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Combinatorics of generalized Tchebycheff polynomials

Dongsu Kim[a], Jiang Zeng[b]

[a]*Department of Mathematics, KAIST, Daejeon 305-701, Republic of Korea*
[b]*Institut Girard Desargues, Université Claude Bernard (Lyon I), France*

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Abstract

By considering a family of orthogonal polynomials generalizing the Tchebycheff polynomials of the second kind we refine the corresponding results of De Sainte-Catherine and Viennot on Tchebycheff polynomials of the second kind (Lecture Notes in Mathematics, vol. 1171, 1985, Springer-Verlag, 120). © 2003 Elsevier Science Ltd. All rights reserved.

1. Introduction

There are a number of integrals of products of classical orthogonal polynomials with respect to their weight functions that have a regular sign pattern (see, e.g. the classical papers by Askey [1, 2]). One way of proving the positivity of those integral expressions—a way that has been very fruitful in the past (see, e.g. [2, 4–9, 11] and the references cited there)—is to construct an adapted combinatorial structure and express the integral as a generating function for that structure. Then proving the positivity of that sign pattern boils down to deriving some specific geometric properties for those structures. Thus, a true analytic result appears as a corollary of a combinatorial property.

The calculation of those integrals can also be viewed as an evaluation of a formal linear functional applied to its associated orthogonal polynomials. Note that when the weight function is unknown the positivity of the evaluation is somehow unexpected. In this paper we want to illustrate this by making a study of “integrals” of products of *general Tchebycheff polynomials* $U_n(x, a)$, which are defined by the following three-term recurrence relation:

$$U_{n+1}(x, a) = xU_n(x, a) - \lambda_n U_{n-1}(x, a), \qquad U_0(x, a) = 1,\ U_1(x, a) = x, \tag{1}$$

E-mail addresses: dskim@math.kaist.ac.kr (D. Kim), zeng@euler.univ-lyon1.fr (J. Zeng).

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where $\lambda_{2k} = a, \lambda_{2k+1} = 1$. Hence $U_n(2x, 1)$ is the Tchebycheff polynomial of the second kind $U_n(x)$ (see [3]). Vauchassade de Chaumont and Viennot [13] have shown how these generalized Tchebycheff polynomials with one parameter a arose naturally from some enumerative problems in molecular biology [13]. It is also known [13, 14] that the polynomials $U_n(x, a)$ are orthogonal with respect to the linear functional $\varphi : \mathbb{R}(a)[x] \to \mathbb{R}(a)$ defined by $\varphi(x^{2n+1}) = 0$ and

$$\varphi(x^{2n}) = \frac{1}{n}\sum_{i=0}^{n-1}\binom{n}{i}\binom{n}{i+1}a^i. \tag{2}$$

In addition, we can readily derive from (1) the following generating function:

$$\sum_{n\geq 0} U_n(x, a)w^n = \frac{1 + xw + aw^2}{(1 + w^2)(1 + aw^2) - x^2w^2}. \tag{3}$$

The aim of this paper is to give a combinatorial interpretation for the following evaluation of φ at a product of generalized Tchebycheff polynomials $U_n(x, a)$:

$$\mathbf{I}((n_1, \ldots, n_k); a) = \varphi(U_{n_1}(x, a) \cdots U_{n_k}(x, a)). \tag{4}$$

In particular, our interpretation will imply that $\mathbf{I}((n_1, \ldots, n_k); a)$ is a polynomial of a with *nonnegative integral* coefficients and generalize a result of De Sainte-Catherine and Viennot on Tchebycheff polynomials of the second kind [5].

We start with some definitions and notations. A *lattice path* of length n is a sequence $\gamma = (p_0, p_1, \ldots, p_n)$ of points $p_i = (x_i, y_i)$ in $\mathbb{N} \times \mathbb{N}$, with steps $s_i = (p_{i-1}, p_i)$ of type *up* ($y_i = y_{i-1} + 1, x_i = x_{i-1} + 1$) or *down* ($y_i = y_{i-1} - 1, x_i = x_{i-1} + 1$). The path is called a Dyck path if $p_0 = (0, 0)$ and $p_n = (n, 0)$. We can identify the path with its step sequence $(s_1, s_2, \ldots, s_n)$, or its step-type sequence $(t(s_1), \ldots, t(s_n))$ with starting point p_0 and ending point p_n, where $t(s_i)$ is u, if s_i is up; d, if it is down. We call x_i (resp. y_i) the *position* (resp. *height*) of p_i. The point p_i is called a *rise* (resp. *fall*) if s_i is up (resp. down); p_i is called a *peak* (resp. *valley*) if s_i is up (resp. down) and s_{i+1} is down (resp. up).

