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Identities Involving Generalized Harmonic Numbers and Other Special Combinatorial Sequences

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Abstract

In this paper, we study the properties of the generalized harmonic numbers $H_{n,k,r}(\alpha,\beta)$. In particular, by means of the method of coefficients, generating functions and Riordan arrays, we establish some identities involving the numbers $H_{n,k,r}(\alpha,\beta)$, Cauchy numbers, generalized Stirling numbers, Genocchi numbers and higher order Bernoulli numbers. Furthermore, we obtain the asymptotic values of some summations associated with the numbers $H_{n,k,r}(\alpha,\beta)$ by Darboux's method and Laplace's method.

1 Introduction

Harmonic numbers are important in various branches of combinatorics and number theory, and they also frequently appear in the analysis of algorithms and expressions for special functions. Recently, many papers have been devoted to the study of harmonic number identities by various methods; see, for instance, [1, 2, 3, 4, 5, 6, 7]. We recall the definition of harmonic numbers $H_n = \sum_{k=1}^{n} \frac{1}{k}$ for $n \geq 0$. The generating function of H_n is $\sum_{n=1}^{\infty} H_n t^n =$

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$-\frac{\ln(1-t)}{1-t}$. In this paper, we discuss a class of generalized harmonic numbers $H_{n,k,r}(\alpha,\beta)$. We refer to Zhao and Wuyungaowa [10] for this topic. The definition of $H_{n,k,r}(\alpha,\beta)$ is

$$\sum_{n=0}^{\infty} H_{n,k,r}(\alpha,\beta)t^n = \frac{(-\ln(1-\alpha t))^r}{(1-\beta t)^k}, \tag{1}$$

where $k \geq 1$ and $r \geq 1$ are integers. Let (α,β) be a pair of real numbers and $(\alpha\beta \neq 0)$. From the generating function of $H_{n,k,r}(\alpha,\beta)$, we know that $H_{n,1,1}(1,1) = H_n (n \geq 0)$.

From (1) we obtain

$$\begin{aligned}
(-\ln(1-\alpha t))^r &= (1-\beta t)^k \sum_{n=0}^{\infty} H_{n,k,r}(\alpha,\beta)t^n \\
&= \sum_{i=0}^{k} \binom{k}{i} (-\beta t)^i \sum_{n=0}^{\infty} H_{n,k,r}(\alpha,\beta)t^n \\
&= \sum_{n=0}^{\infty} \sum_{h=0}^{k} \binom{k}{h} (-\beta)^h H_{n-h,k,r}(\alpha,\beta)t^n.
\end{aligned} \tag{2}$$

For convenience, let us recall some definitions and notations. Denote the generalized Stirling numbers of the first kind by $s(n,k;r)$, and the generalized Stirling numbers of the second kind by $S(n,k;r)$. Denote further $C_n^{(k)}$, $\hat{C}_n^{(k)}$, $B_n^{(r)}$, $G_n^{(x)}$, $G_n^{(k)}$ be the higher order Cauchy numbers of both kinds, higher order Bernoulli numbers, the generalized Genocchi numbers, the higher order Genocchi numbers and the generalized Lah numbers. These numbers satisfy the following generating functions respectively:

$$\sum_{n=k}^{\infty} s(n,k;r) \frac{(-1)^{n-k} t^n}{n!} = \frac{\ln^k(1+t)}{(1+t)^r k!}, \quad k = 0,1,2,\ldots, \tag{3}$$

$$\sum_{n=k}^{\infty} |s(n,k;r)| \frac{t^n}{n!} = \frac{(-\ln(1-t))^k}{(1-t)^r k!}, \quad k = 0,1,2,\ldots, \tag{4}$$

$$\sum_{n=k}^{\infty} S(n,k;r) \frac{t^n}{n!} = \frac{e^{rt}(e^t-1)^k}{k!}, \quad k = 0,1,2,\ldots, \tag{5}$$

$$\sum_{n=0}^{\infty} C_n^{(k)} \frac{t^n}{n!} = \left(\frac{t}{\ln(1+t)}\right)^k, \tag{6}$$

$$\sum_{n=k}^{\infty} \hat{C}_n^{(k)} \frac{t^n}{n!} = \left(\frac{t}{(1+t)\ln(1+t)}\right)^k, \tag{7}$$

$$\sum_{n=0}^{\infty} G_n^{(k)} \frac{t^n}{n!} = \left(\frac{2t}{e^t+1}\right)^k, \tag{8}$$

$$\sum_{n=0}^{\infty} \frac{G_n^{(x)}}{2^n} \frac{t^n}{n!} = \left(\frac{2}{e^t+1}\right)^x, \tag{9}$$

$$\sum_{n=0}^{\infty} B_n^{(r)} \frac{t^n}{n!} = \left(\frac{t}{e^t-1}\right)^r, \tag{10}$$

$$\sum_{n=k}^{\infty} L(n,k;r) \frac{t^n}{n!} = (1+t)^r \frac{1}{k!}\left(\frac{-t}{1+t}\right)^k, \tag{11}$$

$$\sum_{n=0}^{\infty} H_n^{(r)}(z) t^n = \frac{(-\ln(1-t))^{r+1}}{t(1-t)^{1-z}}, \tag{12}$$

$$\sum_{n=1}^{\infty} H_n^{(r)} t^n = \frac{-\ln(1-t)}{(1-t)^r}. \tag{13}$$

Let $[t^n]f(t)$ be the coefficient of t^n in the formal power series of $f(t)$, where $f(t) = \sum_{n=0}^{\infty} f_n t^n$. (See Merlini, Sprugnoli, and Verri [8] for related topics.) If $f(t)$ and $g(t)$ are formal power series, we get the following relations:

$$[t^n](\alpha f(t) + \beta g(t)) = \alpha [t^n] f(t) + \beta [t^n] g(t), \tag{14}$$

