THE COMBINATORICS OF AL-SALAM-CHIHARA q-LAGUERRE POLYNOMIALS

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Abstract. We describe various aspects of the Al-Salam-Chihara q-Laguerre polynomials. These include combinatorial descriptions of the polynomials, the moments, the orthogonality relation and a combinatorial interpretation of the linearization coefficients.

Contents

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1. Introduction

The monic simple Laguerre polynomials $L_n(x)$ may be defined by the explicit formula:

$$L_n(x)=\sum_{k=0}^{n}(-1)^{n-k}\frac{n!}{k!}\binom{n}{k}x^k, \tag{1}$$

or by the three-term recurrence relation

$$L_{n+1}(x)=(x-(2n+1))L_n(x)-n^2L_{n-1}(x). \tag{2}$$

The moments are

$$\mu_n=\mathcal{L}(x^n)=\int_0^\infty x^n e^{-x}dx=n!. \tag{3}$$

The linearization formula reads as follows:

$$L_{n_1}(x)L_{n_2}(x)=\sum_{n_3}C_{n_1\,n_2}^{n_3}L_{n_3}(x),$$

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where

$$C_{n_1\,n_2}^{n_3}=\sum_{s\geq 0}\frac{n_1!\,n_2!\,2^{N_2+n_3-2s}\,s!}{(s-n_1)!(s-n_2)!(s-n_3)!(N_2+n_3-2s)!n_3!}.$$

Equivalently we have

$$\mathcal{L}(L_{n_1}(x)L_{n_2}(x)L_{n_3}(x))=\sum_{s\geq 0}\frac{n_1!\,n_2!\,n_3!\,2^{N_2+n_3-2s}\,s!}{(s-n_1)!(s-n_2)!(s-n_3)!(N_2+n_3-2s)!}. \tag{4}$$

Given positive integers $n_1,n_2,\ldots,n_k$ such that $n=n_1+\cdots+n_k$, let S_i be the consecutive integer segment $\{n_1+\cdots n_{i-1}+1,\ldots,n_1+\cdots+n_i\}$ with $n_0=0$, then $S_1\cup\ldots\cup S_k=[n]$. A permutation σ of $[n]$ is said to be a *generalized derangement* if i and $\sigma(i)$ do not belong to a same segment S_j for all $i\in[n]$. Let $\mathcal{D}_n$ be the set of generalized derangements of $[n]$ then we have

$$\mathcal{L}(L_{n_1}(x)\ldots L_{n_k}(x))=\sum_{\sigma\in\mathcal{D}_n}1. \tag{5}$$

A q-version of (1) was studied by Garsia and Remmel [9] in 1980. Several q-analogues of the moments (2) and recurrence relation(3) were investigated in the last two decades (see [2,18,19]) in order to obtain new *mahonian* statistics on the symmetric groups. On the other hand, in view of the unified combinatorial interpretations of several aspects of Sheffer orthogonal polynomials (moments, polynomials, and the linearization coefficients)(see [14, 20, 22]) it is natural to seek for a q-version of this picture.

As one can expect, the first result in this direction was the linearization formula for q-Hermite polynomials due to Ismail, Stanton and Viennot [12], dated back to 1987. In particular, their formula provides a combinatorial evaluation of the Askey-Wilson integral. However, a similar formula for q-Charlier polynomials was discovered only recently by Anshelevich [1], who used the machinery of q-Levy stochastic processes. Short later, Kim, Stanton and Zeng [15] gave a combinatorial proof of Anshelevich's result.

The object of this paper is to give a q-version of all the above formulas for simple Laguerre polynomials.

2. Al-Salam-Chihara polynomials revisited

The Al-Salam-Chihara polynomials $Q_n(x):=Q_n(x;\alpha,\beta|q)$ may be defined by the recurrence relation [16, Chapter 3]:

$$\begin{cases}Q_0(x)=1, \quad Q_{-1}(x)=0,\\ Q_{n+1}(x)=(2x-(\alpha+\beta)q^n)Q_n(x)-(1-q^n)(1-\alpha\beta q^{n-1})Q_{n-1}(x), \quad n\geq 0.\end{cases} \tag{6}$$

Let $Q_n(x)=2^n p_n(x)$ then

$$xp_n(x)=p_{n+1}(x)+\frac{1}{2}(\alpha+\beta)q^n p_n(x)+\frac{1}{4}(1-q^n)(1-\alpha\beta q^{n-1})p_{n-1}(x). \tag{7}$$

They also have the following explicit expressions:

$$\begin{aligned}Q_n(x;\alpha,\beta|q) &= \frac{(\alpha\beta;\,q)_n}{\alpha^n}\,{}_3\phi_2\left(\begin{matrix}q^{-n},\alpha u,\alpha u^{-1}\\ \alpha\beta,0\end{matrix}\,\middle|\,q;\,q\right)\\ &= (\alpha u;\,q)_n u^{-n}\,{}_2\phi_1\left(\begin{matrix}q^{-n},\beta u^{-1}\\ \alpha^{-1}q^{-n+1}u^{-1}\end{matrix}\,\middle|\,q;\alpha^{-1}qu\right)\\ &= (\beta u^{-1};\,q)_n u^{n}\,{}_2\phi_1\left(\begin{matrix}q^{-n},\alpha u\\ \beta^{-1}q^{-n+1}u\end{matrix}\,\middle|\,q;\beta^{-1}qu^{-1}\right),\end{aligned}$$

where $x=\frac{u+u^{-1}}{2}$ or $x=\cos\theta$ if $u=e^{i\theta}$.

The Al-Salam-Chihara polynomials have the following generating function

$$G(t,x)=\sum_{n=0}^{\infty}Q_n(x;\alpha,\beta|q)\frac{t^n}{(q;\,q)_n}=\frac{(\alpha t,\beta t;\,q)_\infty}{(te^{i\theta},te^{-i\theta};\,q)_\infty}.$$

They are orthogonal with respect to the linear functional $\hat{\mathcal{L}}_q$:

$$\hat{\mathcal{L}}_q(x^n)=\frac{1}{2\pi}\int_0^\pi(\cos\theta)^n\frac{(q,\alpha\beta,e^{2i\theta},e^{-2i\theta};\,q)_\infty}{(\alpha e^{i\theta},\alpha e^{-i\theta},\beta e^{i\theta},\beta e^{-i\theta};\,q)_\infty}d\theta, \tag{8}$$

where $x=\cos\theta$. Note that

$$\hat{\mathcal{L}}_q(Q_n(x)^2)=(q;\,q)_n(\alpha\beta;\,q)_n.$$

Theorem 1. *We have*

$$Q_{n_1}(x)Q_{n_2}(x)=\sum_{n_3\ge 0}C^{n_3}_{n_1,n_2}(\alpha,\beta;\,q)Q_{n_3}(x), \tag{9}$$

where

$$\begin{aligned}C^{n_3}_{n_1,n_2}(\alpha,\beta;\,q) &= (-1)^{N_2+n_3}\frac{(q;\,q)_{n_1}(q;\,q)_{n_2}}{(\alpha\beta;\,q)_{n_3}}\\ &\times\sum_{m_2,m_3}\frac{(\alpha\beta;\,q)_{n_1+m_3}\alpha^{m_2}\beta^{n_3+n_2-n_1-m_2-2m_3}q^{\binom{m_2}{2}+\binom{n_3+n_2-n_1-m_2-2m_3}{2}}}{(q;\,q)_{n_3+n_2-n_1-m_2-2m_3}(q;\,q)_{m_2}(q;\,q)_{m_3+n_1-n_3}(q;\,q)_{m_3+n_1-n_2}(q;\,q)_{m_3}}.\end{aligned}$$

Proof. Clearly $C^{n_3}_{n_1,n_2}(\alpha,\beta;\,q)=\hat{\mathcal{L}}_q(Q_{n_1}(x)Q_{n_2}(x)Q_{n_3}(x))/\hat{\mathcal{L}}_q(Q_{n_3}(x)Q_{n_3}(x))$. Using the Askey-Wilson integral:

$$\frac{(q;\,q)_\infty}{2\pi}\int_0^\pi\frac{(e^{2i\theta},e^{-2i\theta};\,q)_\infty}{\prod_{j=1}^4(t_je^{i\theta},t_je^{-i\theta};\,q)_\infty}d\theta=\frac{(t_1t_2t_3t_4;\,q)_\infty}{\prod_{1\le j<k\le 4}(t_jt_k;\,q)_\infty},$$

one can prove [12, Theorem 3.5] that

$$\begin{aligned}&\hat{\mathcal{L}}_q(G(t_1,x)G(t_2,x)G(t_3,x))\\ &\quad=\frac{(\alpha t_1t_2t_3,\beta qt_1t_2t_3,\alpha\beta q;\,q)_\infty}{(t_1t_2,t_1t_3,t_2t_3;\,q)_\infty}{}_3\phi_2\left(\begin{matrix}t_1t_2, & t_1t_3, & t_2t_3\\ & \alpha t_1t_2t_3, & \beta t_1t_2t_3\end{matrix}\,|q;\alpha\beta\right).\end{aligned}$$

Therefore

$$\sum_{n_1,n_2,n_3} \hat{\mathcal{L}}_q(Q_{n_1}(x)Q_{n_2}(x)Q_{n_3}(x))\frac{t_1^{n_1}}{(q;\,q)_{n_1}}\frac{t_2^{n_2}}{(q;\,q)_{n_2}}\frac{t_3^{n_3}}{(q;\,q)_{n_3}}$$
$$= \sum_{k\geq 0}\frac{(\alpha t_1t_2t_3q^k,\,\beta t_1t_2t_3q^k,\,\alpha\beta;q)_\infty}{(t_1t_2q^k,\,t_1t_3q^k,\,t_2t_3q^k;q)_\infty}\frac{(\alpha\beta)^k}{(q;\,q)_k}. \tag{10}$$

Using the Euler formulas:

$$(t;\,q)_\infty = \sum_{n\geq 0}\frac{(-1)^n q^{\binom{n}{2}}}{(q;\,q)_n}t^n; \qquad \frac{1}{(t;\,q)_\infty} = \sum_{n\geq 0}\frac{1}{(q;\,q)_n}t^n,$$

we can rewrite the sum in (10) as follows:

$$(\alpha\beta;\,q)_\infty \sum_{k\geq 0}\frac{(\alpha\beta)^k}{(q;\,q)_k}\sum_{l_1,l_2\geq 0}\frac{\alpha^{l_1}\beta^{l_2}q^{k(l_1+l_2)}(-t_1t_2t_3)^{l_1+l_2}q^{\binom{l_1}{2}+\binom{l_2}{2}}}{(q;\,q)_{l_1}(q;\,q)_{l_2}}$$
$$\times \sum_{m_1,m_2,m_3\geq 0}\frac{q^{(m_1+m_2+m_3)k}t_1^{m_1+m_2}t_2^{m_1+m_3}t_3^{m_1+m_3}}{(q;\,q)_{m_1}(q;\,q)_{m_2}(q;\,q)_{m_3}}. \tag{11}$$

Substituting

$$\sum_{k\geq 0}\frac{(\alpha\beta q^{l_1+l_2+m_1+m_2+m_3})^k}{(q;\,q)_k} = \frac{1}{(\alpha\beta q^{l_1+l_2+m_1+m_2+m_3};q)_\infty}$$

in (11), we get

$$\sum_{l_1,l_2,m_1,m_2,m_3} t_1^{n_1}t_2^{n_2}t_3^{n_3}\frac{(\alpha\beta)_{n_1+m_3}\alpha^{l_1}\beta^{l_2}q^{\binom{l_1}{2}+\binom{l_2}{2}}}{(q;\,q)_{m_1}(q;\,q)_{m_2}(q;\,q)_{m_3}(q;\,q)_{l_1}(q;\,q)_{l_2}}(-1)^{l_1+l_2}, \tag{12}$$

where $l_1+l_2+m_1+m_2=n_1$, $l_1+l_2+m_1+m_3=n_2$ and $l_1+l_2+m_2+m_3=n_3$.