Let $\mathbf{n} = (n_1, \ldots, n_k)$ be a composition of the positive integer n. Let $[n]$ denote the set $\{1, 2, \ldots, n\}$ and for $j = 1, \ldots, k$, let $S_j \subset [n]$ denote the segment

$$S_j = \{i : n_1 + \cdots + n_{j-1} + 1 \leq i \leq n_1 + \cdots + n_{j-1} + n_j\}.$$

So $[n] = S_1 \cup S_2 \cup \cdots \cup S_k$. The *parity* (resp. *position-parity*) of S_j is defined to be that of n_j (resp. $n_1 + \cdots + n_{j-1}$), for $1 \leq j \leq k$, assuming that $n_0 = 0$. The *local parity* of i in S_j is defined to be that of $i - n_1 - \cdots - n_{j-1}$. Let $\mathrm{DSV}(\mathbf{n})$ be the set of Dyck paths of length n such that all the peak-positions belong to $\{n_1, n_1 + n_2, \ldots, n_1 + \cdots + n_{k-1}\}$.

Given a path γ in $\mathrm{DSV}(\mathbf{n})$, a valley $p_i = (i, y_i)$ of γ is said to be *special*, if y_i is even and i belongs to an even segment of odd position-parity; let $\mathrm{EHR}(\gamma)$ (resp. $\mathrm{SVAL}(\gamma)$) be the number of *even-height rises* (*special valleys*) of γ. The following is our main result.

Theorem 1. *For any composition* $\mathbf{n} = (n_1, \ldots, n_k)$ *of the positive integer* n*, we have*

$$\mathbf{I}(\mathbf{n}; a) = \sum_{\gamma\in \mathrm{DSV}(\mathbf{n})} a^{\mathrm{EHR}(\gamma)+\mathrm{SVAL}(\gamma)}. \tag{5}$$

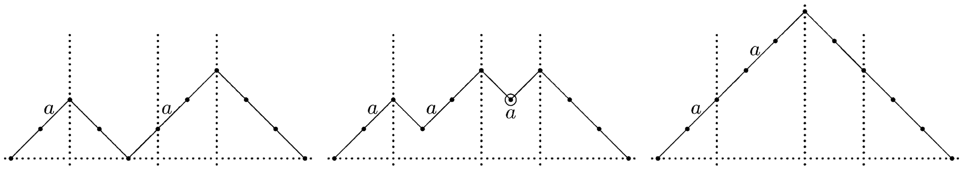

Fig. 1. The Dyck paths of DSV(2, 3, 2, 3).

Some remarks are in order. As we will show later, the weight function in the statement of the above theorem is different, but inspired, from that of the corresponding moments.

If $a = 1$, it can be shown [5] that (4) can be written as

$$\mathbf{I}(\mathbf{n}; 1) = \frac{2}{\pi}\int_{-1}^{1} U_{n_1}(x)\cdots U_{n_k}(x)\sqrt{1-x^2}\mathrm{d}x,$$

so Theorem 1 reduces to a result of De Sainte-Catherine and Viennot [5].

The set DSV($\mathbf{n}$) itself may change if $\mathbf{n}$ is rearranged. Moreover, in $\mathbf{n}$, if all the even integers are to the left of odd ones, as in (2, 2, 3, 3), then there is no even segment of odd position-parity and so no path in DSV($\mathbf{n}$) will have any special valleys.

Example. The set DSV(2, 3, 2, 3) is shown in Fig. 1, where even up-steps and special valleys are each weighted by a. Note that $S_1 = \{1, 2\}$, $S_2 = \{3, 4, 5\}$, $S_3 = \{6, 7\}$, $S_4 = \{8, 9, 10\}$; and only S_3 is an even segment of odd position-parity. Theorem 1 reads $\mathbf{I}(\mathbf{n}; a) = 2a^2 + a^3$ for any rearrangement $\mathbf{n}$ of (2, 3, 2, 3).

If $k = 2$ and $\mathbf{n} = (m, n)$, then to form a Dyck path in DSV(m, n) we must have $m = n$ and the path should start with n up-steps and end with n down-steps, which imply the orthogonality.

Corollary 1. *For any positive integers m, n we have*

$$\varphi(U_m(x, a)U_n(x, a)) = a^{\lfloor n/2\rfloor}\delta_{mn}. \tag{6}$$

For any proposition A we define the characteristic function $\chi(A)$ to be 1, if A is true; 0, if false. The following is the linearization formula for $U_n(x, a)$.