$$[t^n] f(t) = [t^{n-1}] f(t), \tag{15}$$

$$[t^n] f(t) g(t) = \sum_{j=0}^{n} [y^j] f(y) [t^{n-j}] g(t). \tag{16}$$

A *Riordan array* is a pair $(d(t), h(t))$ of formal power series with $h_0 = h(0) = 0$. It defines an infinite lower triangular array $(d_{n,k})_{n,k\in N}$ according to the rule

$$d_{n,k} = [t^n] d(t) (h(t))^k .$$

Hence we write $\{d_{n,k}\} = (d(t), h(t))$. Moreover, if $(d(t), h(t))$ is a Riordan array and $f(t)$ is the generating function of the sequence $\{f_k\}_{k\in N}$, i.e., $f(t) = \sum_{k=0}^{\infty} f_k t^k$, then we have

$$\sum_{k=0}^{\infty} d_{n,k} f_k = [t^n] d(t) f(h(t)) = [t^n] d(t) [f(y) \mid y = h(t)]. \tag{17}$$

Furthermore, based on the generating function (1) we obtain the next three Riordan arrays:

$$\{H_{n,k,r}(\alpha,\beta)\} = \left(\frac{1}{(1-\beta t)^k}, \frac{-\ln(1-\alpha t)}{t}\right), \tag{18}$$

$$\{H_{n,k,r+1}(\alpha,\beta)\} = \left(\frac{-\ln(1-\alpha t)}{(1-\beta t)^k}, \frac{-\ln(1-\alpha t)}{t}\right), \tag{19}$$

$$\{H_{n,k,r}(\alpha,\alpha)\} = \left(\frac{1}{(1-\alpha t)^k}, \frac{-\ln(1-\alpha t)}{t}\right). \tag{20}$$

In this paper, we pay particular attention to the three Riordan arrays above.

2 Identities involving $H_{n,k,r}(\alpha,\beta)$, $s(n,k;r)$, $S(n,k;r)$, $B_n^{(r)}$ and $L(n,k;r)$

Theorem 1. *Let $k,\ r,\ m \geq 1,\ l \geq 0$ be integers. Then*

$$\sum_{j=0}^{n}\sum_{h=0}^{k}\binom{k}{h}(-\beta)^h H_{j-h,k,r}(\alpha,\beta)|s(n-j,m;l)|\frac{\alpha^{n-j}}{(n-j)!} = \frac{1}{m!}H_{n,l,m+r}(\alpha,\alpha). \tag{21}$$

Proof. By applying (2), (4) and (16), we get

$$\begin{aligned}&\sum_{j=0}^{n}\sum_{h=0}^{k}\binom{k}{h}(-\beta)^h H_{j-h,k,r}(\alpha,\beta)|s(n-j,m;l)|\frac{\alpha^{n-j}}{(n-j)!}\\ &= [t^n]\frac{(-\ln(1-\alpha t))^{m+r}}{(1-\alpha t)^l m!} = \frac{1}{m!}H_{n,l,m+r}(\alpha,\alpha)\,.\end{aligned}$$

□

Theorem 2. *Let $n,\ k,\ j \geq 1,\ l \geq 0$ be integers. Then*

$$\sum_{j=m}^{n} H_{n,k,j}(\alpha,\beta)S(j,m;l)\frac{m!}{j!} = \sum_{i=0}^{n-m}\binom{i+k-1}{i}\binom{n-i+l-1}{n-m-i}\beta^i\alpha^{n-i}. \tag{22}$$

Proof. By using (5), (17) and (18), we obtain

$$\begin{aligned}&\sum_{j=m}^{n} H_{n,k,j}(\alpha,\beta)S(j,m;l)\frac{m!}{j!} = [t^n]\frac{1}{(1-\beta t)^k}[(e^y-1)^m e^{yl} \mid y = -\ln(1-\alpha t)]\\ &= \alpha^m[t^{n-m}]\frac{1}{(1-\beta t)^k}\frac{1}{(1-\alpha t)^{l+m}} = \sum_{i=0}^{n-m}\binom{i+k-1}{i}\binom{n-i+l-1}{n-m-i}\beta^i\alpha^{n-i}\,.\end{aligned}$$

□

Corollary 3. *The following relations hold:*

$$\sum_{j=0}^{n}\sum_{h=0}^{k}\binom{k}{h}(-\beta)^h H_{j-h,k,r}(\alpha,\beta)|s(n-j,m)|\frac{\alpha^{n-j}}{(n-j)!} = \frac{\alpha^n}{m!}|s(n,m+r)|\frac{(m+r)!}{n!}, \tag{23}$$

$$\sum_{j=m}^{n} H_{n,k,j}(\alpha,\alpha)S(j,m;l)\frac{m!}{j!} = \binom{n+l+k-1}{n-m}\alpha^n, \tag{24}$$

$$\sum_{j=m}^{n} H(n,r-1)S(j,m;l)\frac{m!}{j!} = \binom{n+l}{n-m}. \tag{25}$$

Proof. Setting $l=0$ in (21), we get (23). Setting $\beta=\alpha$ in (22), we have (24). Setting $\beta=\alpha=k=1$ in (22), we obtain (25). □

Theorem 4. *Let $n,\ k \geq 1$ and $l,\ j \geq 0$ be integers. Then*

$$\sum_{j=m}^{n} H_{n,k,j+1}(\alpha,\beta)S(j,m;l)\frac{m!}{j!} = \sum_{i=0}^{n-m} H_{i,k,1}(\alpha,\beta)\binom{n-i+l-1}{n-m-i}\alpha^{n-i}. \tag{26}$$