Since $l_1+l_2\equiv N_2+n_3 \pmod 2$, extracting the coefficient of $\frac{t_1^{n_1}t_2^{n_2}t_3^{n_3}}{(q;q)_{n_1}(q;q)_{n_2}(q;q)_{n_3}}$ in (12) and dividing by $(q,\alpha\beta;\,q)_{n_3}$ we obtain (9) where l_1 is replaced by m_2. □

3. The new q-Laguerre polynomials

We define the new q-Laguerre polynomials $L_n(x;\,q)$ by re-scaling Al-Salam-Chihara polynomials:

$$L_n(x;\,q) = \left(\frac{\sqrt{y}}{q-1}\right)^n Q_n\left(\frac{(q-1)x+y+1}{2\sqrt{y}};\,\frac{1}{\sqrt{y}},\sqrt{y}q|q\right). \tag{13}$$

It follows from (7) that the polynomials $L_n(x;\,q)$ satisfy the recurrence:

$$L_{n+1}(x;\,q) = (x-y[n+1]_q-[n]_q)L_n(x;\,q) - y[n]_q^2L_{n-1}(x;\,q). \tag{14}$$

We derive then the explicit formula for $L_n(x)$:

$$L_n(x;\,q) = \sum_{k=0}^{n}(-1)^{n-k}\frac{n!_q}{k!_q}\begin{bmatrix}n\\k\end{bmatrix}_q q^{k(k-n)}y^{n-k}\prod_{j=0}^{k-1}\left(x-(1-yq^{-j})[j]_q\right). \tag{15}$$

Thus

$$\begin{aligned}
L_1(x;\, q) &= x - y,\\
L_2(x;\, q) &= x^2 - (1+2y+qy)x + (1+q)y^2,\\
L_3(x;\, q) &= x^3 - (q^2y + 3\,y + q + 2 + 2\,qy)x^2\\
&\quad + (q^3y^2 + yq^2 + q + 2\,qy + 3\,q^2y^2 + 1 + 4\,qy^2 + 2\,y + 3\,y^2)x\\
&\quad - (2\,q^2 + 2\,q + q^3 + 1)y^3
\end{aligned}$$

A combinatorial interpretation of these q-Laguerres polynomials can be derived from the Simion and Stanton's combinatorial model for octabasic Laguerre polynomials [19]. For a subset A of $[n]$, the *functional digraph* of an injection $f : A \to [n]$ consists of disjoint paths and cycles. Each path P is of the form $a_0 \to a_1 \to \cdots \to a_l$, where $f(a_j) = a_{j+1}$ for $0 \le j < l$, with $f^{-1}(a_0)$ empty, and $a_l \in [n] - A$. We put $\mathrm{last}(P) = a_l$ and if $i = a_k \in P$ we write $\mathrm{ind}(i, P) = k$ for the index of i on the path P. For any path P in the digraph and two integers $i < j$, we put

$$n_P(i, j) = |\{a \in P : i < a < j\}|.$$

For $p \in P$ and two integers $i < j$, we define

$$m_P(p; i, j) = |\{a \in P : i < a < j, \mathrm{ind}(p, P) < \mathrm{ind}(a, P)\}|,$$

that is, the number of points on the path "to the right" of p, whose values are strictly between i and j. And finally, for $i \in A$, we denote by $F(i)$ the "first forward iterate" of f which is smaller than i, i.e.,

$$F(i) = \begin{cases} f^p(i), & \text{where } p = \min\{m \ge 1, f^m(i) < i \text{ if such } m \text{ exists}\};\\ i, & \text{if } \{m \ge 1, f^m(i) < i\} \text{ is empty.} \end{cases}$$

For instance, suppose that the path $P = 2 \to 7 \to 1 \to 5 \to 3$ is a connected component of the functional diagraph of f. Then $n_P(1, 4) = |\{2, 3\}| = 2$, $m_P(7; 1, 4) = |\{3\}| = 1$, and $F(2) = F(7) = 1$, $F(1) = 1$, and $F(5) = 3$.

For any $k \in [n]$, let $\alpha(k) = w(k) = 0$ if $k \notin A$, otherwise if k is on a cycle or a path P such that $k > \mathrm{last}(P)$, then $\alpha(k) = 1$ and

$$w(k) = F(k) - 1 - \sum_{\mathrm{last}(Q) > k} n_Q(0, F(k));$$

if k is on a path P such that $k < \mathrm{last}(P)$, then $\alpha(k) = 0$ and

$$w(k) = k - 1 - m_P(k; 0, k) - \sum_{\mathrm{last}(Q) > \mathrm{last}(P)} n_Q(0, k),$$

where Q ranges over all paths in the functional digraphs of f. Let

$$w(A, f) = \sum_{k \in A} w(k) \quad \text{and} \quad \alpha(A, f) = \sum_{k \in A} \alpha(k).$$

Example 1. *Let $n = 9$, $A = \{2, 9\}$ and $\sigma = (6)(4\,7)(3\,5\,1\,8)$ (in cycle notation with maximum at last). Then we have $cyc(\sigma) = 3$ and*

$$w(A, \sigma) = (3 - 1 - 1) + (5 - 1 - 1) + (1 - 1) + (4 - 1 - 2) = 5.$$

Theorem 2. *The q-Laguerre polynomials have the following interpretation:*

$$L_n(x;\, q) = \sum_{A\subset[n], f:A\to[n]} (-1)^{|A|} x^{n-|A|} y^{\alpha(A,f)} q^{w(A,f)},$$

where f is injective.

Proof. This is the $a = 1$, $s = u = 1$ and $r = t = q$ special case of the quadrabasic Laguerre polynomials [19, p.313]. □

Remark 1. *It is easy to see that the constant term $L_n(0)$ is equal to*

$$L_n(0) = (-1)^n y^n n!_q.$$

So the restriction of the statistic on permutations is a Mahonian statistic.

4. Moments of the q-Laguerre polynomials

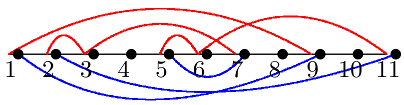
Let $\mathcal{S}_n$ be the set of permutations of $[n] := \{1, 2, \ldots, n\}$. For $\sigma \in \mathcal{S}_n$ the *crossing number* of σ is defined by

$$cr(\sigma) = \sum_{i=1}^{n} \#\{j | j < i \le \sigma(j) < \sigma(i)\} + \sum_{i=1}^{n} \#\{j | j > i > \sigma(j) > \sigma(i)\},$$

while the number of *weak excedances* of σ is defined by

$$wex(\sigma) = \#\{i | 1 \le i \le n \text{ and } i \le \sigma(i)\}.$$

We can depict these statistics by associating with each permutation σ of $[n]$ a diagram by drawing an arc $i \to \sigma(i)$ above (resp. under) the segment $1 \to 2 \to \cdots \to n$ if $i \le \sigma(i)$ (resp. $i > \sigma(i)$). For example, the permutation $\sigma = 9\,3\,7\,4\,6\,11\,5\,8\,1\,10\,2$ can be depicted as follows:



Let $\mu_n^{(\ell)}(y, q)$ be the enumerating polynomial of permutations in $\mathcal{S}_n$ with respect to weak excedances and crossing numbers:

$$\mu_n^{(\ell)}(y, q) := \sum_{\sigma\in S_n} y^{wex(\sigma)} q^{cr(\sigma)}.$$

Randrianarivony [17] and Corteel [3] have proved the following continued fraction expansion:

$$E(y, q, t) := \sum_{n\ge0} \mu_n^{(\ell)}(y, q) t^n = \cfrac{1}{1 - b_0 t - \cfrac{\lambda_1 t^2}{1 - b_1 t - \cfrac{\lambda_2 t^2}{\ddots}}}, \tag{16}$$

where $b_n = y[n + 1]_q + [n]_q$ and $\lambda_n = y[n]_q^2$.

We derive then from the classical theory of orthogonal polynomials the following interpretation of the moments of the q-Laguerre polynomials.

Theorem 3. *The n-th moment of the q-Laguerre polynomials is equal to $\mu_n^{(\ell)}(y,q)$. More precisely, let $\mathcal{L}_q$ be the linear functional defined by $\mathcal{L}_q(x^n)=\mu_n^{(\ell)}(y,q)$, then*

$$\mathcal{L}_q(L_{n_1}(x;\,q)L_{n_2}(x;\,q)) = y^{n_1}(n_1!_q)^2\delta_{n_1\,n_2}. \tag{17}$$

The first values of the moment sequence are as follows:

$$\begin{aligned}
\mu_1^{(\ell)}(y,q) &= y,\\
\mu_2^{(\ell)}(y,q) &= y+y^2,\\
\mu_3^{(\ell)}(y,q) &= y+(3+q)y^2+y^3,\\
\mu_4^{(\ell)}(y,q) &= y+(6+4q+q^2)y+(6+4q+q^2)y^3+y^4.
\end{aligned}$$

Combining the results of Corteel [3], Williams [21, Proposition 4.11] and the classical theory of orthogonal polynomials, one can write the moments of the above q-Laguerre polynomials as a finite double sum (cf. (28)). Here we propose a direct proof of this result. Actually we shall give such a formula for the moments of Al-Salam-Chihara polynomials.

