in the proof of Theorem 2.15. Suppose $0 \leq i \leq n-1$ and $\tau = (\rho, S) \in \mathcal{D}_i$. In forming a member of $\mathcal{D}_{i+1}$ from τ, the number $i+1$ may be added as either a 1-cycle or just after a member of $[i]$ within a cycle of ρ. From the proof of Theorem 2.15 above, there are $\gamma + \gamma'$ choices for the former and $(\alpha + \alpha')i$ choices for the latter. Note that all the members of $\mathcal{D}_n$ may be formed by selecting independently the position of each member of $[n]$ as described starting with the first, which completes the proof in this case.

Now assume $\beta + \beta' = 0$ in general. Let $\mathcal{E}_{n,k,i,j}$ denote the set of configurations $\tau = (\rho, S)$ belonging to $\mathcal{D}_{n,k,i,j}$ considered in the prior proof with the following addition. Suppose $r \in [i]$ and that the r-th cycle of ρ is a 1-cycle. Suppose further that there are exactly t members of $[n]$ which do no start cycles of ρ, lie in cycles to the left of the r-th cycle, and are smaller than the element in the r-th cycle. If the r-th letter of S is a 1, then we may write the number r *below* some 0 in S corresponding to positions in either $[r-1]$ or $[i+1, i+t]$. If the r-th letter of S is a 0, then we may write the number r in one of these same positions but this time *above* the 0. In either case, we may also choose not to write the number r anywhere. Note that once all the possible numbers r have been written, one can have two or more different numbers written above and/or below the same 0 in S, but there can be at most one copy of any such given number throughout all of the positions of S.

We define the weight of $\tau = (p, S) \in \mathcal{E}_{n,k,i,j}$, which we will denote once again by $w = w(\tau)$, by

$$w = \alpha^{j-k} \gamma^{n-j-\ell_0} (\alpha')^{n-i-(j-k)} (\gamma')^{i+j-n-\ell_1} \beta^{\ell_0} (\beta')^{\ell_1},$$

where ℓ_0 and ℓ_1 denote the number of times a member of $[n]$ is written either below or above, respectively, some 0 in S as described. Let $\mathcal{E}_{n,k} = \cup_{i,j} \mathcal{E}_{n,k,i,j}$. Note that the total w-weight of all the members of $\mathcal{E}_{n,k}$ satisfies recurrence (20), upon reasoning as in the proof of Theorem 2.15 and considering two additional cases. Let $\mathcal{E}_n = \cup_{k=0}^{n} \mathcal{E}_{n,k}$ and $\tau = (\rho, S) \in \mathcal{E}_n$. Identify the smallest number $r \in [n]$ written either above or below some letter of S, which we will denote by r_o. We then move r_o to the other position and change the r_o-th letter of S to the other option, leaving ρ unchanged. If τ' denotes the resulting member of $\mathcal{E}_n$, then the mapping $\tau \mapsto \tau'$ is an involution of $\mathcal{E}_n$ which preserves the weight but changes the sign, since $\beta' = -\beta$. The mapping is not defined on those members of $\mathcal{E}_n$ such that $\ell_0 = \ell_1 = 0$, which are synonymous with members of $\mathcal{D}_n$ which have already been counted. $\square$

The solution given in [16, Theorem 9] to (18) when $\alpha' \neq 0$ is algebraic and the explicit formula that results is

$$\left|{n \atop k}\right| = \sum_{i=0}^{n} \sum_{j=0}^{n} \left[{n \atop i}\right] \binom{i}{n-j} \binom{j}{k} \alpha^{j-k} (\gamma \alpha' - \alpha \gamma')^{n-j} (\alpha')^{k-i} (\gamma')^{i+j-n}. \tag{22}$$

One can show bijectively that this equals the expression given in (19).

Proposition 2.20. *If* $\alpha' \neq 0$, *then the formulas for* $\left|{n \atop k}\right|$ *given on the right-hand sides of* (19) *and* (22) *are equal for all* $n, k \geq 0$.