Proof. By applying (5), (17) and (19), we have

$$\sum_{j=m}^{n} H_{n,k,j+1}(\alpha,\beta)S(j,m;l)\frac{m!}{j!} = [t^n]\frac{-\ln(1-\alpha t)}{(1-\beta t)^k}[(e^y-1)^m e^{yl} \mid y = -\ln(1-\alpha t)]$$
$$= \alpha^m [t^{n-m}]\frac{-\ln(1-\alpha t)}{(1-\beta t)^k}\frac{1}{(1-\alpha t)^{l+m}} = \sum_{i=0}^{n-m} H_{i,k,1}(\alpha,\beta)\binom{n-i+l-1}{n-m-i}\alpha^{n-i}.$$

□

Corollary 5. *The following relations hold:*

$$\sum_{j=m}^{n} H_{n,k,j+1}(\alpha,\alpha)S(j,m;l)\frac{m!}{j!} = \alpha^n H_{n-m-1}(1-m-l-k), \tag{27}$$
$$\sum_{j=m}^{n} H_{n,k,j+1}(\alpha,\alpha)S(j,m;l)\frac{m!}{j!} = \alpha^n H_{n-m}^{(m+l+k)}. \tag{28}$$

Proof. Setting $\beta = \alpha$ in (26), we obtain (27) and (28). □

Theorem 6. *Let $k,\ r \geq 1$ and $m,\ l \geq 0$ be integers. Then*

$$\sum_{j=0}^{n}\sum_{h=0}^{k}\binom{k}{h}(-\beta)^h H_{j-h,k,r}(\alpha,\beta)\frac{L(n-j,m;l)}{(n-j)!}(-\alpha)^{n-j}$$
$$= \begin{cases} \sum_{i=r}^{l-m}\binom{l-m}{i}|s(n-m-i,r)|\frac{(-1)^i\alpha^n r!}{m!(n-m-i)!}, & \text{if } l > m;\\ \frac{\alpha^n}{m!}|s(n-m,r)|\frac{r!}{(n-m)!}, & \text{if } l = m;\\ \frac{\alpha^m}{m!}H_{n-m,m-l,r}(\alpha,\alpha), & \text{if } l < m. \end{cases} \tag{29}$$

Proof. By applying (2), (11) and (16), we get

$$\sum_{j=0}^{n}\sum_{h=0}^{k}\binom{k}{h}(-\beta)^h H_{j-h,k,r}(\alpha,\beta)\frac{L(n-j,m;l)}{(n-j)!}(-\alpha)^{n-j}$$
$$= [t^n](-\ln(1-\alpha t))^r\frac{(1-\alpha t)^l}{m!}\left(\frac{\alpha t}{1-\alpha t}\right)^m$$
$$= \begin{cases} \sum_{i=r}^{l-m}\binom{l-m}{i}|s(n-m-i,r)|\frac{(-1)^i\alpha^n r!}{m!(n-m-i)!}, & \text{if } l > m;\\ \frac{\alpha^n}{m!}|s(n-m,r)|\frac{r!}{(n-m)!}, & \text{if } l = m;\\ \frac{\alpha^m}{m!}H_{n-m,m-l,r}(\alpha,\alpha), & \text{if } l < m. \end{cases}$$

□

Theorem 7. *Let $n, \ j, \ k, \ m \geq 1$ be integers. Then*

$$\sum_{j=1}^{n} H_{n,k,j}(\alpha,\alpha)\frac{B_j^{(m)}}{j!} = \begin{cases} \frac{1}{\alpha^m} H_{n+m,k-m,m}(\alpha,\alpha), & \text{if } k > m; \\ \alpha^n |s(n+m,m)| \frac{m!}{(n+m)!}, & \text{if } k = m; \\ \sum_{i=0}^{m-k} \binom{m-k}{i} |s(n+m-i,m)| \frac{(-1)^i \alpha^n m!}{(n+m-i)!}, & \text{if } k < m. \end{cases} \tag{30}$$

Proof. By applying (10), (17) and (20), we get

$$\begin{aligned} \sum_{j=1}^{n} H_{n,k,j}(\alpha,\alpha)\frac{B_j^{(m)}}{j!} &= [t^n]\frac{1}{(1-\alpha t)^k}\left[\left(\frac{y}{e^y-1}\right)^m \mid y = -\ln(1-\alpha t)\right] \\ &= \begin{cases} \frac{1}{\alpha^m} H_{n+m,k-m,m}(\alpha,\alpha), & \text{if } k > m; \\ \alpha^n |s(n+m,m)| \frac{m!}{(n+m)!}, & \text{if } k = m; \\ \sum_{i=0}^{m-k} \binom{m-k}{i} |s(n+m-i,m)| \frac{(-1)^i \alpha^n m!}{(n+m-i)!}, & \text{if } k < m. \end{cases} \end{aligned}$$

□

3 Identities involving $H_{n,k,r}(\alpha,\beta)$, Genocchi numbers and Cauchy numbers

Theorem 8. *Let x be a real number, $k, \ i \geq 1$, $n \geq 0$ be integers. Then*

$$\sum_{i=1}^{n} H_{n,k,i}(\alpha,\beta)\frac{G_i^{(x)}}{2^i i!} = \sum_{h=0}^{x}\sum_{j=0}^{n-h}\binom{n-h-j+k-1}{k-1}\binom{j+x-1}{x-1}\binom{x}{h}\frac{(-1)^h}{2^j}\beta^{n-j-h}\alpha^{j+h}. \tag{31}$$

Proof. By using (9), (17) and (18), we get

$$\begin{aligned} \sum_{i=1}^{n} H_{n,k,i}(\alpha,\beta)\frac{G_i^{(x)}}{2^i i!} &= [t^n]\frac{1}{(1-\beta t)^k}\left[\left(\frac{2}{e^y+1}\right)^x \Bigm| y = -\ln(1-\alpha t)\right] \\ &= [t^n]\frac{1}{(1-\beta t)^k}\frac{1}{(1-\frac{\alpha t}{2})^x}(1-\alpha t)^x \\ &= \sum_{h=0}^{x}\sum_{j=0}^{n-h}\binom{n-h-j+k-1}{k-1}\binom{j+x-1}{x-1}\binom{x}{h}\frac{(-1)^h}{2^j}\beta^{n-j-h}\alpha^{j+h}. \end{aligned}$$