Corollary 2. *For any positive integers m and n,*

$$U_m(x, a)U_n(x, a) = \sum_{k=0}^{\min(m,n)} a^{\lfloor k/2\rfloor+\chi(mn\equiv 0 \bmod 2)\chi(k\equiv 1 \bmod 2)}U_{m+n-2k}(x, a). \tag{7}$$

Proof. Since $U_n(x, a)$ is a polynomial in x of degree n, we have the expansion

$$U_m(x, a)U_n(x, a) = \sum_{l=0}^{m+n} c(m, n, l)U_l(x, a),$$

where $c(m, n, l) \in \mathbb{R}(a)$. From the orthogonality (6) we derive

$$c(m, n, l) = a^{-\lfloor l/2\rfloor}\varphi(U_m(x, a)U_n(x, a)U_l(x, a)).$$

By Theorem 1, $\varphi(U_m(x,a)U_n(x,a)U_l(x,a)) = 0$ if $m+n+l$ is odd or $l > m+n$. Suppose $m+n+l$ is even and set $l = m+n-2k$ with $k \geq 0$, then $\varphi(U_m(x,a)U_n(x,a)U_l(x,a))$ is the generating function of Dyck paths of length $2(m+n-k)$ whose peak-positions are contained in $\{m, m+n\}$. Therefore,

$$c(m,n,l) = \begin{cases} a^{\lfloor (l+k)/2 \rfloor - \lfloor l/2 \rfloor}, & \text{if } m \text{ or } n \text{ is odd, or } k \text{ is even,} \\ a^{\lfloor (l+k+1)/2 \rfloor - \lfloor l/2 \rfloor}, & \text{if } m \text{ and } n \text{ are even, and } k \text{ is odd,} \end{cases}$$
$$= \begin{cases} a^{k/2}, & \text{if } k \text{ is even,} \\ a^{\lfloor k/2 \rfloor}, & \text{if } m, n, k \text{ are odd,} \\ a^{(k+1)/2}, & \text{otherwise.} \end{cases}$$

Combining the above cases, we obtain (7). □

There are several ingredients in our derivation of the combinatorial interpretation for $I(\mathbf{n}; a)$. First, as in [5], we obtain a preliminary combinatorial interpretation of $\mathbf{I}(\mathbf{n}; a)$ with *signed weight function* by interpreting $U_n(x,a)$ as *weighted matching polynomials* of the segment graph on $[n]$ and the moments $\varphi(x^n)$ as generating functions of *noncrossing perfect matchings* of the complete graph on $[n]$. This will be done in the next section. The difficult part is to construct a *weight-preserving sign-reversing involution* or wpsr involution on the underlying set. In Section 3 we describe an involution ψ on the set of weighted paths with fixed starting and ending points, then in Section 4, we apply this involution ψ to prove our main theorem.

2. Matching polynomials and Dyck path complexes

A graph on a finite set S is an ordered pair $G = (S, E)$ where E is a set of some pairs $\{s, t\}$ of elements of S. An element of S is called a vertex of G and that of E an edge of G. A *matching* μ of the graph G is a subset of edges such that no two edges of μ have a common vertex. Given a matching μ of G, a vertex of G is said to be *isolated* if it does not belong to any edge of μ. A perfect matching is a matching without isolated vertices. The complete graph on S is the graph $K_S = (S, E)$, where E is the set of all possible edges $\{s, t\}$ with $s \neq t$. The segment graph on $[n]$ is the graph $\text{Seg}_n = ([n], E)$, where E is the set of all edges $\{i, i+1\}$ for $1 \leq i \leq n-1$. We put weight on the matchings of Seg_n: each isolated point is weighted x and each edge $\{i, i+1\}$ is weighted -1, if i is odd; $-a$, if i is even. According to Viennot's theory [14], it is easy to derive from (1) that the polynomial $U_n(x,a)$ is the generating function of all matchings of Seg_n, i.e. the weighted matching polynomial of Seg_n:

$$U_n(x,a) = \sum_{\mu \in \mathcal{M}(\text{Seg}_n)} (-1)^{|\mu|} a^{\text{EIND}(\mu)} x^{n-2|\mu|}, \tag{8}$$

where $|\mu|$ is the number of edges of μ (thus $n-2|\mu|$ is the number of isolated vertices of μ) and $\text{EIND}(\mu)$ the number of edges $\{i, i+1\}$ with i even. It is possible to give a combinatorial proof of (3) by using the above combinatorial interpretation (8) for $U_n(x,a)$.

Consider the complete graph K_n on $[n]$. A matching μ of K_n is noncrossing iff it has no crossing pairs, i.e. there are no edges $\{i, j\}$ and $\{i', j'\}$ of μ satisfying $i < i' < j < j'$.

Let μ be a noncrossing perfect matching of K_n. The *height* of an edge $\{s, t\}$, $s < t$, in μ is the number of edges $\{i, j\}$ in μ such that $i \le s$ and $j \ge t$. It is known [10, 12] that the number of noncrossing perfect matchings of K_{2n} with i even-height edges is a *Narayana number*, $\frac{1}{n}\binom{n}{i}\binom{n}{i+1}$. Let $\mathrm{EHE}(\mu)$ denote the number of edges of even height in μ. It follows from (2) that

$$\varphi_n(x^n) = \sum_{\mu \in \mathcal{M}_n^*} a^{\mathrm{EHE}(\mu)}, \tag{9}$$

where $\mathcal{M}_n^*$ is the set of noncrossing perfect matchings of K_n.