Proof. We first consider the case when $\gamma = 0$ and show that

$$\sum_{i=0}^{k} \left[{n \atop i}\right] \binom{n-i}{k-i} (\alpha')^{k-i} (\gamma')^i = \sum_{i=0}^{n} \sum_{j=0}^{n} \left[{n \atop i}\right] \binom{i}{n-j} \binom{j}{k} (-1)^{n-j} (\alpha')^{k-i} (\gamma')^i. \tag{23}$$

To do so, given $0 \leq k, i \leq n$ and $\max\{n-i, k\} \leq j \leq n$, let $\mathcal{F}_{n,k,i,j}$ be the set of ordered pairs (ρ, T) in which $\rho \in \mathcal{S}_{n,i}$ and T is a sequence of length n containing k 0's, $n-j$ 1's, and $j-k$ 2's, where all of the 1's lie within the first i positions of T. Note that $|\mathcal{F}_{n,k,i,j}| = \left[{n \atop i}\right] \binom{i}{n-j} \binom{j}{k}$. Define the (signed) weight of $\tau \in \mathcal{F}_{n,k,i,j}$ by

$$u(\tau) = (-1)^{n-j} (\alpha')^{k-i} (\gamma')^i.$$

Since $k - i < 0$ is permissible, we require $\alpha' \neq 0$. Let $\mathcal{F}_{n,k} = \cup_{i,j} \mathcal{F}_{n,k,i,j}$. The right-hand side of (23) then gives the total u-weight of all the members of $\mathcal{F}_{n,k}$.

Given $\tau = (\rho, T) \in \mathcal{F}_{n,k,i,j}$, let r_o be the smallest index r, if it exists, in $[i]$ such that the r-th position of T is either a 1 or a 2. Let τ' be obtained from τ by changing r_o to the other option, leaving the rest of τ unchanged. Note that τ belongs to either $\mathcal{F}_{n,k,i,j+1}$ or $\mathcal{F}_{n,k,i,j-1}$, whence $u(\tau') = -u(\tau)$, and the mapping $\tau \mapsto \tau'$ is an involution, where it is defined. The mapping is

not defined when $j = n$ and the first i positions of T are all 0's. The sign of such members of $\mathcal{F}_{n,k}$ is positive and the left-hand side of (23) gives their total weight according to the number of cycles i of ρ. This completes the case when $\gamma = 0$.

In general, let $\mathcal{G}_{n,k,i,j}$ denote the same set of ordered pairs τ belonging to $\mathcal{F}_{n,k,i,j}$ as described above but where some of the 1's in T may now be marked. If $\tau \in \mathcal{G}_{n,k,i,j}$ has exactly ℓ marked 1's, where $0 \le \ell \le n - j$, then define the weight of τ by

$$u(\tau) = (-1)^{n-j-\ell} \alpha^{n-k-\ell} \gamma^{\ell} (\alpha')^{k-i+\ell} (\gamma')^{i-\ell}.$$

Let $\mathcal{G}_{n,k} = \cup_{i,j} \mathcal{G}_{n,k,i,j}$. Then the right-hand side of (22) is seen to give the total u-weight of all the members $\mathcal{G}_{n,k}$, upon writing the weights as

$$u(\tau) = \alpha^{j-k} (\gamma\alpha')^{\ell} (-\alpha\gamma')^{n-j-\ell} (\alpha')^{k-i} (\gamma')^{i+j-n}.$$

We extend the sign-reversing involution defined above on $\mathcal{F}_{n,k}$ to $\mathcal{G}_{n,k}$ by letting r_o denote the smallest index in $[i]$ corresponding either to a 2 or to a 1 that is not marked and switching to the other option. The set of survivors of this involution are those members $\tau = (\rho, T) \in \mathcal{G}_{n,k}$ in which $\rho \in \mathcal{S}_{n,i}$ for some i and T is a sequence consisting of k 0's, $n - j$ 1's, and $j - k$ 2's for some j, where each 1 is marked and no 2 lies within the first i positions of T. Such members of $\mathcal{G}_{n,k}$ have a positive sign as they contain no unmarked 1's and their total weight is given by the right-hand side of (19) above, which completes the proof. $\square$

3. A q-generalization

In this section, we consider a q-generalization of the numbers $\left|{n \atop k}\right|$ satisfying recurrence (3). Recall that if q is an indeterminate, then $n_q = 1 + q + \cdots + q^{n-1}$ if $n \ge 1$, with $0_q = 0$, and $n_q! = 1_q 2_q \cdots n_q$ if $n \ge 1$, with $0_q! = 1$. We now recall q-generalizations of the two kinds of Stirling numbers.

