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Combinatorics of Riordan arrays with identical A and Z sequences[☆]

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A B S T R A C T

In theory, Riordan arrays can have any A-sequence and any Z-sequence. For examples of combinatorial interest they tend to be related. Here we look at the case that they are identical or nearly so. We provide a combinatorial interpretation in terms of weighted Łukasiewicz paths and then look at several large classes of examples.

1. Introduction

We begin by briefly describing the concept of the Riordan group. Let $\mathcal{F}_n$ for $n = 0, 1, 2, \ldots$ be the set of formal power series defined by

$$\mathcal{F}_n = \{f(z) = f_n z^n + f_{n+1}z^{n+1} + f_{n+2}z^{n+2} + \cdots \,|\, f_n \neq 0,\ f_i \in \mathbb{C}\}.$$

A *Riordan array* denoted $(g(z), f(z))$ is an infinite lower triangular matrix such that the generating function of the kth column for $k = 0, 1, 2, \ldots$ is $g(z)f^k(z)$ where $g(z) \in \mathcal{F}_0$, $g(0) = 1$ and $f(z) \in \mathcal{F}_1$. Given a Riordan array $(g(z), f(z))$ and column vector $h = [h_0, h_1, h_2, \ldots]^T$, then the product of $(g(z), f(z))$ and h gives a column vector whose generating function is $g(z) \cdot h(f(z))$. Simply we write

$$(g(z), f(z))h(z) = g(z) \cdot h(f(z))$$

where $h(z) = \sum_{j\geq 0} h_j z^j$. This property is called the *fundamental theorem for Riordan arrays* and this leads to the matrix multiplication for Riordan arrays:

$$(g(z), f(z))(h(z), \ell(z)) = (g(z) \cdot h(f(z)), \ell(f(z))).$$

The set of all Riordan arrays forms a group under the above operation of a matrix multiplication. This is the *Riordan group* introduced by Shapiro et al. [11]. One subgroup of the Riordan group that we will encounter is the Bell subgroup which consists of the matrices of the form $(g(z), zg(z))$.

A Riordan array $R = (g(z), f(z)) = [r_{n,k}]_{n,k\geq 0}$ can be characterized [5,6,14] by two sequences $A = (a_0, a_1, \ldots)$ and $Z = (z_0, z_1, \ldots)$ such that

$$r_{n+1,0} = \sum_{j\geq 0} z_j r_{n,j} \quad \text{and} \quad r_{n+1,k+1} = \sum_{j\geq 0} a_j r_{n,k+j} \tag{1}$$

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for $n, k \geq 0$. If $A(z)$ and $Z(z)$ are the generating functions for the A- and Z-sequences respectively, then it follows that

$$g(z) = \frac{1}{1 - zZ(f(z))} \quad \text{and} \quad f(z) = zA(f(z)). \tag{2}$$

Riordan arrays arise in the enumeration of lattice paths, e.g. Dyck paths, Motzkin paths, and Schröder paths and so on; see [3,10,11,14]. In particular, a connection between Riordan arrays and 'unweighted' lattice paths having privileged access to the main diagonal has been studied in [6] by means of the A-matrix, which includes the concept of A- and Z-sequences. As suggested by the referee, the concept of consistent Riordan arrays is also related to the concept of lattice paths having unprivileged access to the main diagonal in [6].

The concept of the A- and Z-sequences is often useful for enumeration problems such as deriving combinatorial identities or recursion formulas for entries of a Riordan array. In general, Riordan arrays can have any A-sequence and any Z-sequence, independently. In this paper, we consider Riordan arrays such that the A- and Z-sequences are identical or nearly so.

We begin with a *Dyck path* from $(0, 0)$ to $(2n, 0)$ using the up step $U = (1, 1)$ and the down step $D = (1, -1)$ as its possible steps with the additional constraint that the path cannot go below the x-axis. If we recount the partial Dyck paths that end at $(2n - k, k)$, we get the familiar Catalan triangle matrix:

$$(C(z), zC(z)) = \begin{bmatrix} 1 & & & & \\ 1 & 1 & & & \\ 2 & 2 & 1 & & \\ 5 & 5 & 3 & 1 & \\ 14 & 14 & 9 & 4 & 1 \\ & & \cdots & & \end{bmatrix}$$

where $C(z) = (1 - \sqrt{1 - 4z})/2z$. We note that here both the A- and Z-sequence are $(1, 1, 1, \ldots)$ i.e., $A(z) = Z(z)$.