Definition 1. Let $\mathbf{n} = (n_1, \ldots, n_k)$ be a composition of n. An edge $\{s, t\}$ of K_n is called *homogeneous* (resp. *consecutive*) if both s and t belong to the same segment S_j for some j (resp. $|s - t| = 1$). We define a *matching complex* of type $\mathbf{n}$ to be a noncrossing perfect weighted matching μ of K_n satisfying the following:

1. There are two kinds of edges: red and blue. Blue edges are consecutive and homogeneous.
2. Each red edge is weighted 1, if it is of odd height; a, if of even height. Each blue edge $\{i, i + 1\}$ is weighted -1, if i is of odd local parity; $-a$, if of even local parity.

An edge is said to be an a-edge, if its weight is a or $-a$; 1-edge, if its weight is 1 or -1. Given a matching complex μ, let $\mathrm{EINDB}(\mu)$ denote the number of blue a-edges $\{i, i + 1\}$ in μ, and let $\mathrm{EHRE}(\mu)$ denote the number of red a-edges.

Let $\mathcal{MC}(\mathbf{n})$ be the set of matching complexes of type $\mathbf{n}$. Interpreting blue edges as matchings from the polynomials, and red edges as matchings from the moments, it follows from (4), (8) and (9) that

$$\mathbf{I}(\mathbf{n}; a) = \sum_{\mu \in \mathcal{MC}(\mathbf{n})} (-1)^{b(\mu)} a^{\mathrm{EINDB}(\mu) + \mathrm{EHRE}(\mu)}, \tag{10}$$

where $b(\mu)$ denotes the number of blue edges.

Although it is possible to describe our proof of the main theorem within the model of matching complexes, it seems more convenient to switch to the model of Dyck paths as it will be clear in the next section.

It is well known that the noncrossing perfect matchings of K_n are in one-to-one correspondence with Dyck paths of length n. Indeed, given a noncrossing perfect matching μ of K_n, we can define the step-type sequence $\mathrm{type}(\mu) = t_1 \cdots t_n$ of a Dyck path by setting $t_i = u$, if i is the smaller vertex of an edge in μ; $t_i = d$, otherwise. Obviously the mapping type is a bijection [10]. Thus, using this bijection and suitably weighting path steps we can switch from a matching complex to a *Dyck path complex*, which is defined in the following.

Definition 2. Let $\mathbf{n} = (n_1, \ldots, n_k)$ be a composition of n. A *path complex* of type $\mathbf{n}$ is a pair $\pi = (\gamma, w)$ such that γ is a path with point sequence $(p_0, p_1, \ldots, p_n)$, $p_i = (x_i, y_i)$, and $w = (w_1, \ldots, w_n)$ is the sequence of weights of steps $s_i = (p_{i-1}, p_i)$ satisfying:

- If p_i is not a peak, then $w_i = a$ if s_i is up and y_i is even; 1 if s_i is down or y_i is odd.

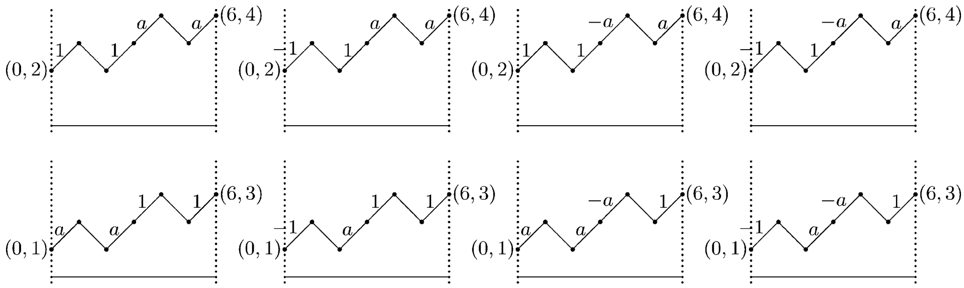

Fig. 2. Paths in $\mathcal{H}(6; 2, 4)$ and $\mathcal{H}(6; 1, 3)$ with step-type sequence (u, d, u, u, d, u).

- If p_i is a peak, then $w_i = 1$ or -1 if y_i is odd and i is of odd local parity; 1 or $-a$ if y_i is odd and i is of even local parity; a or -1 if y_i is even and i is of odd local parity; a or $-a$ if y_i is even and i is of even local parity.

If γ is a Dyck path, the corresponding path complex is called a Dyck path complex. A peak (resp. valley) p_i of γ is said to be *homogeneous* if $i, i+1$ belong to the same S_j for some $j \in [k]$. The weight of p_i is the product of w_i and w_{i+1}. Furthermore p_i is called positive (resp. negative), if its weight is positive (resp. negative).

Let $\mathcal{DC}(\mathbf{n})$ be the set of all Dyck path complexes of type $\mathbf{n}$ whose negative peaks are all homogeneous. For $\pi \in \mathcal{DC}(\mathbf{n})$, let $\mathrm{EPNP}(\pi)$ (resp. $\mathrm{NP}(\pi)$) be the number of *locally-even position negative peaks* (resp. *negative peaks*) of π and let $\mathrm{EHPR}(\pi)$ be the number of *even height positive rises* of π.