Definition 4. *Define the y-versions of the q-Stirling numbers of the second kind by*

$$X^n = \sum_{k=1}^{n} S_q(n,k,y)\prod_{j=0}^{k-1}(X-[j]_q(1-yq^{-j})). \tag{18}$$

The y-versions of q-Stirling numbers of the first kind can be defined by the inverse matrix or equivalently

$$\prod_{j=0}^{n-1}(X-[j]_q(1-yq^{-j})) = \sum_{k=1} s_q(n,k,y)X^k.$$

Remark 2. *We have*

$$S_q(n,k,y)|_{q=1} = S(n,k)(1-y)^{n-k}, \quad S_q(n,k,0)=S_q(n,k),$$

where $S(n,k)$ and $S_q(n,k)$ are, respectively, the Stirling numbers of the second kind and their well-known q-analogues, see [11].

Consider the rescaled Al-Salam-Chihara polynomials $P_n(x)$:

$$\begin{aligned}
P_n(X) &= Q_n(((q-1)X+1/\alpha^2+1)\alpha/2;\,\alpha,\beta|q)\\
&= \alpha^{-n}\sum_{k=0}^{n}\frac{(q^{-n};\,q)_k}{(q;\,q)_k}q^k(\alpha\beta q^k;\,q)_{n-k}(1-q)^k q^{\binom{k}{2}}\alpha^{2k}\\
&\qquad\times\prod_{j=0}^{k-1}\left(X-[i]_q(1-q^{-i}/\alpha^2)\right).
\end{aligned} \tag{19}$$

Theorem 1. *The moments of the rescaled Al-Salam-Chihara polynomials $P_n(X)$ are*

$$\mu_n(\alpha,\beta) = \sum_{k=1}^{n} S_q(n,k,1/\alpha^2)(\alpha\beta;\,q)_k q^{-\binom{k}{2}}(1-q)^{-k}\alpha^{-2k}.$$

Proof. Let $L : X^n \mapsto \mu_n(\alpha,\beta)$ be the linear functional. We check that these moments do satisfy $L(P_n(X)) = 0$ for $n > 0$. Let a_k be the coefficients in front of the product in (19), then we have, using y-Stirling orthogonality,

$$\begin{aligned} L(P_n(X)) &= \sum_{k=0}^{n} a_k \sum_{j=1}^{k} s_q(k,j,1/\alpha^2) \sum_{t=1}^{j} S_q(j,t,1/\alpha^2)(\alpha\beta;\,q)_t q^{-\binom{t}{2}}(1-q)^{-t}\alpha^{-2t} \\ &= \sum_{k=0}^{n} a_k(\alpha\beta;\,q)_k q^{-\binom{k}{2}}(1-q)^{-k}\alpha^{-2k} \\ &= \alpha^{-n}(\alpha\beta;\,q)_n \sum_{k=0}^{n} \frac{(q^{-n};\,q)_k}{(q;\,q)_k} q^k = 0. \end{aligned}$$

Note that the last equality follows by applying the q-binomial formula. □

Theorem 2. *The generating function for the moments* $\mu_n(\alpha,\beta)$ *is*

$$\sum_{n=0}^{\infty} \mu_n(\alpha,\beta)t^n = \sum_{k=0}^{\infty} \frac{(\alpha\beta;\,q)_k q^{-\binom{k}{2}}(1-q)^{-k}\alpha^{-2k}t^k}{\prod_{i=1}^{k}(1-[i]_q t(1-q^{-i}/\alpha^2))}. \tag{20}$$

Proof. By definition (18) we have

$$S_q(n,k,y) = S_q(n-1,k-1,y) + [k]_q(1-yq^{-k})S_q(n-1,k,y).$$

It follows that (18) is equivalent to

$$\sum_{n\ge k} S_q(n,k,y)t^n = \frac{t^k}{\prod_{i=1}^{k}(1-[i]_q t(1-q^{-i}y))}, \tag{21}$$

which yields immediately (20) in view of Theorem 1. □

The moment of q-Charlier polynomials corresponds to the $\beta = 0$, $\alpha = -1/\sqrt{a(1-q)}$ case, while that of q-Laguerre polynomials corresponds to the $\alpha = 1/\sqrt{y}$, $\alpha\beta = q$ case. Therefore,

$$\sum_{n=0}^{\infty} \mu_n^{(c)}(a,q)t^n = \sum_{k=0}^{\infty} \frac{a(qt)^k}{\prod_{i=1}^{k}(q^i - q^i[i]_q t + a(1-q)[i]_q t)}, \tag{22}$$

$$\sum_{n=0}^{\infty} \mu_n^{(\ell)}(y,q)t^n = \sum_{k=0}^{\infty} \frac{k!_q(qty)^k}{\prod_{i=1}^{k}(q^i - q^i[i]_q t + [i]_q ty)}. \tag{23}$$

Theorem 3. *Let* $p = 1/q$. *We have*

$$\sum_{k=0}^{\infty} \frac{(\alpha\beta;\,q)_k q^{-\binom{k}{2}}(1-q)^{-k}\alpha^{-2k}t^k}{\prod_{i=1}^{k}(1-[i]_q t(1-q^{-i}/\alpha^2))} = \sum_{i\ge 0} \frac{c_i(\alpha,\beta)}{1-[i]_q t(1-q^{-i}/\alpha^2)}, \tag{24}$$

where

$$c_i(\alpha,\beta) = \frac{(\alpha\beta;q)_i}{(q;q)_i} \frac{q^{i-i^2}\alpha^{-2i}}{(q^{1-2i}/\alpha^2;q)_i} \frac{(p^{1+i}\alpha\beta/\alpha^2;p)_\infty}{(p^{1+2i}/\alpha^2;p)_\infty}.$$

Proof. Note the following partial fraction decomposition formula:

$$\frac{t^k}{(1-a_1t)(1-a_2t)\dots(1-a_kt)} = \frac{(-1)^k}{a_1\cdots a_k} + \sum_{i=1}^{k} \frac{a_i^{-1}\prod_{j=1,j\neq i}^{k}(a_i-a_j)^{-1}}{1-a_it}.$$

Therefore

$$\frac{t^k}{\prod_{i=1}^{k}(1-[i]_qt(1-q^{-i}/\alpha^2))} = \sum_{i=0}^{k} \frac{\gamma_k(i)}{1-[i]_qt(1-q^{-i}/\alpha^2)}, \tag{25}$$

where

$$\gamma_k(i) = \frac{1}{k!_q}\begin{bmatrix}k\\i\end{bmatrix}_q \frac{\alpha^{2(k-i)}q^{\binom{k}{2}+k-i^2}}{(q^{1-2i}/\alpha^2;q)_i(q^{1+2i}\alpha^2;q)_{k-i}} \qquad (0\le i\le k).$$

Substituting this in (24) yields

$$\begin{aligned} c_i(\alpha,\beta) &= \sum_{k\ge i}\frac{(\alpha\beta;q)_k}{(q;q)_k}\begin{bmatrix}k\\i\end{bmatrix}_q \frac{q^{k-i^2}\alpha^{-2i}}{(q^{1-2i}/\alpha^2;q)_i(q^{1+2i}\alpha^2;q)_{k-i}} \\ &= \frac{(\alpha\beta;q)_i}{(q;q)_i}\frac{q^{i-i^2}\alpha^{-2i}}{(q^{1-2i}/\alpha^2;q)_i}\sum_{k\ge 0}\frac{(\alpha\beta q^i;q)_k}{(q;q)_k}\frac{q^k}{(q^{1+2i}\alpha^2;q)_k}. \end{aligned}$$

The theorem follows then by applying the ${}_1\Phi_1$ summation formula (see [10, II.5]). ☐

By partial fraction decomposition (see [21, Theorem 4.12]), we get

$$\sum_{n=0}^{\infty}\mu_n^{(c)}(a,q)t^n = \sum_{i\ge 0}\frac{a^i(1-a(1-q)p^{2i})/(a(1-q)p^i;p)_\infty}{i!_q q^{i^2}(q^i-q^i[i]_qt+a[i]_qt(1-q))}, \tag{26}$$

$$\sum_{n=0}^{\infty}\mu_n^{(\ell)}(y,q)t^n = \sum_{i\ge 0}\frac{y^i(q^{2i}-y)}{q^{i^2}(q^i-q^i[i]t+[i]ty)}. \tag{27}$$

Note that (27) yields the following polynomial formula in y for $\mu_n^{(\ell)}(y,q)$:

$$\mu_n^{(\ell)}(y,q) = \sum_{k=1}^{n}\sum_{i=0}^{k-1}(-1)^i[k-i]_q^n q^{k(i-k)}\left(\binom{n}{i}q^{k-i}+\binom{n}{i-1}\right)y^k, \tag{28}$$

while (26) does not yield such a polynomial formula in a for $\mu_n^{(c)}(a,q)$.

On the other hand, it follows from (25) and (21) that

$$S_q(n,k,y) = \frac{q^{-\binom{k}{2}}}{k!_q}\sum_{i=1}^{k}\begin{bmatrix}k\\i\end{bmatrix}_q y^{i-k}q^{k^2-i^2}\frac{([i]_q(1-q^{-i}y))^n}{(q^{1-2i}y;q)_i(q^{1+2i}/y;q)_{k-i}}. \tag{29}$$

Using Theorem 1 and the above explicit formula for q-Stirling numbers we can also write the moments $\mu_n(\alpha,\beta)$ as a double sum.

5. Linearization coefficients of the q-Laguerre polynomials

The following is our main result of this section.

Theorem 5. *The linearization coefficients of the q-Laguerre polynomials are*

$$\mathcal{L}_q(L_{n_1}(x;\, q)\dots L_{n_k}(x;\, q)) = \sum_{\sigma\in\mathcal{D}(n_1,\dots,n_k)} y^{wex(\sigma)}q^{cr(\sigma)}. \tag{30}$$

A proof à la Viennot (cf. [12, 15]) of (30) would use the combinatorial interpretations for the moments and q-Laguerre polynomials to rewrite the left-hand side of (30) and then construct an adequate *killing involution* on the resulting set. For the time being we do not have such a proof to offer, instead we provide an inductive proof.

We first show that the above result is true for $(n_1,\dots,n_k) = (1,\dots,1)$.

Lemma 6. *Let* $d_n(y,q) = \sum_{\sigma\in\mathcal{D}_n} y^{wex(\sigma)}q^{cr(\sigma)}$. *Then* $\mathcal{L}_q((x-y)^n) = d_n(y,q)$.