A Riordan array $(g(z), f(z))$ is said to be *consistent* if $A(z) = Z(z)$. It follows from (2) that $(g(z), f(z))$ is consistent if and only if $g(z) = \frac{1}{1-f(z)}$ or $g(z) = 1 + g(z)f(z)$ where $g'(0) \neq 0$. Hence a consistent Riordan array can be expressed as either $(g(z), \frac{g(z)-1}{g(z)})$ or $(\frac{1}{1-f(z)}, f(z))$. In particular, $(\frac{1}{1-f(z)}, f(z))^{-1} = (1 - z, \bar{f}(z))$ where $\bar{f}(z)$ is the compositional inverse of $f(z)$ i.e., $f(\bar{f}(z)) = \bar{f}(f(z)) = z$.

One purpose of this paper is to show that every consistent Riordan array with nonnegative integer coefficients has a certain combinatorial interpretation. Specifically, in Section 2 we first provide a combinatorial interpretation in terms of weighted Łukasiewicz paths. Then several large classes of examples such as k-Dyck paths, colored Schröder paths, colored Motzkin paths, and ordered trees are discussed in Section 3.

2. Combinatorial interpretations for consistent Riordan arrays

Throughout this paper, a *Łukasiewicz path* (or simply *L-path*) is a lattice path that starts at the origin, cannot go below the x-axis and has as possible steps $S_{-r} = (1, -r)(r \geq -1)$. It is known that the number of L-paths from $(0, 0)$ to (n, k) is $\eta_{n,k} := \frac{k+1}{n+1}\binom{2n-k}{n}$, which is the (n, k)-entry of the Catalan triangle $(C(z), zC(z))$ above. Let us consider an L-path P with weights. The weight of the step S_{-r} is denoted by $\omega(S_{-r})$ and the weight of P is defined as the product of weights of the steps used and is denoted by $\omega(P)$.

Theorem 2.1. *Let $R = [r_{n,k}]_{n,k\geq 0}$ be a Riordan array with $A = (a_0, a_1, \ldots)$. Then R is consistent if and only if $r_{n,k}$ is the sum of weights of weighted L-paths from the origin to the point (n, k) using the weights*

$$\omega(S_{-r}) = \begin{cases} a_r & \text{if } S_{-r} \text{ touches the } x\text{-axis,} \\ a_{r+1} & \text{otherwise} \end{cases} \tag{3}$$

where $r \geq 0$ and $\omega(S_1) = a_0$.

Proof. Let $A(z) = Z(z) = \sum_{n\geq 0} a_n z^n$. Let us define $\sigma(n, k)$ to be the sum of weights of weighted L-paths from $(0, 0)$ to (n, k) using the weights given by (3) where $\sigma(0, 0) := 1$. It is enough to consider the final steps of the paths that go to the point (n, k). If $k = 0$, the final step S_{-r} $(r = 0, 1, 2, \ldots)$ meets the x-axis. Hence from (3) we have

$$\sigma(n + 1, 0) = \sum_{j\geq 0} \omega(S_{-j})\sigma(n, j) = \sum_{j\geq 0} a_j \sigma(n, j).$$

Let $k \geq 1$. We note that when $k = 1$ the paths coming from the x-axis have S_1 as their final step. Otherwise, the final steps of paths do not meet the x-axis. Since $\omega(S_1) = a_0$ for all $k \geq 1$, from (3) we have

$$\begin{aligned} \sigma(n + 1, k) &= \omega(S_1)\sigma(n, k - 1) + \sum_{j\geq 0} \omega(S_{-j})\sigma(n, k + j) \\ &= a_0\sigma(n, k - 1) + \sum_{j\geq 0} a_{j+1}\sigma(n, k + j) = \sum_{j\geq 0} a_j\sigma(n, k + j - 1). \end{aligned}$$

By substituting a_j for z_j in (1), we see that $r_{n,k}$ and $\sigma(n,k)$ satisfy the same recurrence relation with the same initial condition $r_{0,0} = \sigma(0,0) = 1$.