□

Theorem 9. *Let x be a real number, $k, \ j \geq 1$ be integers. Then*

$$\sum_{j=1}^{\infty} H_{n,k,j}(\alpha,\alpha)\frac{G_j^{(x)}}{2^j j!} = \begin{cases} \sum_{i=0}^{n} \binom{i+k-x-1}{i}\binom{x+n-i-1}{n-i}\frac{\alpha^n}{2^{n-i}}, & \text{if } k > x; \\ \binom{n+k-1}{n}\frac{\alpha^n}{2^n}, & \text{if } k = x; \\ \sum_{i=0}^{x-k}\binom{x-k}{i}\binom{x+n-i-1}{n-i}\frac{(-1)^i\alpha^n}{2^{n-i}}, & \text{if } k < x. \end{cases} \tag{32}$$

Proof. By using (9), (17) and (20), we have

$$\sum_{j=1}^{\infty} H_{n,k,j}(\alpha,\alpha)\frac{G_j^{(x)}}{2^j j!} = [t^n]\frac{1}{(1-\alpha t)^k}\left[\left(\frac{2}{e^y+1}\right)^x \;\middle|\; y=-\ln(1-\alpha t)\right]$$

$$= \begin{cases} \sum_{i=0}^{n}\binom{i+k-x-1}{i}\binom{x+n-i-1}{n-i}\frac{\alpha^n}{2^{n-i}}, & \text{if } k>x; \\ \binom{n+k-1}{n}\frac{\alpha^n}{2^n}, & \text{if } k=x; \\ \sum_{i=0}^{x-k}\binom{x-k}{i}\binom{x+n-i-1}{n-i}\frac{(-1)^i\alpha^n}{2^{n-i}}, & \text{if } k<x. \end{cases}$$

□

Corollary 10. *The following relations hold:*

$$\sum_{j=1}^{n} H_{n,k,j}(\alpha,\alpha)\frac{G_j}{2^j j!} = \sum_{i=0}^{n}\binom{i+k-2}{i}\frac{\alpha^n}{2^{n-i}}, \tag{33}$$

$$\sum_{j=1}^{n} H(n,j-1)\frac{G_j}{2^j j!} = \frac{1}{2^n}. \tag{34}$$

Proof. Setting $x=1$ in (32), we have (33). Setting $x=k=1$ in (32), we obtain (34). □

Theorem 11. *Let $k,\ l,\ m\geq 1$ be integers. Then*

$$\sum_{l=1}^{n} H_{n,k,l}(\alpha,\beta)\frac{G_l^{(m)}}{l!} = \sum_{j=0}^{m}\sum_{i=0}^{n-j} H_{i,k,m}(\alpha,\beta)\binom{m}{j}\binom{n-i-j+m-1}{m-1}\frac{(-1)^j\alpha^{n-i}}{2^{n-i-j}}. \tag{35}$$

Proof. By applying (8), (17) and (18), we have

$$\sum_{l=1}^{n} H_{n,k,l}(\alpha,\beta)\frac{G_l^{(m)}}{l!} = [t^n]\frac{1}{(1-\beta t)^k}\left[\left(\frac{2y}{e^y+1}\right)^m \;\middle|\; y=-\ln(1-\alpha t)\right]$$

$$= [t^n]\frac{(-\ln(1-\alpha t))^m}{(1-\beta t)^k}\frac{1}{(1-\frac{\alpha t}{2})^m}(1-\alpha t)^m$$

$$= \sum_{j=0}^{m}\sum_{i=0}^{n-j} H_{i,k,m}(\alpha,\beta)\binom{m}{j}\binom{n-i-j+m-1}{m-1}\frac{(-1)^j\alpha^{n-i}}{2^{n-i-j}}.$$

□

Theorem 12. *Let $n,\ k,\ j,\ m\geq 1$ be integers. Then*

$$\sum_{j=1}^{n} H_{n,k,j}(\alpha,\alpha)\frac{G_j^{(m)}}{j!} = \begin{cases} \sum_{i=0}^{n} H_{n-i,k-m,m}(\alpha,\alpha)\binom{i+m-1}{m-1}\frac{\alpha^i}{2^i}, & \textit{if } k>m; \\ \sum_{i=m}^{n}|s(i,m)|\frac{m!}{i!}\binom{n-i+m-1}{m-1}\frac{\alpha^n}{2^{n-i}}, & \textit{if } k=m; \\ \sum_{i=0}^{m-k}\sum_{j=0}^{n-i}\binom{m-k}{i}\binom{j+m-1}{m-1}|s(n-i-j,m)|\frac{(-1)^i\alpha^n m!}{2^j(n-i-j)!}, & \textit{if } k<m. \end{cases} \tag{36}$$

Proof. By using (8), (17) and (20), we have

$$\sum_{j=1}^{n} H_{n,k,j}(\alpha,\alpha)\frac{G_j^{(m)}}{j!} = [t^n]\frac{1}{(1-\alpha t)^k}\left[\left(\frac{2y}{e^y+1}\right)^m \,\middle|\, y=-\ln(1-\alpha t)\right]$$

$$= \begin{cases} \sum_{i=0}^{n} H_{n-i,k-m,m}(\alpha,\alpha)\binom{i+m-1}{m-1}\frac{\alpha^i}{2^i}, & \text{if } k>m; \\ \sum_{i=m}^{n} |s(i,m)|\frac{m!}{i!}\binom{n-i+m-1}{m-1}\frac{\alpha^n}{2^{n-i}}, & \text{if } k=m; \\ \sum_{i=0}^{m-k}\sum_{j=0}^{n-i}\binom{m-k}{i}\binom{j+m-1}{m-1}|s(n-i-j,m)|\frac{(-1)^i\alpha^n m!}{2^j(n-i-j)!}, & \text{if } k<m. \end{cases}$$