Proof. Note that

$$\mathcal{L}_q((x-y)^n) = \sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}y^{n-k}\mu_k^{(\ell)}(y,q).$$

By binomial inversion, it suffices to prove that

$$\mu_n^{(\ell)}(y,q) = \sum_{k=0}^{n}\binom{n}{k}y^k d_{n-k}(y,q).$$

But the latter identity is obvious. □

The invariance of $\sum_{\sigma\in\mathcal{D}(n_1,n_2,\dots,n_k)} y^{wex(\sigma)}q^{cr(\sigma)}$ by permutating the $n_i's$ is a direct consequence of Theorem 5, but for our proof we need to first establish this property.

Theorem 7. *For any permutation* $\gamma\in\mathcal{S}_k$ *we have*

$$\sum_{\sigma\in\mathcal{D}(n_1,n_2\dots,n_k)} y^{wex(\sigma)}q^{cr(\sigma)} = \sum_{\sigma\in\mathcal{D}(n_{\gamma(1)},n_{\gamma(2)},\dots,n_{\gamma(k)})} y^{wex(\sigma)}q^{cr(\sigma)}.$$

Since the two cyclic permutations $(1,2)$ and $(1,2,3,\dots,k)$ generate the symmetric group $\mathcal{S}_k$, Theorem 7 is a corollary of the following two lemmas (proved in the next two sections).

Lemma 8.

$$\sum_{\sigma\in\mathcal{D}(n_1,n_2,\dots,n_k)} y^{wex(\sigma)}q^{cr(\sigma)} = \sum_{\sigma\in\mathcal{D}(n_2,n_3,\dots,n_k,n_1)} y^{wex(\sigma)}q^{cr(\sigma)}.$$

Lemma 9.

$$\sum_{\sigma\in\mathcal{D}(n_1,n_2,\dots,n_k)} y^{wex(\sigma)}q^{cr(\sigma)} = \sum_{\sigma\in\mathcal{D}(n_2,n_1,n_3\dots,n_k)} y^{wex(\sigma)}q^{cr(\sigma)}.$$

Proof of Theorem 4. Writing (14) as

$$(x-y)L_n(x) = L_{n+1}(x) + (yq+1)[n]_q L_n(x) + y[n]_q^2 L_{n-1}(x),$$

we derive that

$$\sum_{\sigma\in\mathcal{D}(1,n,n_2,\ldots,n_k)} w(\pi) = \sum_{\sigma\in\mathcal{D}(n+1,n_2,\ldots,n_k)} w(\pi) + (yq+1)[n]_q \sum_{\sigma\in\mathcal{D}(n,n_2,\ldots,n_k)} w(\pi) + y[n]_q^2 \sum_{\sigma\in\mathcal{D}(n-1,n_2,\ldots,n_k)} w(\pi), \tag{31}$$

where $w(\pi) = y^{wex(\sigma)}q^{cr(\sigma)}$. In view of Lemma 6 it suffices to prove (31).

We distinguish four cases for permutations $\pi \in \mathcal{D}(1, n, n_2, \ldots, n_k)$.

a) $\pi(1), \pi^{-1}(1) \in \{2, \ldots, n+1\}$. Let $\pi(1) = i$ and $\pi(j) = 1$ with $i, j \in \{2, \ldots, n+1\}$. Then we define the mapping $\pi \to \pi' \in \mathcal{D}(n-1, n_2, \ldots, n_k)$ by deleting 1 and j and adding the edge $\pi^{-1}(j) \to i$ if $i \neq j$. Clearly

$$w(\pi) = yq^{(i-1)+(j-1)-2}w(\pi').$$

Summing over all $i, j \in \{2, \ldots, n+1\}$ yields the generating function:

$$y[n]_q^2 \sum_{\sigma\in\mathcal{D}(n-1,n_2,\ldots,n_k)} y^{wex(\sigma)}q^{cr(\sigma)}.$$

b) $\pi(1) \in \{2, \ldots, n+1\}$ and $\pi^{-1}(1) > n+1$. We define the mapping $\pi \to \pi' \in \mathcal{D}(n, n_2, \ldots, n_k)$ by deleting $i := \pi(1)$ and replace the two edges $1 \to \pi(1) \to \pi^2(1)$ by $1 \to \pi^2(1)$. Clearly $w(\pi) = yq^{i-1}w(\pi')$. Summing over all $i = 2, \ldots, n+1$ yields the generating function:

$$qy[n]_q \sum_{\sigma\in\mathcal{D}(n,n_2,\ldots,n_k)} y^{wex(\sigma)}q^{cr(\sigma)}.$$

c) $\pi^{-1}(1) \in \{2, \ldots, n+1\}$ and $\pi(1) > n+1$. We define the mapping $\pi \to \pi' \in \mathcal{D}(n, n_2, \ldots, n_k)$ by deleting $i := \pi^{-1}(1)$ and replace the two edges $1 \leftarrow \pi^{-1}(1) \leftarrow \pi^{-2}(1)$ by $1 \leftarrow \pi^{-2}(1)$. Clearly $w(\pi) = q^{i-2}w(\pi')$. Summing over all $i = 2, \ldots, n+1$ yields the generating function:

$$[n]_q \sum_{\sigma\in\mathcal{D}(n,n_2,\ldots,n_k)} y^{wex(\sigma)}q^{cr(\sigma)}.$$

d) $\pi(1) > n+1$ and $\pi^{-1}(1) > n+1$. Clearly we can consider π as a permutation in $\mathcal{D}(n+1, n_2, \ldots, n_k)$. The generating function is

$$\sum_{\sigma\in\mathcal{D}(n+1,n_2,\ldots,n_k)} y^{wex(\sigma)}q^{cr(\sigma)}.$$

Summing up we obtain (31). □

When $k = 2$ Theorem 4 reduces to the orthogonality of the q-Laguerre polynomials (17). When $k = 3$ we can derive the following explicit formula from Theorem 1.

Theorem 10. *We have*

$$\mathcal{L}_q(L_{n_1}(x;q)L_{n_2}(x;q)L_{n_3}(x;q)) = \sum_s \frac{n_1!_q\, n_2!_q\, n_3!_q\, s!_q\, y^s}{(n_1+n_2+n_3-2s)!_q(s-n_3)!_q(s-n_2)!_q(s-n_1)!_q} \times \sum_k \begin{bmatrix} n_1+n_2+n_3-2s \\ k \end{bmatrix}_q y^k q^{\binom{k+1}{2}+\binom{n_1+n_2+n_3-2s-k}{2}}.$$

Proof. By Theorem 1 with $a = \frac{1}{\sqrt{y}}$ and $b = \sqrt{y}q$ we have

$$\begin{aligned}
&\mathcal{L}_q(L_{n_1}(x;\,q)L_{n_2}(x;\,q)L_{n_3}(x;\,q))\\
&= \mathcal{L}_q(L_{n_3}(x;\,q)^2)\left(\frac{\sqrt{y}}{q-1}\right)^{n_1+n_2-n_3} C^{n_3}_{n_1,n_2}(a,b;\,q)\\
&= \sum_{m_2,m_3} \frac{n_1!_q\, n_2!_q\, n_3!_q\, (n_1+m_3)!_q\, y^{n_2+n_3-m_2-m_3}\, q^{\binom{m_2}{2}+\binom{n_3+n_2-n_1-m_2-2m_3+1}{2}}}{(n_3+n_2-n_1-m_2-2m_3)!_q\, m_2!_q\, (m_3+n_1-n_3)!_q\, (m_3+n_1-n_2)!_q\, m_3!_q}.
\end{aligned}$$

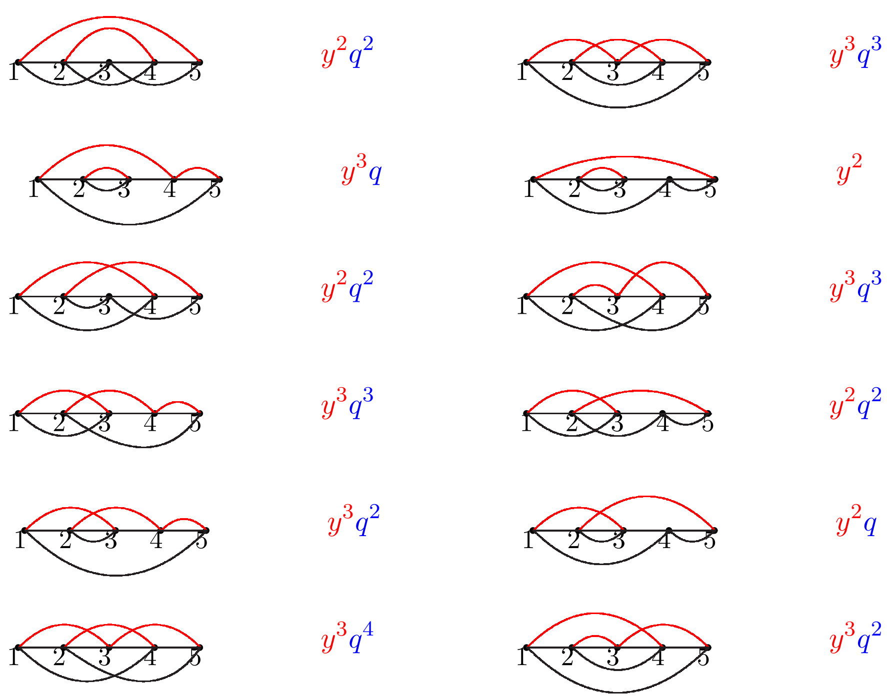
Substituting $s = n_1 + m_3$ and $k = n_3 + n_2 - n_1 - m_2 - 2m_3$ in the last sum yields the desired formula. □

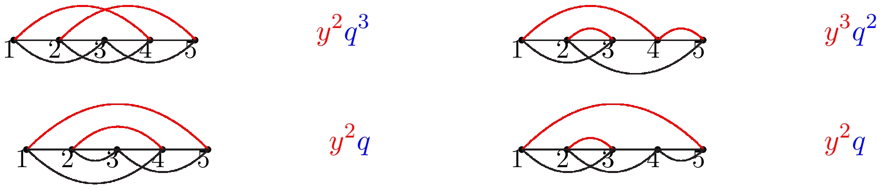
Remark 3. *It would be interesting to give a combinatorial proof of the above result as in [12, 15]. When $q = 1$ such a proof was given in [23].*

We end this section with an example. If $\mathbf{n} = (2, 2, 1)$, by Theorem 8 we have

$$\begin{aligned}
\mathcal{L}_q(L_2(x;\,q)L_2(x;\,q)L_1(x;\,q)) &= \sum_s \frac{2!_q 2!_q 1!_q s!_q y^s}{(5-2s)!_q (s-1)!_q (s-2)!_q (s-2)!_q}\\
&\qquad \times \sum_{k\geq 0} \begin{bmatrix} 5-2s \\ k \end{bmatrix}_q y^k q^{\binom{k+1}{2}+\binom{5-2s-k}{2}}\\
&= (1+q)^3(1+qy)y^2. \qquad (32)
\end{aligned}$$

On the other hand, the sixteen derangements, depicted by their diagrams and the corresponding weights are tabulated as follows:





Summing up we get $\sum_{\sigma\in D(2,2,1)} y^{wex\,\sigma}q^{cr\,\sigma} = y^2(1+qy)(1+q)^3$, which coincides with (32).