It is easy to show that the converse holds for $k \geq 0$. Hence the proof is complete. □

Even at this point there are two ways to develop examples. We can start with an $A = Z$-sequence and use it to put appropriate weights on L-paths or we can use any $g(z)$ together with $g(z) - 1$ as our leftmost columns to develop the rest of the array.

To compute the row sums of any Riordan array we multiply by the column vector $[1, 1, 1, \ldots]^T$ which has the generating function $1/(1-z)$. It is convenient to switch freely between a sequence, the sequence as a column vector, and its generating function. In the case of consistent arrays we get

$$\left(g(z), \frac{g(z)-1}{g(z)}\right)\frac{1}{1-z} = g(z)\cdot\left(\frac{1}{1-(\frac{g(z)-1}{g(z)})}\right) = g^2(z). \tag{4}$$

It follows that the row sums of a consistent array $(g(z), \frac{g(z)-1}{g(z)})$ are the elements of the $k = 1$ column of the Riordan matrix $(g(z), zg(z))$ except for a zero as the initial term.

By Theorem 2.1, $g^2(z)$ represents the generating function for the sum of weights of L-paths ending at some point on the line $x = n$ using the weights given by (3) where $(a_0, a_1, \ldots)$ is the sequence with the generating function $\frac{z}{h(z)}$ for the solution $h(z)$ of $g(h(z)) = \frac{1}{1-z}$. Incidentally, $g^2(z)$ is also the generating function counting all points on the x-axis for all such weighted L-paths.

For an integer k, a Riordan array (g, f) is said to be *k-consistent* if $A(z) = Z(z) - k$ i.e., $A = (a_0, a_1, \ldots)$ and $Z = (k + a_0, a_1, \ldots)$. It follows from (2) that $(g(z), f(z))$ is k-consistent if and only if

$$g(z) = \frac{1}{1-kz-f(z)} \quad \text{or} \quad f(z) = \frac{(1-kz)g(z)-1}{g(z)}. \tag{5}$$

For instance, let us consider the k-consistent array $(g(z), f(z))$ with $A = (1, 1, 1, \ldots)$ and $Z = (k + 1, 1, 1, \ldots)$. Then we have

$$g(z) = \frac{C(z)}{1-kzC(z)} \quad \text{and} \quad f(z) = zC(z).$$

In particular, we obtain the k-consistent Riordan arrays for $k = -1, 0, 1, 2$ respectively:

$$(F(z), zC(z)), \qquad (C(z), zC(z)), \qquad (C^2(z), zC(z)), \qquad (B(z)C(z), zC(z))$$

where $C(z)$, $F(z)$ and $B(z)$ are the generating functions for the Catalan numbers, the Fine numbers (A000957, [13]) [2] and the central binomial coefficients, respectively.

A similar combinatorial interpretation as in Theorem 2.1 is given for a k-consistent Riordan array. Also see [8].

Theorem 2.2. *Let $R = [r_{i,j}^{(k)}]_{i,j\geq 0}$ be a k-consistent Riordan array with $A = (a_0, a_1, \ldots)$. Then $r_{i,j}^{(k)}$ is the sum of weights of weighted L-paths from the origin to the point (i, j) using the weights*

$$\omega(S_{-r}) = \begin{cases} k\delta_{r,0} + a_r & \text{if } S_{-r} \text{ touches the } x\text{-axis,} \\ a_{r+1} & \text{otherwise} \end{cases}$$

where $r \geq 0$, $\omega(S_1) = a_0$ and $\delta_{r,0}$ denotes the Kronecker delta.

The following theorem shows that 0-consistent and 1-consistent Riordan arrays may have different combinatorial interpretations by means of lattice paths.