Carlitz [4] considered the q-generalization of $\left[{n \atop k}\right]$ satisfying the recurrence

$$\left[{n \atop k}\right]_q = \left[{n-1 \atop k-1}\right]_q + (n-1)_q \left[{n-1 \atop k}\right]_q, \quad n, k \ge 1,$$

with $\left[{n \atop 0}\right]_q = \delta_{n,0}$ and $\left[{0 \atop k}\right]_q = \delta_{0,k}$ if $n, k \ge 0$. Given $\sigma \in \mathcal{S}_{n,k}$, let us denote here by $inv_1(\sigma)$ the number of inversions in the word obtained by expressing σ in standard cycle form and then erasing the parentheses which enclose the cycles. For example, if $\sigma = 651423 = (163)(25)(4) \in \mathcal{S}_{6,3}$, then $inv_1(\sigma) = 1 + 1 + 4 = 6$, the number of inversions in 163254. Carlitz [4] observed that $\left[{n \atop k}\right]_q$ is the distribution polynomial for the inv_1 statistic on $\mathcal{S}_{n,k}$, that is,

$$\left[{n \atop k}\right]_q = \sum_{\sigma \in \mathcal{S}_{n,k}} q^{inv_1(\sigma)}.$$

We also recall the q-generalization of $\left\{{n \atop k}\right\}$ satisfying the recurrence

$$\left\{{n \atop k}\right\}_q = \left\{{n-1 \atop k-1}\right\}_q + k_q \left\{{n-1 \atop k}\right\}_q, \quad n, k \ge 1,$$

with $\left\{{n \atop 0}\right\}_q = \delta_{n,0}$ and $\left\{{0 \atop k}\right\}_q = \delta_{0,k}$ if $n, k \ge 0$. The $\left\{{n \atop k}\right\}_q$ were first considered by Carlitz [5] who derived an elegant explicit formula for them. Later, they were given various combinatorial interpretations by Carlitz [4], Milne [11], Sagan [14], and Wachs and White [19].

We now recall a partition statistic considered in [11] and [14]. Given $\pi = B_1/B_2 \cdots /B_k \in \mathcal{P}_{n,k}$, expressed in standard form, let us denote here by $inv_2(\pi)$ the number of ordered pairs (a, B_j), where $a \in B_i$, $i < j$, and $a > \min B_j$ (sometimes called *block inversions*). For example, the partition

$$\pi = 17/24/358/69 = B_1/B_2/B_3/B_4$$

has inversions $(7, B_2)$, $(4, B_3)$, $(7, B_3)$, $(7, B_4)$ and $(8, B_4)$, so $inv_2(\pi) = 5$. The distribution of the inv_2 statistic on $\mathcal{P}_{n,k}$ is $\left\{{n \atop k}\right\}_q$ since it is seen to satisfy the same recurrence and initial conditions, as observed in [11].

We now state our generalization of Theorem 2.1. One can provide a combinatorial proof for it by replacing normal counting in the proof of Theorem 2.1 above with q-counting in the appropriate places.