6. Proof of Lemma 8

For each fixed $k \in [n]$ define the two subsets of $\mathcal{S}_n$:

$$^{k}\mathcal{S}_n = \{\sigma \in S_n \,|\, \sigma(i) > k \quad \text{for } 1 \le i \le k\},$$
$$\mathcal{S}_n^k = \{\sigma \in S_n \,|\, \sigma(n+1-i) < n+1-k \quad \text{for } 1 \le i \le k\}.$$

We first construct a simple bijection $\Phi_k : {}^{k}\mathcal{S}_n \to \mathcal{S}_n^k$. Let $\sigma \in \mathcal{S}_n^k$. For $1 \le i \le n$ we define $\sigma'(i) := \Phi_k(\sigma)(i)$ as follows:

$$\sigma'(i) = \begin{cases} \sigma(i+k)-k, & \text{if } 1 \le i \le n-k \text{ and } \sigma(i+k) > k;\\ \sigma(i+k)+n-k, & \text{if } 1 \le i \le n-k \text{ and } \sigma(i+k) \le k;\\ \sigma(i+k-n)-k, & \text{if } n-k+1 \le i \le n. \end{cases}$$

We can illustrate the map by the diagrams of permutations.

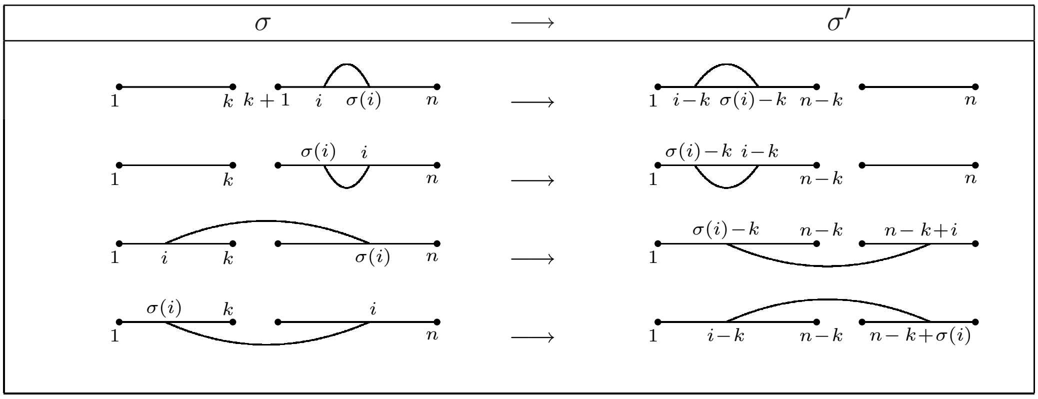
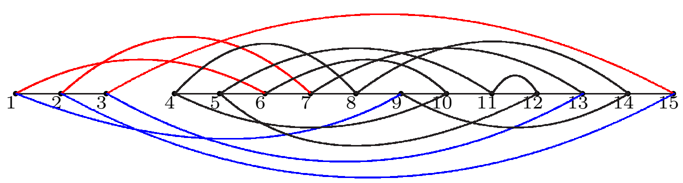
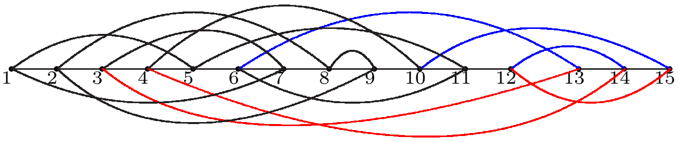


Table 1. The mapping $\Phi_k : \sigma \to \sigma'$.

For example, consider the permutation $\sigma \in {}^{3}\mathcal{S}_{15}$, whose diagram is given below.



Then the diagram of $\Phi_3(\sigma)$ is given by



The main properties of Φ_k are summarized in the following result.

Lemma 11. *For each positive integer $k \in [n]$, the map $\Phi_k : {}^k\mathcal{S}_n \to \mathcal{S}_n^k$ is a bijection such that for any $\sigma \in {}^k\mathcal{S}_n$ there holds*

$$(wex, cr)\Phi_k(\sigma) = (wex, cr)\sigma. \tag{33}$$

Now, Lemma 8 is an immediate consequence of Lemma 11. Let $n = n_1 + n_2 + \cdots + n_k$. Then $\mathcal{D}(n_1, n_2, \ldots, n_k) \subseteq {}^{n_1}\mathcal{S}_n$. By definition of Φ_{n_1}, for any $\sigma \in {}^{n_1}\mathcal{S}_n$ and $i \in [n - n_1]$ satisfying $\sigma(i + n_1) > n_1$, we have $i - \Phi_{n_1}(\sigma)(i) = i + n_1 - \sigma(i + n_1)$, so $\Phi_{n_1}(\mathcal{D}(n_1, n_2, \ldots, n_k)) \subseteq \mathcal{D}(n_2, n_3, \ldots, n_k, n_1)$. Since the cardinality of $\mathcal{D}(n_1, n_2, \ldots, n_k)$ is invariant by permutations of the n_i's and Φ_{n_1} is bijective, we have $\Phi_{n_1}(\mathcal{D}(n_1, n_2, \ldots, n_k)) = \mathcal{D}(n_2, n_3, \ldots, n_k, n_1)$. The result follows then by applying (33).

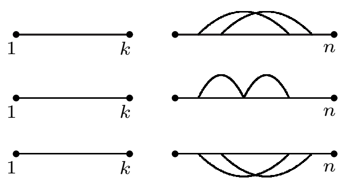
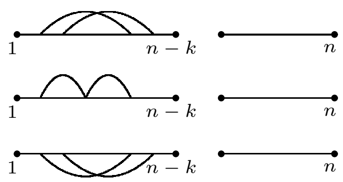
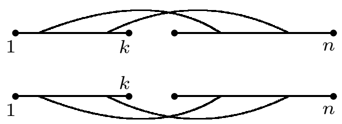
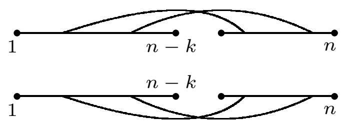
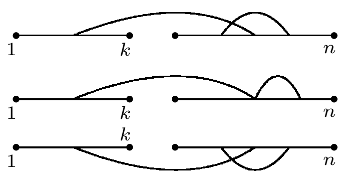
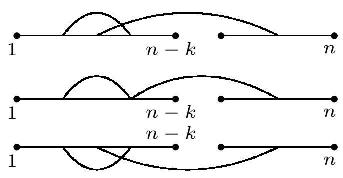
i	$L_i(\sigma)$	$R_i(\sigma')$
1		
2		
3		

TABLE 2. Forms of crossings in $L_i(\sigma)$ and $R_i(\sigma')$.

Proof of Lemma 11 It is easy to see that Φ_k is a bijection. Let $\sigma \in {}^k\mathcal{S}_n$ and $\sigma' = \Phi_k(\sigma)$. The equality $wex(\sigma') = wex(\sigma)$ follows directly from the definition of Φ_k. It then remains to prove that $cr(\sigma') = cr(\sigma)$. We first decompose the crossings of σ and σ' into three subsets. Set

$$L_1(\sigma) = \{(i,j) \mid k < i < j \leq \sigma(i) < \sigma(j) \quad \text{or} \quad i > j > \sigma(i) > \sigma(j) > k\},$$
$$L_2(\sigma) = \{(i,j) \mid i < j \leq k < \sigma(i) < \sigma(j) \quad \text{or} \quad i > j > k \geq \sigma(i) > \sigma(j)\},$$
$$L_3(\sigma) = \{(i,j) \mid i \leq k < j \leq \sigma(i) < \sigma(j) \quad \text{or} \quad i > j > \sigma(i) > k \geq \sigma(j)\},$$

and

$$R_1(\sigma') = \{(i,j) \mid i < j \leq \sigma'(i) < \sigma'(j) \leq n-k \quad \text{or} \quad n-k \geq i > j > \sigma'(i) > \sigma'(j)\},$$
$$R_2(\sigma') = \{(i,j) \mid i < j \leq n-k < \sigma'(i) < \sigma'(j) \quad \text{or} \quad i > j > n-k \geq \sigma'(i) > \sigma'(j)\},$$
$$R_3(\sigma') = \{(i,j) \mid i < j \leq \sigma'(i) \leq n-k < \sigma'(j) \quad \text{or} \quad i > n-k \geq j > \sigma(i) > \sigma(j)\}.$$

The "forms" of the crossings in the L_i's and R_i's is given in Table 2. Clearly, we have $cr(\sigma) = \sum_{i=1}^{3} |L_i(\sigma)|$ and $cr(\sigma') = \sum_{i=1}^{3} |R_i(\sigma')|$ since $\sigma \in {}^k\mathcal{S}_n$ and $\sigma' \in \mathcal{S}_n^k$.