Theorem 2.3. *Let $g_{i,j}$ be the number of lattice paths from $(0, 0)$ to $((m+1)i - j, mj)$ using the step set $\{(1, 1), (1, -m), (m + 1, 0)\}$ for a nonnegative integer m, where the paths cannot go below the x-axis. Then the number array $[g_{i,j}]_{i,j\geq 0}$ is a 1-consistent Riordan array given by $(G(z), zG^m(z))$, where $G(z) = \sum_{i\geq 0} g_{i,0}z^i$. However, if we do not allow flat steps on the x-axis then the array is a 0-consistent Riordan array given by $(\frac{1}{1-zG^m(z)}, zG^m(z))$, where we refer to the step $(m + 1, 0)$ as a flat step.*

Proof. Let us consider a lattice path whose first return to the x-axis is at $((m+1)\ell, 0)$ $(\ell \geq 1)$. Since $g_{i,0}$ counts lattice paths ending at $((m + 1)i, 0)$ and $G(z) = \sum_{i\geq 0} g_{i,0}z^i$, we have $G(z) = 1 + zG(z) + zG^{m+1}(z)$ as illustrated in Fig. 1.

Let $j \geq 1$. Since $g_{i,j}$ counts lattice paths ending at $((m + 1)i - j, mj)$ and the jth column generating function is $\sum_{i\geq 0} g_{i,j}z^i$, a similar diagram to that in Fig. 1 shows that $\sum_{i\geq 0} g_{i,j}z^i = z^jG^{mj+1}(z)$. Thus

$$g_{i,j} = [z^i]z^jG^{mj+1}(z) = [z^i]G(z)\cdot(zG^m(z))^j.$$

Since $zG^m(z) = \frac{(1-z)G(z)-1}{G(z)}$, it follows from (5) that $[g_{i,j}]_{i,j\geq 0} = (G(z), zG^m(z))$ is a 1-consistent Riordan array.

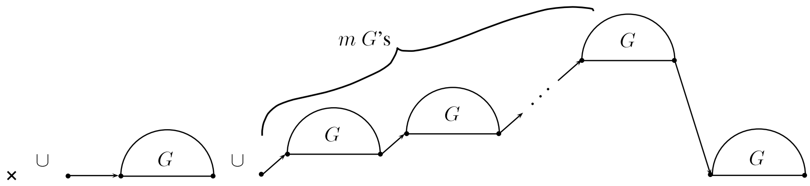

Fig. 1. A lattice path with a first return at $((m+1)i, 0)$.

Now let $g^*_{i,j}$ be the number of lattice paths ending at $((m+1)i-j, mj)$ with no flat step $(m+1, 0)$ on the x-axis, and let $g(z) = \sum_{i\geq 0} g^*_{i,0} z^i$. Since every lattice path with no flat step on the x-axis starts with the step $(1, 1)$, we have $g(z) = 1 + zg(z)G^m(z)$. For $j \geq 1$, by a similar argument to the previous case we have $\sum_{i\geq 0} g^*_{i,j} z^i = g(z) \cdot (zG^m(z))^j$. Since $g(z) = \frac{1}{1-zG^m(z)}$, it follows from (5) that $[g^*_{i,j}]_{i,j\geq 0} = (\frac{1}{1-zG^m(z)}, zG^m(z))$ is a 0-consistent array. $\square$

For instance, let $m = 1$. Since the step set is $\{(1, 1), (1, -1), (2, 0)\}$, $g_{i,0}$ counts the number of Schröder paths from $(0, 0)$ to $(2i, 0)$, and $G(z) = r(z) = \sum_{i\geq 0} g_{i,0} z^i = \frac{1-z-\sqrt{1-6z+z^2}}{2z}$ is the generating function for the large Schröder numbers. Thus the 1-consistent and 0-consistent Riordan arrays start as

$$(r(z), zr(z)) = \begin{bmatrix} 1 & & & & \\ 2 & 1 & & & \\ 6 & 4 & 1 & & \\ 22 & 16 & 6 & 1 & \\ 90 & 68 & 30 & 8 & 1 \\ & & \cdots & & \end{bmatrix}, \qquad (s(z), zr(z)) = \begin{bmatrix} 1 & & & & \\ 1 & 1 & & & \\ 3 & 3 & 1 & & \\ 11 & 11 & 5 & 1 & \\ 45 & 45 & 23 & 7 & 1 \\ & & \cdots & & \end{bmatrix}$$