□

Theorem 13. *Let $n,\ k,\ r,\ m\ge 1$ be integers. Then*

$$\sum_{j=0}^{n}\sum_{h=0}^{k}\binom{k}{h}(-\beta)^h H_{j-h,k,r}(\alpha,\beta)\frac{(-\alpha)^{n-j}C_{n-j}^{(m)}}{(n-j)!} = \begin{cases} \alpha^n|s(n-m,r-m)|\frac{(r-m)!}{(n-m)!}, & \textit{if } r>m; \\ \alpha^m\delta_{n,m}, & \textit{if } r=m; \\ \frac{(-1)^{n-r}\alpha^n}{(n-r)!}C_{n-r}^{(m-r)}, & \textit{if } r<m. \end{cases} \tag{37}$$

Proof. By using (2), (6) and (16), we have

$$\sum_{j=0}^{n}\sum_{h=0}^{k}\binom{k}{h}(-\beta)^h H_{j-h,k,r}(\alpha,\beta)\frac{(-\alpha)^{n-j}C_{n-j}^{(m)}}{(n-j)!} = [t^n](-\ln(1-\alpha t))^r\left(\frac{-\alpha t}{\ln(1-\alpha t)}\right)^m$$

$$= \begin{cases} \alpha^n|s(n-m,r-m)|\frac{(r-m)!}{(n-m)!}, & \text{if } r>m; \\ \alpha^m\delta_{n,m}, & \text{if } r=m; \\ \frac{(-1)^{n-r}\alpha^n}{(n-r)!}C_{n-r}^{(m-r)}, & \text{if } r<m. \end{cases}$$

□

Theorem 14. *Let $n,\ k,\ r,\ m\ge 1$ be integers. Then*

$$\sum_{j=0}^{n}\sum_{h=0}^{k}\binom{k}{h}(-\beta)^h H_{j-h,k,r}(\alpha,\beta)\frac{(-\alpha)^{n-j}\hat{C}_{n-j}^{(m)}}{(n-j)!}$$
$$= \begin{cases} \alpha^m H_{n-m,m,r-m}(\alpha,\alpha), & \textit{if } r>m; \\ \alpha^n\binom{n-1}{n-m}, & \textit{if } r=m; \\ \sum_{i=m-r}^{n-r}\hat{C}_{n-r-i}^{(m-r)}\frac{(-1)^{n-r-i}\alpha^n}{(n-r-i)!}\binom{i+r-1}{r-1}, & \textit{if } r<m. \end{cases} \tag{38}$$

Proof. The proof of (38) is similar to that of (37), and it is omitted here. □

Theorem 15. *Let $k,\ r\ge 1$ be integers. Then*

$$\sum_{j=0}^{n} H_{j,k,r}(\alpha,\beta)\frac{(-\alpha)^{n-j}C_{n-j}}{(n-j)!} = \alpha H_{n-1,k,r-1}(\alpha,\beta), \tag{39}$$

$$\sum_{j=0}^{n} H_{j,k,r}(\alpha,\beta)\frac{(-\alpha)^{n-j}\hat{C}_{n-j}}{(n-j)!} = \sum_{i=0}^{n-1} H_{i,k,r-1}(\alpha,\beta)\alpha^{n-i}. \tag{40}$$

Proof. By applying (1), (6) and (16), we get

$$\sum_{j=0}^{n} H_{j,k,r}(\alpha,\beta)\frac{(-\alpha)^{n-j}C_{n-j}}{(n-j)!} = [t^n]\frac{(-\ln(1-\alpha t))^r}{(1-\beta t)^k}\frac{\alpha t}{-\ln(1-\alpha t)} = \alpha H_{n-1,k,r-1}(\alpha,\beta)\,.$$

Hence (39) holds. The proof of (40) is similar to that of (39), and it is omitted here. □

Corollary 16. *Let $n,\ r \geq 2$. Then*

$$\sum_{j=r}^{n} \frac{H(j,r-1)(-\alpha)^{n-j}C_{n-j}}{(n-j)!} = \alpha^n H(n-1,r-2), \tag{41}$$

$$\sum_{j=r}^{n} \frac{H(j,r-1)(-\alpha)^{n-j}\hat{C}_{n-j}}{(n-j)!} = \alpha^n \sum_{i=r}^{n-1} H(i-1,r-2). \tag{42}$$

Proof. Setting $\beta=\alpha$ and $k=1$ in (39) and (40), we obtain (41) and (42). □

4 Asymptotics

In this section, we give the asymptotic expansion of certain sums involving $H_{n,k,r}(\alpha,\beta)$. We first recall three lemmas.

A singularity of $f(z)$ at $|z|=w$ is called *algebraic* if $f(z)$ can be written as the sum of a function analytic in a neighborhood of w and a finite number of terms of the form

$$(1-\frac{z}{w})^{\alpha}g(z), \tag{43}$$

where $g(z)$ is analytic near w, $g(w)\neq 0$ and $\alpha \notin \{0,1,2,\ldots\}$.