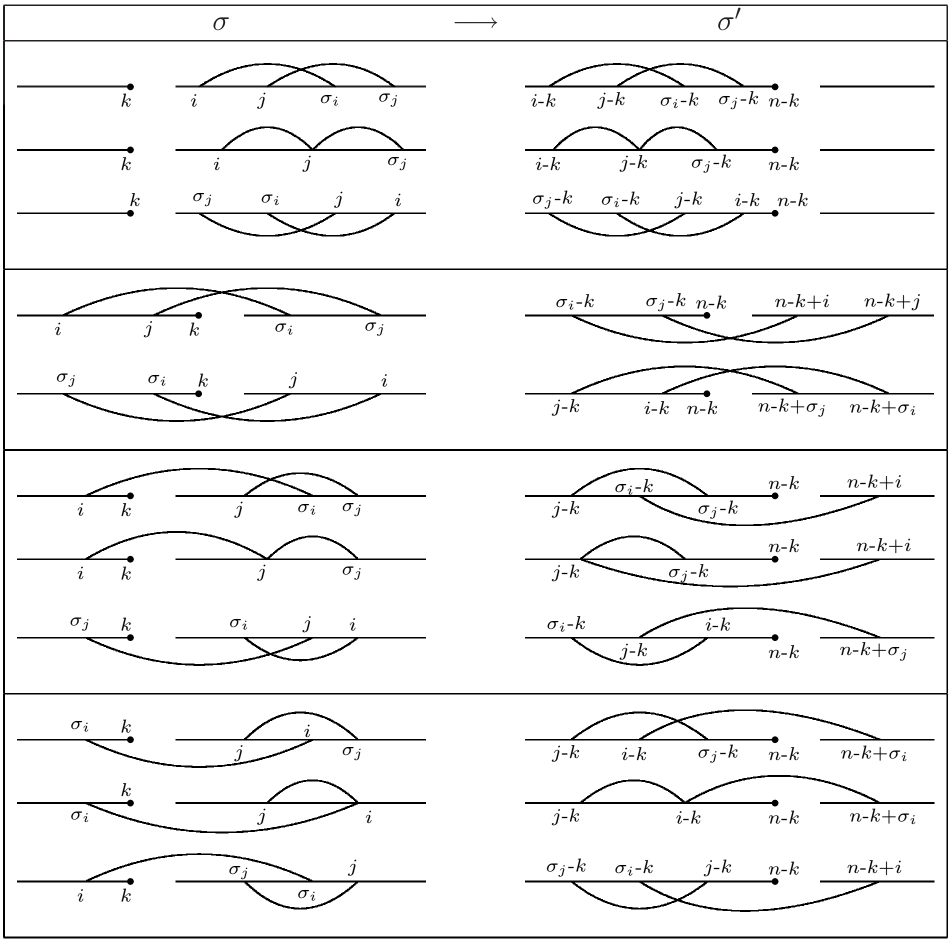


TABLE 3. Effects of the mapping Φ_k on the crossings of σ and σ'.

By the definition of Φ_k, it is readily seen (see Row 1 in Table 3) that $(i,j) \in L_1(\sigma)$ if and only if $(i-k, j-k) \in R_1(\sigma')$, and thus $|L_1(\sigma)| = |R_1(\sigma')|$. Similarly, we have (see Row 2 in Table 3) that $|L_2(\sigma)| = |R_2(\sigma')|$. It then remains to prove that $|L_3(\sigma)| = |R_3(\sigma')|$. Let

$$L_4(\sigma) = \{(i,j) \mid \sigma(i) \le k < j < i \le \sigma(j) \quad \text{or} \quad i \le k < \sigma(j) < \sigma(i) < j\}.$$

Then it is not difficult to show (see Row 4 of Table 3) that $|R_3(\sigma')| = |L_4(\sigma)|$. It then suffices to prove that $|L_3(\sigma)| = |L_4(\sigma)|$.

Suppose $\sigma([1,k]) = \{i_1, i_2, \dots, i_k\}$ and $\sigma^{-1}([1,k]) = \{j_1, j_2, \dots, j_k\}$. Then by definition of $L_3(\sigma)$ and $L_4(\sigma)$ we have

$$|L_3(\sigma)| = \sum_{s=1}^{k} |\{\ell \mid k < \ell \le i_s < \sigma(\ell)\}| + |\{\ell \mid \ell > j_s > \sigma(\ell) > k\}|, \tag{34}$$

$$|L_4(\sigma)| = \sum_{s=1}^{k} |\{\ell \mid \ell > i_s > \sigma(\ell) > k\}| + |\{\ell \mid k < \ell < j_s \le \sigma(\ell)\}|. \tag{35}$$

For any integer $i \in [n]$ set $A_i(\sigma) = \{j \mid j < i < \sigma(j)\}$. Note that it is easily seen that

$$|A_i(\sigma)| = |\{j \mid j < i < \sigma(j)\}| = |\{j \mid j > i > \sigma(j)\}| = |A_i(\sigma^{-1})|. \tag{36}$$

Let $s \in [k]$. By elementary manipulations we get

$$\begin{aligned} |\{\ell \mid k < \ell \le i_s < \sigma(\ell)\}| &= |\{\ell \mid \ell \le i_s < \sigma(\ell)\}| - |\{\ell \mid \ell \le k < i_s < \sigma(\ell)\}| \\ &= |A_{i_s}(\sigma)| + \chi(i_s < \sigma(i_s)) - |\{\ell \mid \ell \le k < i_s < \sigma(\ell)\}| \\ &= |A_{i_s}(\sigma)| + \chi(i_s < \sigma(i_s)) - |\{t \mid i_t > i_s\}|. \end{aligned} \tag{37}$$

By a similar reasoning, we obtain the following identities:

$$|\{\ell \mid \ell > j_s > \sigma(\ell) > k\}| = |A_{j_s}(\sigma^{-1})| - |\{t \mid j_t > j_s\}|. \tag{38}$$

$$|\{\ell \mid \ell > i_s > \sigma(\ell) > k\}| = |A_{i_s}(\sigma^{-1})| - |\{t \mid j_t > i_s\}| \tag{39}$$

$$|\{\ell \mid k \le \ell < j_s \le \sigma(\ell)\}| = |A_{j_s}(\sigma)| + \chi(k < \sigma^{-1}(j_s) < j_s) - |\{t \mid i_t > j_s\}|. \tag{40}$$

Inserting (37) and (38) in (34) and using (36), we get

$$|L_3(\sigma)| = \sum_{s=1}^{k} |A_{i_s}(\sigma)| + |A_{j_s}(\sigma)| + \chi(i_s < \sigma(i_s)) - |\{t \mid i_t > i_s\}| - |\{t \mid j_t > j_s\}|. \tag{41}$$

Similarly, inserting (39) and (40) in (35) and using (36), we get

$$|L_4(\sigma)| = \sum_{s=1}^{k} |A_{i_s}(\sigma)| + |A_{j_s}(\sigma)| + \chi(k < \sigma^{-1}(j_s) < j_s) - |\{t \mid j_t > i_s\}| - |\{t \mid i_t > j_s\}|. \tag{42}$$

Since the i_t's and j_t's are distinct we have $\sum_{s=1}^{k} |\{t \mid i_t > i_s\}| = \sum_{s=1}^{k} |\{t \mid j_t > j_s\}| = \binom{k}{2}$ and thus

$$\sum_{s=1}^{k} |\{t \mid i_t > i_s\}| + |\{t \mid j_t > j_s\}| = k(k-1). \tag{43}$$

On the other hand,

$$\begin{aligned}\sum_{s=1}^{k} |\{t \mid j_t > i_s\}| + |\{t \mid i_t > j_s\}| &= \sum_{s=1}^{k} |\{t \mid i_t \neq j_s\}| \\ &= k^2 - \sum_{s=1}^{k} \chi(j_s \in \{i_1, i_2, \ldots, i_k\}) \\ &= k^2 - \sum_{s=1}^{k} \chi(\sigma^{-1}(j_s) \leq k\}). \end{aligned} \tag{44}$$

Also, it follows from the definition of the j_t's that for any $s \in [k]$, we have $j_s > k$ and $\sigma^{-1}(j_s) \neq j_s$, and thus

$$\chi(k < \sigma^{-1}(j_s) < j_s) + \chi(\sigma^{-1}(j_s) > j_s) = \chi(\sigma^{-1}(j_s) > k). \tag{45}$$

Inserting (53), (54) and (45) in (41) and (42) lead to $|L_3(\sigma)| = |L_4(\sigma)|$ as desired.

7. Proof of Lemma 9

For any two integers n_1, n_2 satisfying $N_2 := n_1 + n_2 \leq n$ we denote by $\mathcal{S}_n^{(n_1,n_2)}$ the set of permutations σ in $\mathcal{S}_n$ such that

$$(i, \sigma(i)) \notin [1, n_1]^2 \cup [n_1 + 1, N_2]^2.$$

In other words, any two integers in $[1, n_1]$ or $[n_1 + 1, N_2]$ are not connected by an arc in its graph.

We now construct a map $\Gamma^{(n_1,n_2)} : \sigma \mapsto \sigma'$ from $\mathcal{S}_n^{(n_1,n_2)}$ to $\mathcal{S}_n^{(n_2,n_1)}$ as follows. For $i = 1, \ldots, n$,

(1) If $i > N_2$ and $\sigma(i) > N_2$, set $\sigma'(i) = \sigma(i)$.

(2) Suppose

$$\{(i, \sigma(i)) \mid i < \sigma(i) \leq N_2\} = \{(i_1, N_2 + 1 - j_1), (i_2, N_2 + 1 - j_2), \ldots, (i_p, N_2 + 1 - j_p)\}$$
$$\{(\sigma(i), i) \mid \sigma(i) < i \leq N_2\} = \{(k_1, N_2 + 1 - \ell_1), (k_2, N_2 + 1 - \ell_2), \ldots, (k_q, N_2 + 1 - \ell_q)\}.$$

Then set $\sigma'(j_s) = N_2 + 1 - i_s$ and $\sigma'(N_2 + 1 - k_t) = \ell_t$ for any $s \in [p]$ and $t \in [q]$.

(3) Let

$$C = \{i \in [1, N_2]\ ;\ \sigma(i) > N_2\} \quad \text{and} \quad D = \{i \in [1, N_2]\ ;\ \sigma^{-1}(i) > N_2\}.$$

It is easy to see that $|C|$=$|D|$. Suppose $C = \{c_1, c_2, \ldots, c_u\}_<$, $D = \{d_1, d_2, \ldots, d_u\}_<$, $\sigma(C) = \{r_1, r_2, \ldots, r_u\}_<$ and $\sigma^{-1}(D) = \{s_1, s_2, \ldots, s_u\}_<$. Let $\alpha, \beta \in \mathcal{S}_u$ be the (unique) permutations satisfying $\sigma(c_i) = r_{\alpha(i)}$ and $\sigma^{-1}(d_i) = s_{\beta(i)}$ for each $1 \leq i \leq u$. Let

$$E = [1, N_2] \setminus \{j_1, \ldots, j_p, N_2 + 1 - k_1, \ldots, N_2 + 1 - k_q\}$$
$$F = [1, N_2] \setminus \{N_2 + 1 - i_1, \ldots, N_2 + 1 - i_p, \ell_1, \ldots, \ell_q\}.$$

Clearly, we have $|E| = |C|$ and $|F| = |D|$. Suppose $E = \{e_1, \ldots, e_u\}_<$ and $F = \{f_1, \ldots, f_u\}_<$. Then set $\sigma'(e_i) = r_{\alpha(i)}$ and $\sigma'(s_i) = f_{\beta(i)}$ for each $1 \leq i \leq u$.