Now we can restate (10) as follows:

$$I(\mathbf{n}; a) = \sum_{\pi \in \mathcal{DC}(\mathbf{n})} (-1)^{\mathrm{NP}(\pi)} a^{\mathrm{EPNP}(\pi)+\mathrm{EHPR}(\pi)}. \tag{11}$$

Comparing (11) with (5), we need to find a wpsr involution on $\mathcal{DC}(\mathbf{n})$ whose fixed points are in one-to-one correspondence with paths in $\mathrm{DSV}(\mathbf{n})$.

3. Fundamental involution

If the composition $\mathbf{n}$ reduces to (n) a path complex of type $\mathbf{n}$ is simply called a weighted path of length n. Note that the peaks of any weighted path are homogeneous. Let $\mathcal{H}(n; k, l)$ be the set of weighted paths of length n whose underlying paths go from $(0, k)$ to (n, l). Clearly one should require that $n \geq |k - l|$ and $n \equiv k + l \mod 2$ in order to have at least one such path in $\mathcal{H}(n; k, l)$.

Example. All weighted paths in $\mathcal{H}(6; 2, 4)$ and $\mathcal{H}(6; 1, 3)$ whose underlying paths have step-type sequence (u, d, u, u, d, u) are shown in Fig. 2, where weight 1 of each down-step is omitted.

Since a lattice path is determined by its step-type sequence when the starting and ending points are fixed, we can encode any weighted path $\pi \in \mathcal{H}(n; k, l)$ by a *biword*

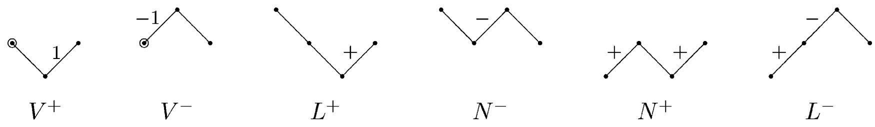

Fig. 3. Patterns V^+, V^-, L^+, N^-, N^+ and L^- ($\odot$ is the starting point).

$\pi = \begin{pmatrix} t_1 & \cdots & t_n \\ w_1 & \cdots & w_n \end{pmatrix}$, where $(t_1, \ldots, t_n)$ is the step-type sequence and $(w_1, \ldots, w_n)$ the weight sequence. We say that a biword τ is a *pattern* of π if it is a subword of π, i.e. $\pi = \alpha\tau\beta$ for some biwords α and β, where the composition of words is concatenation.

In order to describe our involution we shall need the following patterns:

$$V^+ = \begin{pmatrix} d & u \\ 1 & 1 \end{pmatrix}, \qquad L^+ = \begin{pmatrix} d & d & u \\ 1 & 1 & + \end{pmatrix}, \qquad N^+ = \begin{pmatrix} u & d & u \\ + & 1 & + \end{pmatrix},$$
$$V^- = \begin{pmatrix} u & d \\ -1 & 1 \end{pmatrix}, \qquad N^- = \begin{pmatrix} d & u & d \\ 1 & - & 1 \end{pmatrix}, \qquad L^- = \begin{pmatrix} u & u & d \\ + & - & 1 \end{pmatrix},$$

where '+' means 1 or a and '−' means -1 or $-a$, moreover, we require that the negative peak in the pattern N^- is of height at least 1. The patterns are illustrated in Fig. 3.

Definition 3. Define $\mathcal{H}^*(n; k, l)$ to be the set of all weighted paths $\pi \in \mathcal{H}(n; k, l)$ satisfying one of the following conditions:

(i) π starts with $(n-k+l)/2$ positive up-steps and ends with $(n+k-l)/2$ down-steps.
(ii) π has s, $s > 0$, negative valleys of height 0, say $p_{r_1}, p_{r_2}, \ldots, p_{r_s}$ with $r_1 < r_2 \cdots < r_s$; it starts with $(r_1-k)/2$ positive up-steps and is followed by $(r_1+k)/2$ down-steps; for each i, $1 \le i \le s-1$, the portion of π from p_{r_i} to $p_{r_{i+1}}$ is one negative peak followed by $(r_{i+1}-r_i-2)/2$ positive up-steps and $(r_{i+1}-r_i-2)/2$ down-steps; the portion of π after p_{r_s} is one negative peak followed by $(n-r_s-2+l)/2$ positive up-steps and $(n-r_s-2-l)/2$ down-steps.

Example. Two weighted paths of type (ii) in $\mathcal{H}^*(29; 3, 2)$ are illustrated in Fig. 4, where π_1 and π_2 coincide starting from the eighth step. Note that π_1 has four negative valleys of height 0 with abscissas 9, 11, 19, 25 respectively, while π_2 has one more such valley, $(3, 0)$.