Lemma 17. *(See [8].) Suppose that $f(z)$ is analytic for $|z|<R$, $R>0$ and has only algebraic singularities on $|z|=R$. Let a be the minimum of $\Re(\alpha)$ for the terms of the form at the singularity of $f(z)$ on $|z|=R$, and let w_j, α_j and $g_j(z)$ be the w,α and $g(z)$ for those terms of the form (43) for which $Re(\alpha)=a$. Then, as $n\to\infty$,*

$$[z^n]f(z) = \sum_j \frac{g_j(w_j)n^{-\alpha_j-1}}{\Gamma(-\alpha_j)w_j^n} + o(R^{-n}n^{-a-1}).$$

Lemma 18. *(see [11]) Let α be a real number and*

$$L(z) = \ln(\frac{1}{1-z})\,.$$

When $n\to\infty$,

$$[z^n](1-z)^{\alpha}L^k(z) \sim \frac{1}{\Gamma(-\alpha)}n^{-\alpha-1}\ln^k n\,, \quad (\alpha \notin \{0,1,2,\ldots\})$$
$$[z^n](1-z)^{m}L^k(z) \sim (-1)^m k m! n^{-m-1}\ln^{k-1} n\,, \quad (m\in\mathbb{Z}_{\geq 0},\ k\in\mathbb{Z}_{\geq 1}).$$

Lemma 19. *(see [11]) Suppose that* $a(z) = \sum a_n z^n$ *and* $b(z) = \sum b_n z^n$ *are power series with radii of convergence* $\alpha > \beta \geq 0$*, respectively. Suppose that* $\frac{b_{n-1}}{b_n} \to \beta$ *as* $n \to \infty$*. If* $a(\beta) \neq 0$ *and* $\sum c_n z^n = a(z)b(z)$*, then*

$$c_n \sim a(\beta) b_n \quad \text{as } n \to \infty.$$

Theorem 20. *Let* $k,\ r \geq 1$*. As* $n \to \infty$*, we get*

$$\sum_{h=0}^{n} \binom{k}{h} (-\beta)^h H_{n-h,k,r}(\alpha, \beta) \sim \frac{\alpha^n r}{n} \ln^{r-1} n\,.$$

Proof. By Eq. (2) and Lemma 18, we get

$$\sum_{h=0}^{k} \binom{k}{h} (-\beta)^h H_{n-h,k,r}(\alpha, \beta) = [t^n](-\ln(1-\alpha t))^r \sim \frac{\alpha^n r}{n} \ln^{r-1} n\,.$$

□

Theorem 21. *Let* $k,\ r,\ m \geq 1$*. As* $n \to \infty$*, we have*

$$\sum_{j=0}^{n} \sum_{h=0}^{k} \binom{k}{h} (-\beta)^h H_{j-h,k,r}(\alpha, \beta) |s(n-j, m)| \frac{\alpha^{n-j}}{(n-j)!} \sim \frac{\alpha^n (m+r)}{m! n} (\ln n)^{m+r-1}\,.$$

Proof. By Eq. (23) and Lemma 18, we obtain

$$\sum_{j=0}^{n} \sum_{h=0}^{k} \binom{k}{h} (-\beta)^h H_{j-h,k,r}(\alpha, \beta) |s(n-j, m)| \frac{\alpha^{n-j}}{(n-j)!} = \frac{1}{m!} [t^n](-\ln(1-\alpha t))^{m+r}$$
$$\sim \frac{\alpha^n (m+r)}{m! n} (\ln n)^{m+r-1}\,.$$

□

Theorem 22. *Let* $k,\ j,\ m \geq 1$ *and* $l \geq 0$*. As* $n \to \infty$*, we have*

$$\sum_{j=1}^{n} H_{n,k,j}(\alpha, \alpha) S(j, m; l) \frac{m!}{j!} \sim \frac{\alpha^n n^{k+m+l-1}}{\Gamma(k+l+m)}\,.$$

Proof. By Eq. (24) and Lemma 17, we obtain

$$\sum_{j=1}^{n} H_{n,k,j}(\alpha, \alpha) S(j, m; l) \frac{m!}{j!} = \alpha^m [t^n] \frac{t^m}{(1-\alpha t)^{k+l+m}} \sim \frac{\alpha^n n^{k+m+l-1}}{\Gamma(k+l+m)}\,.$$

□

Theorem 23. *Let* $k,\ r \geq 1$*. As* $n \to \infty$*, we get*

$$\sum_{j=0}^{n} \sum_{h=0}^{k} \binom{k}{h} (-\beta)^h H_{j-h,k,r}(\alpha, \beta) \binom{n-j+l}{n-j} \beta^{n-j} \sim \frac{\alpha^n n^l}{l!} \ln^r n\,.$$

Proof. It is well known that

$$\sum_{n=0}^{\infty}\binom{n+l}{n}t^n=\frac{1}{(1-t)^{l+1}}. \tag{44}$$

By Lemma 18 and Eq. (44), we get

$$\sum_{j=0}^{n}\sum_{h=0}^{k}\binom{k}{h}(-\beta)^h H_{j-h,k,r}(\alpha,\beta)\binom{n-j+l}{n-j}\alpha^{n-j}=[t^n]\frac{(-\ln(1-\alpha t))^r}{(1-\alpha t)^{l+1}}\sim\frac{\alpha^n n^l}{l!}\ln^r n\,.$$

□

Theorem 24. *Let $j,\ k,\ m\geq 1$. As $n\to\infty$, we have*

$$\sum_{j=1}^{n}H_{n,k,j}(\alpha,\alpha)\frac{B_j^{(m)}}{j!}\sim\begin{cases}(-1)^{m-k}\alpha^n m(m-k)!(n+m)^{k-m-1}(\ln(n+m))^{m-1}, & \textit{if } m-k\in\mathbb{Z}_{\geq 0};\\ \frac{\alpha^n(n+m)^{k-m-1}}{\Gamma(m-k)}(\ln(n+m))^m, & \textit{if } m-k\notin\mathbb{Z}_{\geq 0}.\end{cases}$$

Proof. By the proof of Eq. (28) and Lemma 18, we obtain

$$\sum_{j=1}^{n}H_{n,k,j}(\alpha,\alpha)\frac{B_j^{(m)}}{j!}=[t^n]\frac{1}{(1-\alpha t)^k}\left[\left(\frac{y}{e^y-1}\right)^m\mid y=-\ln(1-\alpha t)\right]$$

$$\sim\begin{cases}(-1)^{m-k}\alpha^n m(m-k)!(n+m)^{k-m-1}(\ln(n+m))^{m-1}, & \text{if } m-k\in\mathbb{Z}_{\geq 0};\\ \frac{\alpha^n(n+m)^{k-m-1}}{\Gamma(m-k)}(\ln(n+m))^m, & \text{if } m-k\notin\mathbb{Z}_{\geq 0}.\end{cases}$$