We can illustrate the map through the diagrams of permutations. See Table 4.

For example, if we consider the permutation in $\mathcal{S}_{15}^{(3,4)}$ whose diagram is given by

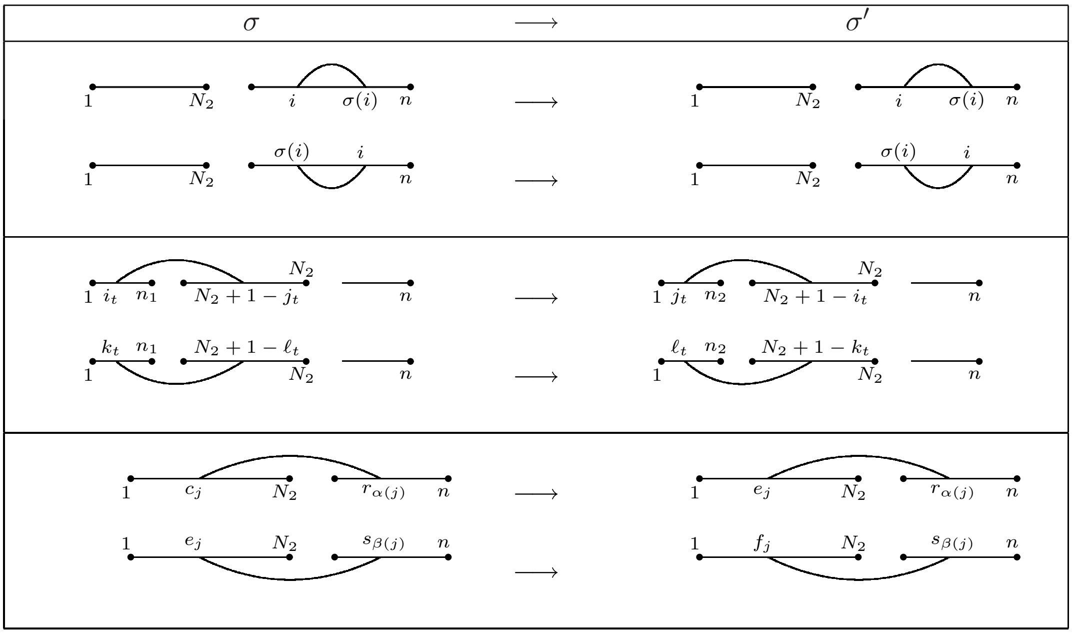
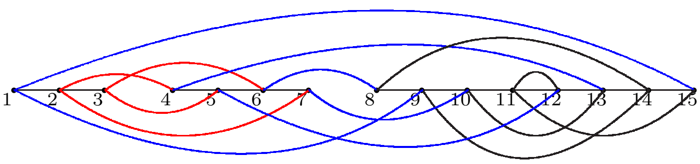
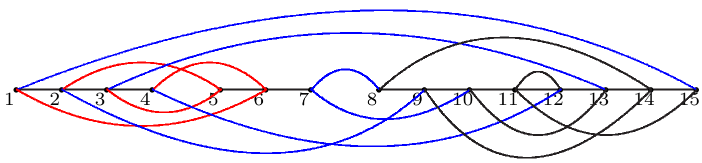


TABLE 4. The mapping $\Gamma^{(n_1,n_2)} : \sigma \mapsto \sigma'$



then the diagram of $\Gamma^{(n_1,n_2)}(\sigma)$ is given by



It is not hard to check that $\Gamma^{(n_1,n_2)}$ is well defined from $\mathcal{S}_n^{(n_1,n_2)}$ to $\mathcal{S}_n^{(n_2,n_1)}$. Since each step of the construction of $\Gamma^{(n_1,n_2)}$ is reversible, the map $\Gamma^{(n_1,n_2)}$ is bijective. Actually we can prove, the details are left to the reader, that $(\Gamma^{(n_1,n_2)})^{-1} = \Gamma^{(n_2,n_1)}$.

Lemma 12. *For each positive integers n_1, n_2, n, with $N_2 \le n$, the map $\Gamma^{(n_1,n_2)}$ is a bijection from $\mathcal{S}_n^{(n_1,n_2)}$ to $\mathcal{S}_n^{(n_2,n_1)}$ such that for each $\sigma \in \mathcal{S}_n^{(n_1,n_2)}$, we have*

$$(wex, cr)\Gamma^{(n_1,n_2)}(\sigma) = (wex, cr)\sigma. \tag{46}$$

As an immediate consequence, we obtain Lemma 9. Let $n = n_1 + n_2 + \cdots + n_k$. Then $\mathcal{D}(n_1, n_2, \ldots, n_k) \subseteq \mathcal{S}_n^{(n_1,n_2)}$. By definition of $\Gamma^{(n_1,n_2)}$, for any $\sigma \in \mathcal{S}_n^{(n_1,n_2)}$ and $i > N_2$ satisfying

$\sigma(i) > N_2$, we have

$$i - \Gamma^{(n_1,n_2)}(\sigma)(i) = i - \sigma(i),$$

so $\Gamma^{(n_1,n_2)}(\mathcal{D}(n_1, n_2, \ldots, n_k)) \subseteq \mathcal{D}(n_2, n_3, \ldots, n_k, n_1)$. Since the cardinality of $\mathcal{D}(n_1, n_2, \ldots, n_k)$ doesn't depend of the order of the n_i's and $\Gamma^{(n_1,n_2)}$ is a bijection, we have

$$\Gamma^{(n_1,n_2)}(\mathcal{D}(n_1, n_2, \ldots, n_k)) = \mathcal{D}(n_2, n_3, \ldots, n_k, n_1).$$

Lemma 9 then follows from (46).

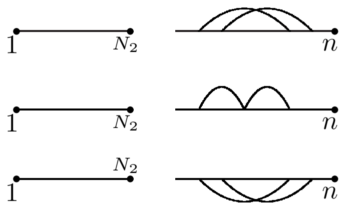
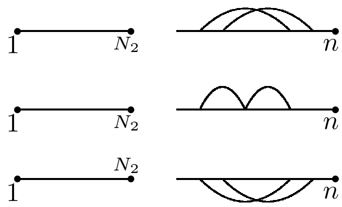
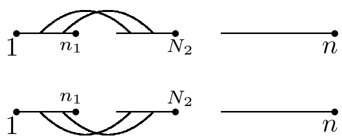
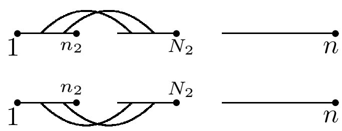
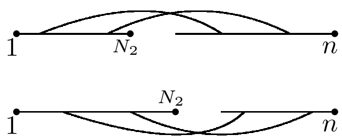
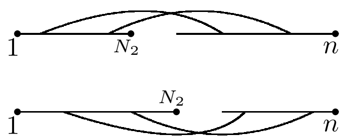
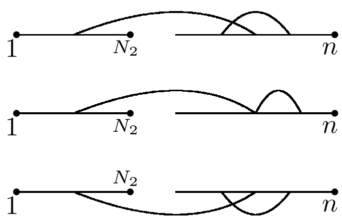
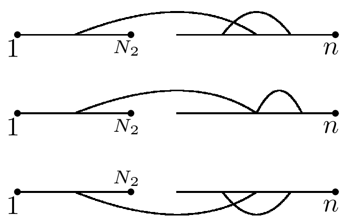
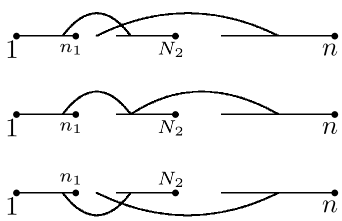
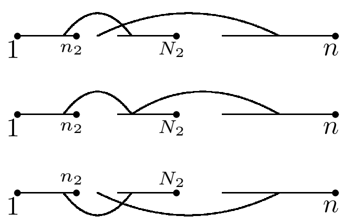
i	$G_i^{(n_1,n_2)}(\gamma)$	$G_i^{(n_2,n_1)}(\gamma)$
1		
2		
3		
4		
5		

TABLE 5. Forms of the crossings in $G_i^{(n_1,n_2)}(\gamma)$ and $G_i^{(n_2,n_1)}(\gamma)$.

Proof of Lemma 12 It was shown above that $\Gamma^{(n_1,n_2)}$ is bijective. Let $\sigma \in \mathcal{S}_n^{(n_1,n_2)}$ and $\sigma' := \Gamma^{(n_1,n_2)}(\sigma)$. The equality $wex(\sigma') = wex(\sigma)$ is an immediate consequence of the definition of $\Gamma^{(n_1,n_2)}$. It then remains to prove that $cr(\sigma') = cr(\sigma)$. The idea is the same than for the proof of Lemma 8. We first decompose the number of crossings of σ and σ'. For each permutation $\gamma \in \mathcal{S}_n$, set

$$\begin{aligned}
G_1^{(n_1,n_2)}(\gamma) &= \{(i,j) \mid N_2 < i < j \le \gamma(i) < \gamma(j) \quad \text{or} \quad i > j > \gamma(i) > \gamma(j) > N_2\},\\
G_2^{(n_1,n_2)}(\gamma) &= \{(i,j) \mid i < j < \gamma(i) < \gamma(j) \le N_2 \quad \text{or} \quad N_2 \ge i > j > \gamma(i) > \gamma(j)\},\\
G_3^{(n_1,n_2)}(\gamma) &= \{(i,j) \mid i < j \le N_2 < \gamma(i) < \gamma(j) \quad \text{or} \quad i > j > N_2 \ge \gamma(i) > \gamma(j)\},\\
G_4^{(n_1,n_2)}(\gamma) &= \{(i,j) \mid i \le N_2 < j \le \gamma(i) < \gamma(j) \quad \text{or} \quad i > j > \gamma(i) > N_2 \ge \gamma(j)\},\\
G_5^{(n_1,n_2)}(\gamma) &= \{(i,j) \mid i < j \le \gamma(i) \le N_2 < \gamma(j) \quad \text{or} \quad i > N_2 \ge j > \gamma(i) > \gamma(j)\}.
\end{aligned}$$

Clearly, for any $\gamma \in \mathcal{S}_n^{(n_1,n_2)}$, we have $cr(\gamma) = \sum_{i=1}^5 |G_i^{(n_1,n_2)}(\gamma)|$. In particular,

$$cr(\sigma) = \sum_{i=1}^5 |G_i^{(n_1,n_2)}(\sigma)| \quad \text{and} \quad cr(\sigma') = \sum_{i=1}^5 |G_i^{(n_2,n_1)}(\sigma')|. \tag{47}$$

The "forms" of the crossings in the $G_i^{(n_1,n_2)}$'s and $G_i^{(n_2,n_1)}$'s are given in Table 5. By the definition of $\Gamma^{(n_1,n_2)}$, it is readily seen (see Row 1 in Table 6) that $G_1^{(n_1,n_2)}(\sigma) = G_1^{(n_2,n_1)}(\sigma')$ and thus $|G_1^{(n_1,n_2)}(\sigma)| = |G_1^{(n_2,n_1)}(\sigma')|$. By similar considerations we can prove (see Table 6) that $|G_i^{(n_1,n_2)}(\sigma)| = |G_i^{(n_2,n_1)}(\sigma')|$ for $i = 2, 3, 4$. It then suffices to prove that $|G_5^{(n_1,n_2)}(\sigma)| = |G_5^{(n_2,n_1)}(\sigma')|$ which will follow from the following lemma.