The following lemma gives a characterization of the set $\widetilde{\mathcal{H}}^*(n; k, l) = \mathcal{H}(n; k, l) \backslash \mathcal{H}^*(n; k, l)$.

Lemma 1. *Assume that k is odd. Then* $\pi \in \widetilde{\mathcal{H}}^*(n; k, l)$ *if and only if it contains one of the patterns* V^+, V^-, N^+, N^-, L^+ *and* L^-.

Proof. The condition is clearly sufficient. It remains to show the 'only if' part.

Note that if p_i, $i > 1$, is a negative peak of height 1 then p_{i-1} must be a negative valley of height 0, and if p_i, $i < n-1$, is a negative valley of height 0 then p_{i+1} must be a negative peak of height 1.

Let π be a path in $\widetilde{\mathcal{H}}^*(n; k, l)$, then it must contain one of the following patterns: (1) a negative peak of height greater than 1, (2) a valley of height greater than 0, (3) a positive

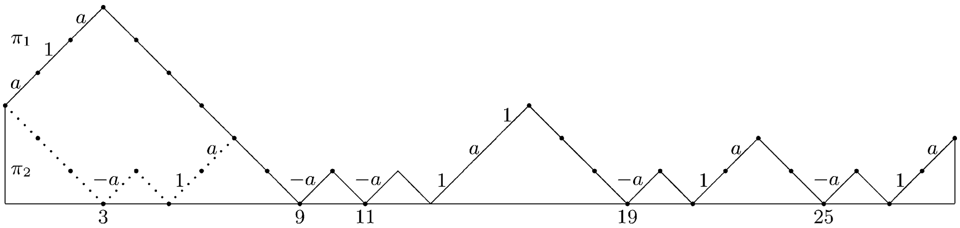

Fig. 4. Two paths π_1 and π_2 in $\mathcal{H}^*(29; 3, 2)$.

valley of height 0 preceded by two positive steps. Now, in case (1), it has a portion of type V^-, N^- or L^-; in case (2), π has a valley p_i of height greater than 0, which entails the following four cases:

(i) if π has a positive valley p_1, then it has a portion of type V^+;
(ii) if p_i is negative, then it has a negative peak p_{i+1} of height greater than 1;
(iii) if $i > 1$ and p_i is positive and w_{i-1} is positive, then it has a portion of type N^+ or L^+;
(iv) if $i > 1$ and p_i is positive and w_{i-1} is negative, then it has a negative peak p_{i-1} of height greater than 1;

in case (3), it has a pattern N^+ or L^+. $\square$

The following result is crucial for the proof of our main theorem.

Proposition 1. *Suppose that* $n \geq |k-l|$ *and* $n \equiv k+l \pmod 2$. *Then there is a weight-preserving sign-reversing involution* ψ *on* $\mathcal{H}(n; k, l)$ *which has no fixed point, if* $n > k+l$*; exactly one fixed point, if* $n \leq k+l$. *In the latter case, if k is even the unique path starts with* $(n+k-l)/2$ *down-steps and ends with* $(n-k+l)/2$ *up-steps; if k is even the unique path starts with* $(n-k+l)/2$ *up-steps and ends with* $(n+k-l)/2$ *down-steps.*

Proof. We construct a wpsr involution ψ on $\mathcal{H}(n; k, l)$ as follows: let $\pi \in \mathcal{H}(n; k, l)$.

1. If k is even, we distinguish two cases.
   - π contains a peak: Let p_i be the leftmost one, we define $\psi(\pi)$ by changing the ith weight of π to its opposite (this is possible because i and y_i have the same parity when k is even).
   - π contains no peak: Then π must start with $(n+k-l)/2$ down-steps and end with $(n-k+l)/2$ up-steps, clearly $n \leq k+l$. Define $\psi(\pi) = \pi$.
2. If k is odd, we distinguish three cases.
   - $\pi \in \widetilde{\mathcal{H}}^*(n; k, l)$: If $\pi = V^+\beta$ or $V^-\beta$ then set $\psi(\pi) = \begin{pmatrix} t_2 & t_1 \\ -w_1 & w_2 \end{pmatrix}\beta$; else if π contains a pattern $\tau = L^+, N^+, L^-$ or N^- then find the leftmost such a portion, i.e. find the smallest i such that $\pi = \alpha \begin{pmatrix} t_i & t_{i+1} & t_{i+2} \\ w_i & w_{i+1} & w_{i+2} \end{pmatrix} \beta = \alpha\tau\beta$

and define $\psi(\pi) = \alpha \begin{pmatrix} t_i & t_{i+2} & t_{i+1} \\ w_i & -w_{i+1} & w_{i+2} \end{pmatrix} \beta$. Clearly ψ has no fixed points in this case.