respectively, where $s(z) = \frac{1}{1-zr(z)}$ is the generating function for the small Schröder numbers. In fact, the number of Schröder paths from $(0, 0)$ to $(5, 1)$ is $16 = [z^3]r(z) \cdot (zr(z))^1$. Eliminating the paths which do have a flat step on the x-axis leaves $11 = [z^3]s(z) \cdot (zr(z))^1$.

Here are three more examples of consistent arrays $(g(z), f(z))$. Familiar sequences show up but in unfamiliar arrays.

$$\left(\frac{1}{1-zm(z)}, zm(z)\right) = \begin{bmatrix} 1 & & & & & \\ 1 & 1 & & & & \\ 2 & 2 & 1 & & & \\ 5 & 5 & 3 & 1 & & \\ 13 & 13 & 9 & 4 & 1 & \\ 35 & 35 & 26 & 14 & 5 & 1 \\ & & \cdots & \cdots & & \end{bmatrix}, \qquad (1+zm(z), zR(z)) = \begin{bmatrix} 1 & & & & & \\ 1 & 1 & & & & \\ 1 & 1 & 1 & & & \\ 2 & 2 & 1 & 1 & & \\ 4 & 4 & 3 & 1 & 1 & \\ 9 & 9 & 6 & 4 & 1 & 1 \\ & & \cdots & \cdots & & \end{bmatrix},$$

$$\left(\frac{1}{\sqrt{1-2z-3z^2}}, z(1+2zm(z))\right) = \begin{bmatrix} 1 & & & & & \\ 1 & 1 & & & & \\ 3 & 3 & 1 & & & \\ 7 & 7 & 5 & 1 & & \\ 19 & 19 & 15 & 7 & 1 & \\ 51 & 51 & 45 & 27 & 9 & 1 \\ & & \cdots & \cdots & & \end{bmatrix}, \tag{6}$$

where $m(z) = \frac{1-z-\sqrt{1-2z-3z^2}}{2z^2}$ and $R(z) = \frac{1+zm(z)}{1+z}$ are generating functions for the Motzkin numbers and Riordan numbers respectively.

Many questions now present themselves. First might be to give combinatorial interpretations to all the entries in these arrays, not just the two left hand columns with the generating functions $g(z)$ and $g(z)f(z)$. Even the entries in the next column with generating function $g(z)f^2(z)$ leads to some sequences that were not listed in Sloane's EIS [13]. This is answered in part in the next section starting with L-paths using the a_i as weights and then converting these weighted paths to k-Dyck paths. A further bijection leads to 2-paths which are just Dyck paths. Also there are generalizations of several of these examples in the next section. One of these are the α-Motzkin paths with $A(z) = 1 + \alpha z + z^2$ and another is the α-Schröder paths with $A(z) = \frac{1+\alpha z}{1-z} = 1 + (\alpha+1)z + (\alpha+1)z^2 + (\alpha+1)z^3 + \cdots$.

Remark. We can think about more of a combinatorial approach and the key is that $g(z) = \frac{1}{1-f(z)}$ so that the kth column of our $(g(z), f(z))$ represents trees (or paths) consisting of a green tree on the left and then k planted red trees on the right. We get as many examples as we want. For instance if $f(z) = z + z^2$ then we get a Fibonacci consistent array and each planted

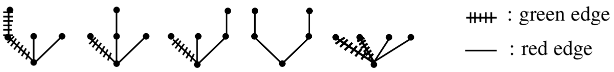

Fig. 2. The five ordered trees with 4 edges, 2 planted red trees.

subtree is just one or two edges with no branching other than at the root. The first few entries are as follows:

$$\left(\frac{1}{1-z-z^2}, z+z^2\right) = \begin{bmatrix} 1 & & & & & \\ 1 & 1 & & & & \\ 2 & 2 & 1 & & & \\ 3 & 3 & 3 & 1 & & \\ 5 & 5 & 5 & 4 & 1 & \\ 8 & 8 & 8 & 8 & 5 & 1 \\ & & \cdots & \cdots & & \end{bmatrix}$$

and the five ordered trees which are counted by the $(4, 2)$-entry are illustrated in Fig. 2.