Lemma 13. *Let n_1, n_2, n be positive integers with $N_2 \le n$ and $\gamma \in \mathcal{S}_n^{(n_1,n_2)}$. Suppose that*

$$\begin{aligned}
B(\gamma) &:= \{(i,\gamma(i)) \mid i < \gamma(i) \le N_2\} = \{(i_1,j_1),(i_2,j_2),\dots,(i_p,j_p)\}\\
B(\gamma^{-1}) &= \{(\gamma(i),i) \mid \gamma(i) < i \le N_2\} = \{(k_1,\ell_1),(k_2,\ell_2),\dots,(k_q,\ell_q)\},
\end{aligned}$$

with $i_1 < i_2 < \cdots < i_p$ and $k_1 < k_2 < \cdots < k_q$. Then we have

$$|G_5^{(n_1,n_2)}(\gamma)| = \sum_{r=1}^p (j_r - i_r) + \sum_{r=1}^q (\ell_r - k_r - 1) - \binom{p+q}{2}. \tag{48}$$

Suppose

$$\begin{aligned}
B(\sigma) &= \{(i_1, N_2+1-j_1),\dots,(i_p, N_2+1-j_p)\}\\
B(\sigma^{-1}) &= \{(k_1, N_2+1-\ell_1),\dots,(k_q, N_2+1-\ell_q)\},
\end{aligned}$$

then, by construction of σ', we have

$$\begin{aligned}
B(\sigma') &= \{(j_1, N_2+1-i_1),\dots,(j_p, N_2+1-i_p)\}\\
B(\sigma'^{-1}) &= \{(\ell_1, N_2+1-k_1),\dots,(\ell_q, N_2+1-k_q)\}.
\end{aligned}$$

By symmetry, the identity (48) is also valid on $\mathcal{S}_n^{(n_2,n_1)}$. Applying (48) to σ' and σ lead to $|G_5^{(n_1,n_2)}(\sigma)| = |G_5^{(n_1,n_2)}(\sigma')|$. This conclude the proof of Lemma 12. It then remains to prove Lemma 13.

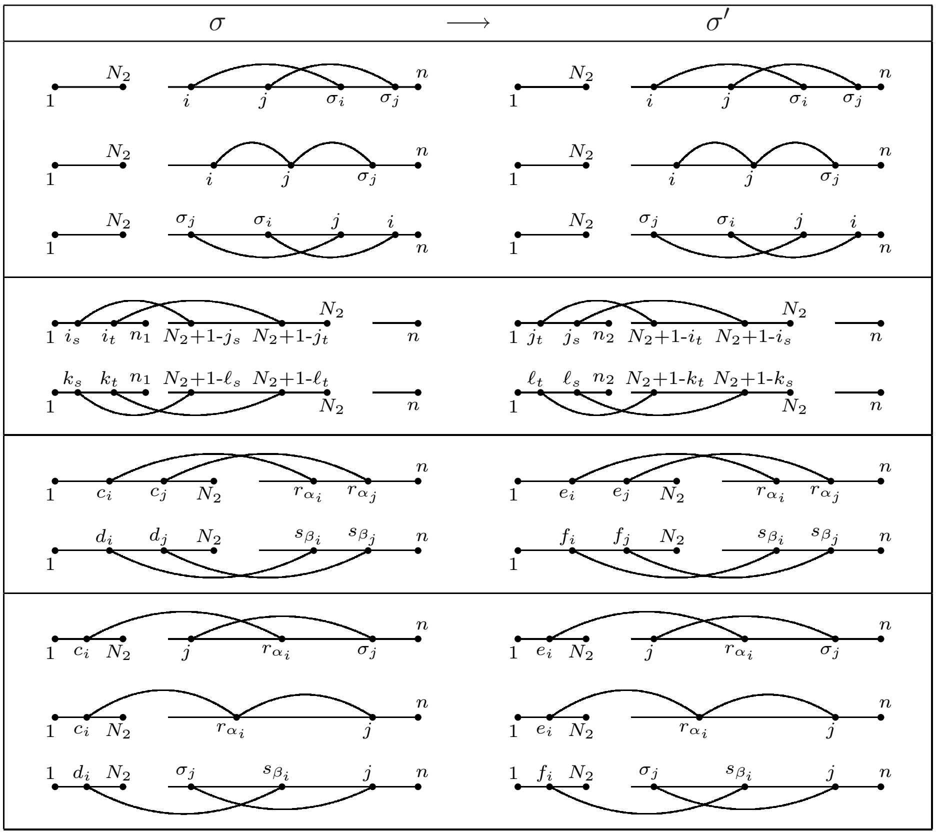


TABLE 6. Effects of the mapping $\Gamma^{(n_1,n_2)}$ on the crossings of σ and σ'.

Proof of Lemma 13 By definition of $G_5^{(n_1,n_2)}(\gamma)$, we have

$$|G_5^{(n_1,n_2)}(\gamma)| = |\{(i,j) \mid i < j < \gamma(i) \le N_2 < \gamma(j)\}| + |\{(i,j) \mid \gamma(j) < \gamma(i) < j \le N_2 < i\}| \\ + |\{i \mid i < \gamma(i) \le N_2 < \gamma^2(i)\}|. \tag{49}$$

Now, by elementary manipulations and the definition of $B(\gamma)$ we get

$$|\{(i,j) \mid i < j < \gamma(i) \le N_2 < \gamma(j)\}| = \sum_{r=1}^{p} |\{x \mid i_r < x < j_r \le N_2 < \gamma(x)\}| \\ = \sum_{r=1}^{p} |\{x \mid i_r < x < j_r\}| - |\{x \mid i_r < x < j_r\,,\ \gamma(x) \le N_2\}|.$$

But for any $r \in [1,p]$, we have $|\{x \mid i_r < x < j_r\}| = j_r - i_r - 1$ and

$$\begin{aligned} &|\{x \mid i_r < x < j_r\,,\, \gamma(x) \leq N_2\}| \\ =&|\{x \mid i_r < x < j_r\,,\, x < \gamma(x) \leq N_2\}| + |\{x \mid i_r < x < j_r\,,\, \gamma(x) < x \leq N_2\}| \\ =&|\{t \mid i_r < i_t < j_r\}| + |\{t \mid i_r < \ell_t < j_r\}| \quad \text{by definition of } B(\gamma) \text{ and } B(\gamma^{-1}), \\ =&|\{t \mid i_r < i_t\}| + |\{t \mid \ell_t < j_r\}|, \end{aligned}$$

since by definition of $\mathcal{S}_n^{(n_1,n_2)}$ we have that for any integers r and t, $i_r \leq n_1$, $k_t \leq n_1$, $j_r > n_1$ and $\ell_t > n_1$, and thus $i_r < j_t$ and $i_r < \ell_t$.

Summing over all r yields

$$|\{(i,j) \mid i < j < \gamma(i) \leq N_2 < \gamma(j)\}| = \sum_{r=1}^{p}(j_r - i_r - 1) - \sum_{r=1}^{p}|\{t \mid i_r < i_t\}| - \sum_{r=1}^{p}|\{t \mid \ell_t < j_r\}|. \quad (50)$$

Since $|\{(i,j) \mid \gamma(j) < \gamma(i) < j \leq N_2 < i\}| = |\{(i,j) \mid i < j < \gamma^{-1}(i) \leq N_2 < \gamma^{-1}(j)\}|$, it follows from (50) that

$$|\{(i,j) \mid \gamma(j) < \gamma(i) < j \leq N_2 < i\}| = \sum_{r=1}^{q}(\ell_r - k_r - 1) - \sum_{r=1}^{q}|\{t \mid k_r < k_t\}| - \sum_{r=1}^{q}|\{t \mid j_t < \ell_r\}|. \quad (51)$$

Remarking that $|\{i \mid i < \gamma(i) \leq N_2 < \gamma^2(i)\}| = |\{t \mid \gamma(j_t) > N_2\}|$ and inserting (50) and (51) in (49) lead to

$$\begin{aligned} |G_5^{(n_1,n_2)}(\gamma)| = &\sum_{r=1}^{p}(j_r - i_r - 1) + \sum_{r=1}^{q}(\ell_r - k_r - 1) + |\{t \mid \gamma(j_t) > N_2\}| - \sum_{r=1}^{p}|\{t \mid i_r < i_t\}| \quad (52) \\ &- \sum_{r=1}^{p}|\{t \mid \ell_t < j_r\}| - \sum_{r=1}^{q}|\{t \mid k_r < k_t\}| - \sum_{r=1}^{q}|\{t \mid j_t < \ell_r\}|. \end{aligned}$$

Since the i_r's and k_r's are distinct we have

$$\sum_{r=1}^{p}|\{t \mid i_r < i_t\}| = \binom{p}{2} \quad \text{and} \quad \sum_{r=1}^{q}|\{t \mid k_r < k_t\}| = \binom{q}{2}. \quad (53)$$

On the other hand,

$$\begin{aligned} \sum_{r=1}^{p}|\{t \mid \ell_t < j_r\}| + \sum_{r=1}^{q}|\{t \mid j_t < \ell_r\}| &= \sum_{r=1}^{p}|\{t \mid \ell_t \neq j_r\}| \\ &= pq - |\{t \mid j_t \in \{\ell_1, \ell_2, \ldots, \ell_q\}| \\ &= pq - \sum_{s=1}^{k}|\{t \mid \gamma(j_t) \leq N_2\}|, \quad (54) \end{aligned}$$

where the last identity follows from the definitions of $B(\gamma)$ and $B(\gamma^{-1})$. Inserting (53) and (54) in (52) lead to (48). This concludes the proof of Lemma 13. □

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