- $\pi \in \mathcal{H}^*(n; k, l)$ with $n > k + l$: Then π is one of the following forms:

$$U = \binom{u}{a} \alpha \binom{d}{1}^k \binom{d}{1} \beta, \qquad D = \binom{d}{1}^k \binom{u}{-a} \binom{d}{1} \alpha' \beta',$$

  where α and α' are either empty or a sequence of positive up-steps; and β and β' are either empty or start with a down-step. In either case, define $\psi_o(\pi)$ to be the path of the other form with $\alpha = \alpha'$ and $\beta = \beta'$. Clearly ψ has no fixed points in this case.
- $\pi \in \mathcal{H}^*(n; k, l)$ with $n \le k + l$: Clearly, π is the unique path of type (i) in the definition of $\mathcal{H}^*(n; k, l)$ because a path of type (ii) must have a length greater than $k + l$. Define $\psi(\pi) = \pi$.

Summarizing all the above yields the proposition. □

Example. The involution ψ on $\mathcal{H}^*(29; 3, 2)$ is illustrated in Fig. 4, where $\psi : \pi_1 \mapsto \pi_2$.

By computing the number of *even up-steps* of the weighted path fixed by ψ, we derive from the above proposition that the generating function of $\mathcal{H}(n; k, l)$ is 0 if $n > k + l$, $a^{\lceil (n-k+l)/4 \rceil}$ if $n \le k + l$ with k even and l odd, $a^{\lfloor (n-k+l)/4 \rfloor}$ otherwise.

4. Proof of the main theorem

For any sequence $\mathbf{m} = (m_0, m_1, \ldots, m_k)$ of nonnegative integers with $m_0 = m_k = 0$, let $\mathcal{DC}(\mathbf{n}; \mathbf{m})$ denote the set of all Dyck path complexes in $\mathcal{DC}(\mathbf{n})$ whose underlying Dyck paths have the ordinate m_i at abscissa $n_1 + \cdots + n_i$ for $i = 1, 2, \ldots, k$. Similarly we can define the subset $\mathrm{DSV}(\mathbf{n}; \mathbf{m})$ of $\mathrm{DSV}(\mathbf{n})$. Thus we have the following decompositions:

$$\mathcal{DC}(\mathbf{n}) = \bigcup_{\mathbf{m}} \mathcal{DC}(\mathbf{n}; \mathbf{m}), \qquad \mathrm{DSV}(\mathbf{n}) = \bigcup_{\mathbf{m}} \mathrm{DSV}(\mathbf{n}; \mathbf{m}).$$

We now define a wpsr involution on $\mathcal{DC}(\mathbf{n}; \mathbf{m})$. Any Dyck path complex π in $\mathcal{DC}(\mathbf{n}; \mathbf{m})$ will be identified with a k-tuple $(\pi_1, \ldots, \pi_k)$ of weighted paths, where $\pi_i \in \mathcal{H}(n_i; m_{i-1}, m_i)$ is the restriction of π on the segment S_i for $1 \le i \le k$.

For $\pi = (\pi_1, \ldots, \pi_k)$ in $\mathcal{DC}(\mathbf{n}; \mathbf{m})$, if $\psi(\pi_i) = \pi_i$ for all i, set $\Psi(\pi) = \pi$; otherwise, set $\Psi(\pi) = (\pi_1, \ldots, \psi(\pi_i), \ldots, \pi_k)$, where i is the smallest integer such that $\psi(\pi_i) \ne \pi_i$. Clearly Ψ is a wpsr involution on $\mathcal{DC}(\mathbf{n}; \mathbf{m})$. According to Proposition 1, there is a unique fixed point $\pi_\circ$ in $\mathcal{DC}(\mathbf{n}; \mathbf{m})$ if $|m_{i-1} - m_i| \le n_i \le m_{i-1} + m_i$ for all i; no fixed point, otherwise. Moreover, if $\pi_\circ = (\pi_1, \ldots, \pi_k)$ with $\pi_i \in \mathcal{H}(n_i; m_{i-1}, m_i)$, then each π_i is the path described by Proposition 1, i.e. if m_{i-1} is even (resp. odd) then π_i starts with $(n_i + m_{i-1} - m_i)/2$ down-steps (resp. up-steps) and ends with $(n_i - m_{i-1} + m_i)/2$

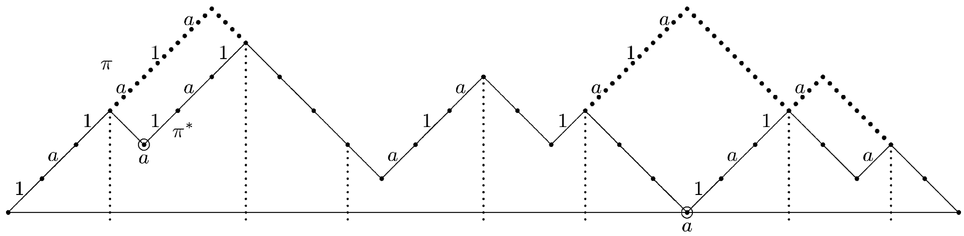

Fig. 5. Bijection $\Phi : \pi \mapsto \pi^*$ with $(\mathbf{n};\mathbf{m}) = ((3, 4, 3, 4, 3, 6, 3, 2); (0, 3, 5, 2, 4, 3, 3, 2, 0))$.