3. Consistent Riordan arrays with special *A*-sequences

In this section, we will look at other combinatorial interpretations for the consistent Riordan arrays for some special A-sequences.

3.1. *k*-Dyck path interpretation

A *k-Dyck path* for $k \in \mathbb{N}$ is a lattice path from $(0, 0)$ to $(kn, 0)$ using the steps $U = (1, 1)$, $D_k = (1, 1-k)$ which is not allowed to go below the x-axis. It is well known that the number of k-Dyck paths is $\frac{1}{(k-1)n+1}\binom{kn}{n}$, $n \geq 0$, the sequence of *k-ary numbers*. Its generating function $B_k(z)$ satisfies the functional equation $B_k(z) = 1 + zB_k^k(z)$. It can be shown [4] that the following identity is valid for all real numbers r:

$$B_k^r(z) = \sum_{n\geq 0} \frac{r}{kn+r}\binom{kn+r}{n} z^n. \tag{7}$$

Let $\mathcal{B}_k = [b_{i,j}^{(k)}]_{i,j\geq 0} = (B_k(z), 1 - 1/B_k(z))$. Then $\mathcal{B}_k$ is the consistent array with $A(z) = Z(z) = 1/(1-z)^{k-1} = \sum_{j\geq 0}\binom{j+k-2}{j} z^j$. Further, $\mathcal{B}_k$ can be factored as

$$\mathcal{B}_k = (B_k(z), zB_k(z))(1, zB_{k-1}(z)).$$

By Theorem 2.1, $b_{i,j}^{(k)}$ is the sum of weights of L-paths from the origin to the point (i, j) using the weights

$$\omega(S_{-r}) = \begin{cases} \dbinom{k+r-2}{r} & \text{if } S_{-r} \text{ touches the } x\text{-axis,} \\ \dbinom{k+r-1}{r+1} & \text{otherwise} \end{cases} \tag{8}$$

where $\omega(S_1) = 1$. It follows from (7) and $B_k(z) = 1 + zB_k^k(z)$ that

$$b_{i,j}^{(k)} = [z^i]B_k(z)\left(1 - \frac{1}{B_k(z)}\right)^j = [z^{i-j}]B_k^{(k-1)j+1}(z) = \frac{(k-1)j+1}{(k-1)i+1}\binom{ki-j}{i-j}. \tag{9}$$

For $k = 2$ there is a classic bijection [15] from L-paths to Dyck paths. Replace each S_{-r} by UD^{r+1}. In fact, in this case, since

$$\omega(S_{-r}) = \begin{cases} \dbinom{2+r-2}{r} = 1 & \text{if } S_{-r} \text{ touches the } x\text{-axis,} \\ \dbinom{2+r-1}{r+1} = 1 & \text{otherwise,} \end{cases}$$

the regular unweighted L-paths are counted by $B_2(z) = C(z)$ as expected.

In the following theorem, for any $k \geq 2$ we obtain a bijection between weighted L-paths and sets of partial k-Dyck paths which have no restriction on the end points.