Theorem 3.1. *Suppose p and q are indeterminates. Let $\left|{n \atop k}\right|_{p,q}$ denote the array defined by the recurrence*

$$\left|{n \atop k}\right|_{p,q} = (\alpha(n-1)_p + \beta k_q + \gamma) \left|{n-1 \atop k}\right|_{p,q} + (\beta' k_q + \gamma') \left|{n-1 \atop k-1}\right|_{p,q} + [n = k = 0], \quad n, k \ge 0. \tag{24}$$

Then $\left|\begin{smallmatrix} n \\ k \end{smallmatrix}\right|_{p,q}$ is given explicitly by

$$\left|\begin{matrix} n \\ k \end{matrix}\right|_{p,q} = \prod_{i=1}^{k}(\beta' i_q + \gamma') \sum_{i=0}^{n}\sum_{j=0}^{n}\left[\begin{matrix} n \\ i \end{matrix}\right]_p\binom{i}{j}\left\{\begin{matrix} j \\ k \end{matrix}\right\}_q \alpha^{n-i}\beta^{j-k}\gamma^{i-j}. \tag{25}$$

Furthermore, using the notation from the proof of Theorem 2.1, we have

$$\left|\begin{matrix} n \\ k \end{matrix}\right|_{p,q} = \sum_{\lambda=(\rho,\omega)\in\mathcal{B}_{n,k}} p^{inv_1(\rho)}q^{inv_2(\omega)}v(\lambda). \tag{26}$$

In (26), it is understood that inv_2, which was introduced as a statistic on ordinary set partitions, is extended to ordered set partitions in the obvious way. Choosing certain values of the parameters in (25) gives previously considered q-generalizations. For example, choosing $\alpha = \gamma' = 1$, with all other parameters zero, gives the q-Stirling numbers $\left[\begin{smallmatrix} n \\ k \end{smallmatrix}\right]_q$, and, similarly, one can obtain $\left\{\begin{smallmatrix} n \\ k \end{smallmatrix}\right\}_q$. Choosing $\alpha = \beta = \gamma = 0$, and taking $k = n$, gives a generalized q-factorial. By suitably modifying the arguments, it is possible to obtain generalizations of some of the identities satisfied by $\left|\begin{smallmatrix} n \\ k \end{smallmatrix}\right|$, a couple of which are as follows.

Proposition 3.2. *If $n, k \geq 0$, then*

$$\left|\begin{matrix} n+k+1 \\ k \end{matrix}\right|_{p,q} = \sum_{i=0}^{k}(\alpha(n+i)_p + \beta i_q + \gamma)\left|\begin{matrix} n+i \\ i \end{matrix}\right|_{p,q}\prod_{j=i+1}^{k}(\beta' j_q + \gamma'), \tag{27}$$

where $\left|\begin{smallmatrix} n \\ k \end{smallmatrix}\right|_{p,q}$ is given by (24).

Proposition 3.3. *If $n, k, m \geq 0$, then*

$$\left|\begin{matrix} n+m \\ k \end{matrix}\right|_{p,q} = \sum_{i=0}^{n}\sum_{j=0}^{m}\left|\begin{matrix} m \\ j \end{matrix}\right|_{p,q}\binom{n}{i}\left|\begin{matrix} n-i \\ k-j \end{matrix}\right|_{p,q}\prod_{\ell=m}^{m+i-1}(\alpha\ell_p + \beta j_q), \tag{28}$$

where $\left|\begin{smallmatrix} n \\ k \end{smallmatrix}\right|_{p,q}$ is given by (24) with $\beta' = 0$.

4. Conclusion

Here, we have provided combinatorial proofs of explicit formulas, which had been shown earlier using algebraic methods, for arrays $\left|\begin{smallmatrix} n \\ k \end{smallmatrix}\right|$ satisfying two kinds of general recurrences. Using combinatorial reasoning, we were also able to provide bijective proofs of some prior identities as well as deduce some new ones. In addition, we introduced a polynomial generalization by considering the joint distribution of two statistics counting different kinds of inversions on the underlying structure associated with the first explicit formula. While a combinatorial interpretation may be given for the numbers satisfying recurrence (1) in its full generality (see proof of Theorem 2.19 above), it does not seem to yield a general explicit formula such as (4). Such a formula may be given for $\left|\begin{smallmatrix} n \\ k \end{smallmatrix}\right|$ in the case when $\frac{\alpha}{\beta} = \frac{\alpha'}{\beta'} + 1$, see [16, Theorem 18], and it would be interesting to have a combinatorial proof. Finally, we seek further identities for the sequences defined by (3) or (18), such as orthogonality relations.

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