□

Theorem 25. *Let j be fixed and x be a real number. As $n\to\infty$, we have*

$$\sum_{j=1}^{n}H_{n,k,j}(\alpha,\alpha)\frac{G_j^{(x)}}{2^j j!}\sim\begin{cases}\frac{2^x\alpha^n n^{k-x-1}}{\Gamma(k-x)}, & \textit{if } k>x;\\ \frac{\alpha^n n^{k-1}}{2^n\Gamma(k)}, & \textit{if } k=x;\\ \frac{(-1)^{x-k}\alpha^n n^{x-1}}{2^n\Gamma(x)}, & \textit{if } k<x.\end{cases}$$

Proof. By the proof of (32) and Lemma 17, we obtain

$$\sum_{j=1}^{n}H_{n,k,j}(\alpha,\alpha)\frac{G_j^{(x)}}{2^j j!}=[t^n]\begin{cases}\frac{1}{(1-\alpha t)^{k-x}}\frac{1}{(1-\frac{\alpha t}{2})^x}, & \text{if } k>x;\\ (1-\frac{\alpha t}{2})^x, & \text{if } k=x;\\ \frac{(1-\alpha t)^{x-k}}{(1-\frac{\alpha t}{2})^x}, & \text{if } k<x.\end{cases}\sim\begin{cases}\frac{2^x\alpha^n n^{k-x-1}}{\Gamma(k-x)}, & \text{if } k>x;\\ \frac{\alpha^n n^{k-1}}{2^n\Gamma(k)}, & \text{if } k=x;\\ \frac{(-1)^{x-k}\alpha^n n^{x-1}}{2^n\Gamma(x)}, & \text{if } k<x.\end{cases}$$

□

Theorem 26. *Let $j,\ k,\ m\geq 1$, j be fixed. As $n\to\infty$, we have*

$$\sum_{j=1}^{n}H_{n,k,j}(\alpha,\alpha)\frac{G_j^{(m)}}{j!}\sim\begin{cases}\frac{\alpha^n 2^m n^{k-m-1}\ln^{m+1}n}{\Gamma(k-m)}, & \textit{if } k>m;\\ \frac{2^m\alpha^n m}{n}\ln^{m-1}n, & \textit{if } k=m;\\ \frac{(-1)^{m-k}2^m\alpha^n(m-k)!}{n^{m+1-k}}\ln^m n, & \textit{if } k<m.\end{cases}$$

Proof. By (36), we see that

$$\sum_{j=1}^{n} H_{n,k,j}(\alpha,\alpha)\frac{G_j^{(m)}}{j!} = [t^n]\begin{cases} \frac{1}{(1-\alpha t)^{k-m}}\frac{1}{(1-\frac{\alpha t}{2})^m}(-\ln(1-\alpha t))^m, & \text{if } k>m;\\ \frac{1}{(1-\frac{\alpha t}{2})^m}(-\ln(1-\alpha t))^m, & \text{if } k=m;\\ (1-\alpha t)^{m-k}\frac{1}{(1-\frac{\alpha t}{2})^m}(-\ln(1-\alpha t))^m, & \text{if } k<m. \end{cases}$$

Let $k=m$, and set

$$a(t)=\frac{1}{(1-\frac{\alpha t}{2})^m}=\sum_{n=0}^{\infty}\binom{n+m-1}{m-1}(\frac{\alpha t}{2})^n \text{ and } b(t)=(-\ln(1-\alpha t))^m=\sum_{n=m}^{\infty}|s(n,m)|\frac{m!}{n!}\alpha^n t^n\,.$$

in Lemma 19. Since $\frac{|s(n-1,m)|\frac{m!}{(n-1)!}}{|s(n,m)|\frac{m!}{n!}}\to\frac{1}{\alpha}$ as $n\to\infty$, $a(\frac{1}{\alpha})=2^m\neq 0$. Then we have

$$\sum_{j=1}^{n} H_{n,k,j}(\alpha,\alpha)\frac{G_j^{(m)}}{j!}\sim\begin{cases} \frac{\alpha^n 2^m n^{k-m-1}\ln^{m+1} n}{\Gamma(k-m)}, & \text{if } k>m;\\ \frac{2^m\alpha^n m}{n}\ln^{m-1} n, & \text{if } k=m;\\ \frac{(-1)^{m-k}2^m\alpha^n(m-k)!}{n^{m+1-k}}\ln^m n, & \text{if } k<m. \end{cases}$$

This gives the result for the case $k=m$. The proofs of the cases $k>m$ and $k<m$ are similar to that of $k=m$. □

In the final result of this section, we give the asymptotic expansion of certain sums for binomial coefficients and $H_{n,k,r}(\alpha,\beta)$ by Laplace's method.