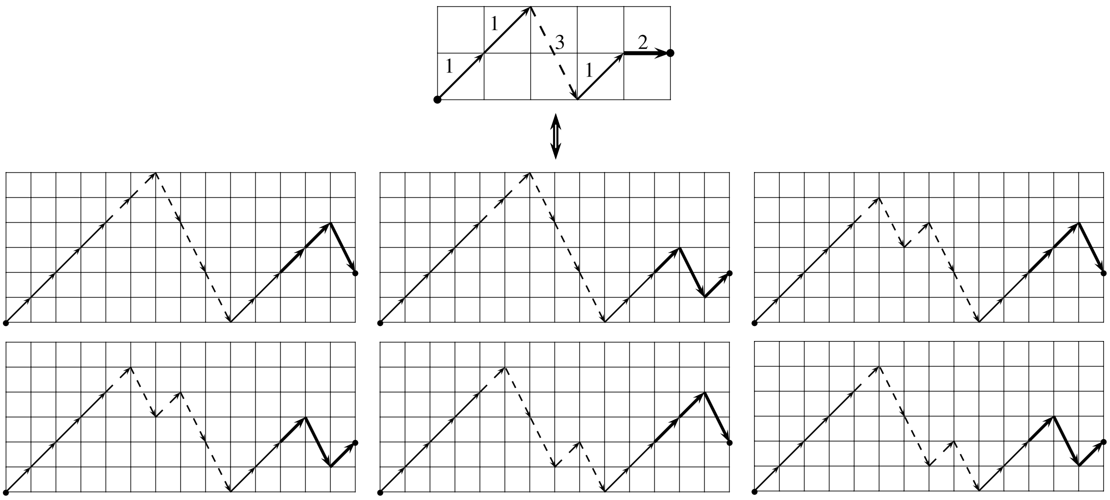

Fig. 3. The L-path with $\omega(L) = 6$ and the corresponding six 3-Dyck paths from $(0, 0)$ to $(14, 2)$.

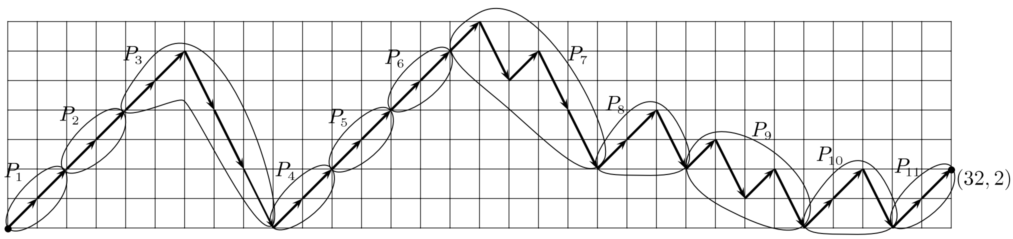

Fig. 4. The composition of 3-Dyck path D by disjoint subpaths.

Theorem 3.1. *Let $k \geq 2$. There is a bijection between the set of weighted L-paths ending at the point (i, j) using the weights given by (8) and the collection of sets of partial k-Dyck paths ending at the point $(ki - j, (k - 1)j)$.*

Proof. Let $\mathcal{L}$ be the set of weighted L-paths from the origin to (i, j) using the weights (8) with $|\mathcal{L}| = \eta$, where $\eta := \eta_{i,j} = \frac{j+1}{i+1}\binom{2i-j}{i}$. We first define the rule Ω_1 changing a weighted L-path into a set of partial k-Dyck paths.

(i) For a $L \in \mathcal{L}$, we replace S_1 with the steps $U^{k-1} = \underbrace{U \cdots U}_{k-1}$.

(ii) For $r \geq 0$ if S_{-r} meets the x-axis then we replace S_{-r} with $(k + r)$ steps $UP_{k+r-2}D_k$ where P_{k+r-2} is a path consisting of $(k - 2)$ U's and r D_k's. This yields $\binom{k+r-2}{r}$ k-Dyck paths.

(iii) If S_{-r} does not meet the x-axis then we replace S_{-r} with $(k + r)$ steps UP^*_{k+r-1} where P^*_{k+r-1} is a path consisting of $(k - 2)$ U's and $(r + 1)$ D_k's.

It is easy to show that all the partial k-Dyck paths obtained from the L-path $L \in \mathcal{L}$ ending at (i, j) have the same end point $(ki - j, (k - 1)j)$. After Ω_1 is applied the number of such partial k-Dyck paths coincides with the weight of L.

Let us illustrate with a small example that will show the ideas with less clutter. For $k = 3$, each L-path with the weights given by (8) now corresponds to the set of partial 3-Dyck paths. For instance we start with a L-path L with $\omega(L) = 6$ ending at $(5, 1)$. Then by the rule Ω_1, we obtain a set of 6 partial 3-Dyck paths ending at $(14, 2)$, see Fig. 3.