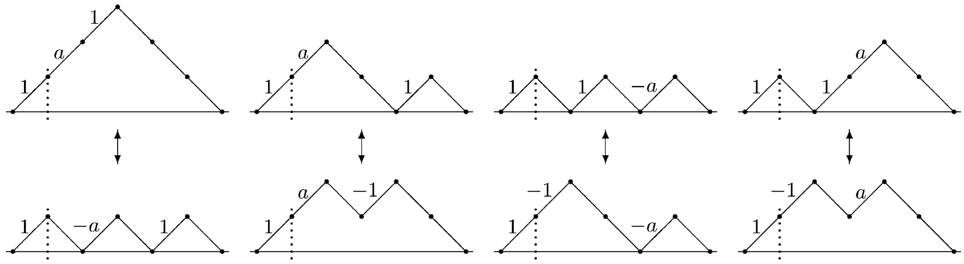

Fig. 6. Involution Ψ on the path complexes of $\mathcal{DC}(1, 5)$ with weight a or $-a$.

up-steps (resp. down-steps). Let $F(\mathbf{n};\mathbf{m})$ denote the fixed point set of Ψ. Clearly, we have $\text{EHPR}(\pi) = \text{EHR}(\pi)$ for all $\pi \in F(\mathbf{n};\mathbf{m})$. It follows from (11) that

$$I(\mathbf{n}; a) = \sum_{\mathbf{m}} \sum_{\pi \in F(\mathbf{n};\mathbf{m})} a^{\text{EHR}(\pi)}. \tag{12}$$

Note that a path complex in $F(\mathbf{n};\mathbf{m})$ has a homogeneous peak in S_i if m_{i-1} is odd and $n_i > |m_{i-1} - m_i|$, while no path in $\text{DSV}(\mathbf{n};\mathbf{m})$ has any homogeneous peak. To complete the proof, we construct a bijection $\Phi : \pi \mapsto \pi^*$ from $F(\mathbf{n};\mathbf{m})$ to $\text{DSV}(\mathbf{n};\mathbf{m})$ by setting $\pi^* = (\pi_1^*, \dots, \pi_k^*)$, where $\pi_i^* = \pi_i$, if m_{i-1} is even; π_i^* is the path starting with $(n_i + m_{i-1} - m_i)/2$ down-steps and ending with $(n_i - m_{i-1} + m_i)/2$ up-steps, in other words, the step-type sequence of π_i^* is the reverse of that of π_i, if m_{i-1} is odd. Clearly, the resulting path π^* is an element of $\text{DSV}(\mathbf{n};\mathbf{m})$ and the map is a bijection.

Now we compare the weights of π and π^*. For each i, if π^* has a special valley in S_i, i.e. both m_{i-1} and $(n_i - m_{i-1} + m_i)/2$ are odd and n_i is even, then π_i^* has one more even positive up-step than π_i; otherwise, π_i^* has the same number of them as π_i. So if we weight each special valley with a, then Φ becomes weight-preserving, i.e. $\text{EHR}(\pi) = \text{EHR}(\pi^*) + \text{SVAL}(\pi^*)$. Our main theorem follows from (12).

Example. An example of $\Phi : \pi \mapsto \pi^*$ from $F(\mathbf{n};\mathbf{m})$ to $\text{DSV}(\mathbf{n};\mathbf{m})$ is illustrated in Fig. 5 for $\mathbf{n} = (3, 4, 3, 4, 3, 6, 3, 2)$ and $\mathbf{m} = (0, 3, 5, 2, 4, 3, 3, 2, 0)$, where the two special valleys are circled. While the involution Ψ on the path complexes of $\mathcal{DC}(1, 5)$ with weight a or $-a$ is illustrated in Fig. 6.

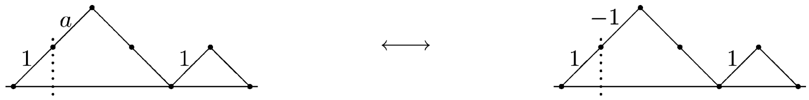

Fig. 7. De Sainte-Catherine and Viennot's involution with $\mathbf{n} = (1, 5)$.

Remark. When $a = 1$, De Sainte-Catherine and Viennot [5] have given a sign-reversing involution on $\mathcal{DC}(\mathbf{n})$, which consists of changing the leftmost homogeneous negative peak to a positive peak and vice versa. As shown in Fig. 7, for $a \neq 1$, their involution does not preserve weights. While it is obvious from the algebraic expression (4) that $I(\mathbf{n}; a)$ is symmetric by permuting n_i in $\mathbf{n}$, this is not clear from our combinatorial interpretation, but Guoniu Han (private communication) has recently constructed an involution on DSV($\mathbf{n}$) exhibiting this symmetry.

Acknowledgements

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