Theorem 27. *Let $k,\ r\geq 1,\ 0<\beta,\ (b+1)^{b+1}\neq\beta b^b,\ (b+1)^{b+1}>\alpha b^b$ and b be a positive integer. As $k\to\infty$, we get*

$$\sum_{n=0}^{\infty}\frac{H_{n,k,r}(\alpha,\beta)}{[(b+1)n+1]\binom{(b+1)n}{n}} \sim\left(\frac{(b+1)^{b+1}}{(b+1)^{b+1}-\beta b^b}\right)^{k-\frac{1}{2}}\left(\ln\frac{(b+1)^{b+1}}{(b+1)^{b+1}-\alpha b^b}\right)^r\sqrt{\frac{2\pi(b+1)^{b-2}}{\beta k b^{b-1}}}. \tag{45}$$

Proof. From Tiberiu [9], we know that the inverse of a binomial coefficient is related to an integral as follows:

$$\binom{n}{m}^{-1}=(n+1)\int_0^1 t^m(1-t)^{n-m}dt. \tag{46}$$

Owing to (46), we have

$$\begin{aligned}\sum_{n=0}^{\infty}\frac{H_{n,k,r}(\alpha,\beta)}{[(b+1)n+1]\binom{(b+1)n}{n}} &= \sum_{n=0}^{\infty}H_{n,k,r}(\alpha,\beta)\int_0^1 t^n(1-t)^{nb}dt\\ &= \int_0^1\sum_{n=0}^{\infty}H_{n,k,r}(\alpha,\beta)(t(1-t)^b)^n dt=\int_0^1\frac{(-\ln(1-\alpha t(1-t)^b))^r}{(1-\beta t(1-t)^b)^k}dt\\ &= \int_0^1(-\ln(1-\alpha t(1-t)^b)^r e^{-k\ln(1-\beta t(1-t)^b)}dt\,.\end{aligned}$$

Let $\varphi(t) = (-\ln(1-\alpha t(1-t)^b)^r$ and $h(t) = -\ln(1-\beta t(1-t)^b)$. Then $h(t)$ reaches a maximum at $t = \frac{1}{b+1}$, $h'(\frac{1}{b+1}) = 0$ and $h''(\frac{1}{b+1}) < 0$. By applying Laplace's method, we have

$$\sum_{n=0}^{\infty} \frac{H_{n,k,r}(\alpha,\beta)}{[(b+1)n+1]\binom{(b+1)n}{n}} = \int_0^1 (-\ln(1-\alpha t(1-t)^b)^r e^{-k\ln(1-\beta t(1-t)^b} dt$$
$$\sim \varphi(\frac{1}{b+1}) e^{kh(\frac{1}{b+1})} \sqrt{\frac{-2\pi}{kh''(\frac{1}{b+1})}}.$$

□

Corollary 28. *Let $k,\ r \geq 1$, $\alpha < 4$, $0 < \beta$, $\beta \neq 4$ and b be a positive integer. As $k \to \infty$, we get*

$$\sum_{n=0}^{\infty} \frac{H_{n,k,r}(\alpha,\beta)}{(2n+1)\binom{2n}{n}} \sim \left(\frac{4}{4-\beta}\right)^{k-\frac{1}{2}} \left(\ln \frac{4}{4-\alpha}\right)^r \sqrt{\frac{\pi}{k\beta}}. \tag{47}$$

Proof. Setting $b = 1$ in (45), we obtain (47). □

Theorem 29. *Let $k,\ r \geq 1$, $0 < \alpha$, $(b+1)^{b+1} \neq -\beta b^b$ and b be a positive integer. As $r \to \infty$, we get*

$$\sum_{n=0}^{\infty} \frac{(-1)^n H_{n,k,r}(\alpha,\beta)}{[(b+1)n+1]\binom{(b+1)n}{n}}$$
$$\sim (-1)^r \left(\frac{(b+1)^{b+1}}{(b+1)^{b+1}+\beta b^b}\right)^k \left(\ln \frac{(b+1)^{b+1}+\alpha b^b}{(b+1)^{b+1}}\right)^{r+\frac{1}{2}} \sqrt{\frac{2\pi(b+1)^{b-2}((b+1)^{b+1}+\alpha b^b)}{\alpha r b^{b-1}(b+1)^{b+1}}}. \tag{48}$$

Proof. Owing to (46), we get

$$\sum_{n=0}^{\infty} \frac{(-1)^n H_{n,k,r}(\alpha,\beta)}{[(b+1)n+1]\binom{(b+1)n}{n}} = \sum_{n=0}^{\infty} (-1)^n H_{n,k,r}(\alpha,\beta) \int_0^1 t^n (1-t)^{nb} dt$$
$$= \int_0^1 \sum_{n=0}^{\infty} H_{n,k,r}(\alpha,\beta)(-t(1-t)^b)^n dt = \int_0^1 \frac{(-\ln(1+\alpha t(1-t)^b))^r}{(1+\beta t(1-t)^b)^k} dt$$
$$= (-1)^r \int_0^1 \frac{e^{r\ln\ln(1+\alpha t(1-t)^b}}{(1+\beta t(1-t)^b)^k} dt.$$

Let $\varphi(t) = \frac{1}{(1+\beta t(1-t)^b)^k}$ and $h(t) = \ln\ln(1+\alpha t(1-t)^b)$. Then $h(t)$ reaches the maximum at $t = \frac{1}{b+1}$, $h'(\frac{1}{b+1}) = 0$ and $h''(\frac{1}{b+1}) < 0$. By applying Laplace's method, we have

$$\sum_{n=0}^{\infty} \frac{(-1)^n H_{n,k,r}(\alpha,\beta)}{[(b+1)n+1]\binom{(b+1)n}{n}} = \int_0^1 (-\ln(1-\alpha t(1-t)^b)^r e^{-k\ln(1-\beta t(1-t)^b} dt$$
$$\sim \varphi(\frac{1}{b+1}) e^{rh(\frac{1}{b+1})} \sqrt{\frac{-2\pi}{rh''(\frac{1}{b+1})}}.$$

□

Corollary 30. *Let $k,\ r \geq 1,\ 0 < \alpha,\ \beta \neq -4$. As $r \to \infty$, we get*

$$\sum_{n=0}^{\infty} \frac{(-1)^n H_{n,k,r}(\alpha,\beta)}{(2n+1)\binom{2n}{n}} \sim (-1)^r \left(\frac{4}{4+\beta}\right)^k \left(\ln \frac{4+\alpha}{4}\right)^{r+\frac{1}{2}} \sqrt{\frac{\pi(4+\alpha)}{4\alpha r}}. \tag{49}$$

Proof. Setting $b = 1$ in (48), we derive (49). □

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