Let $D(L)$ be the set of all partial k-Dyck paths obtained from $L \in \mathcal{L}$ by the rule Ω_1 and $\mathcal{D}$ be the set of all partial k-Dyck paths ending at $(ki - j, (k - 1)j)$.

We claim that $\mathcal{D}$ is completely partitioned into η sets $D(L)$ where $L \in \mathcal{L}$ and $|\mathcal{L}| = \eta$. To see this, we first consider the inverse of the rule Ω_1. Given a partial k-Dyck path $D \in \mathcal{D}$ of length $ki - j$, let $D = s_1 s_2 \cdots s_{ki-j}$ where $s_\ell \in \{U, D_k\}$. Then divide D into disjoint subpaths $P_1, P_2, \ldots$ starting with U and containing exactly $k - 1$ U's, see Fig. 4.

Replace P_ℓ with S_{1-r} where r is the number of down steps in P_ℓ. Finally, assign a weight on each step S_{1-r} according to (8). Then we obtain the weighted L-path $L^{(D)}$ corresponding to the k-Dyck path $D \in \mathcal{D}$, see Fig. 5.

As with the previous figure, it can easily be shown that the resulting weighted L-path $L^{(D)}$ ends at the point (i, j).

Let us now define the relation $\sim$ on the set $\mathcal{D}$ by $D \sim D'$ if and only if $L^{(D)} = L^{(D')}$. It is easy to show that $\sim$ is an equivalence relation on $\mathcal{D}$. Thus $D(L)$ coincides with the equivalence class containing a k-Dyck path D such that $L = L^{(D)}$, and so $\mathcal{D} = \bigcup_{L \in \mathcal{L}} D(L)$. Hence a map $\varphi : \mathcal{L} \to \{D(L) | L \in \mathcal{L}\}$ defined by $\varphi(L) = D(L)$ gives a bijection. This completes the proof. $\square$

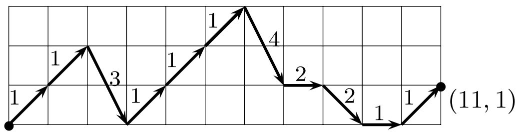

Fig. 5. The weighted L-path $L^{(D)}$ corresponding to D in Fig. 4.

By Theorems 2.1 and 3.1, we have the following corollary.

Corollary 3.2. *Let $b_{i,j}^{(k)}$ be the same as in (9). Then $b_{i,j}^{(k)}$ counts the number of partial k-Dyck paths from $(0, 0)$ to $(ki-j, (k-1)j)$. Further,*

$$\sum_{j=0}^{n} \frac{(k-1)j+1}{(k-1)n+1} \binom{kn-j}{n-j} = \frac{2}{kn+2} \binom{kn+2}{n}. \tag{10}$$

Proof. The identity (10) immediately follows from (4) and (7). □

3.2. Colored Schröder path interpretation

Rogers [10] studied two kinds of Schröder polynomials $r_n(\alpha)$ and $s_n(\alpha)$ obtained from the weighted Schröder paths. Specifically for an integer $\alpha \geq 0$, $r_n(\alpha)$ is the sum of weights of weighted Schröder paths ending at $(2n, 0)$ with the steps $U = (1, 1)$, $D = (1, -1)$ and $F = (2, 0)$ weighted by 1, 1 and α respectively, and $s_n(\alpha)$ is the sum of weights of the same weighted Schröder paths with no flat steps on the x-axis. It is known that

$$R(\alpha) = 1 + \alpha z R(\alpha) + z(R(\alpha))^2 \quad \text{and} \quad S(\alpha) = 1 - \alpha z S(\alpha) + z(1+\alpha)(S(\alpha))^2 \tag{11}$$

where $R(\alpha) = \sum_{n\geq 0} r_n(\alpha) z^n$ and $S(\alpha) = \sum_{n\geq 0} s_n(\alpha) z^n$.

Theorem 3.3. *For any complex number α, the Riordan array $(S(\alpha), zR(\alpha))$ is a consistent array with $A = (1, \alpha+1, \alpha+1, \ldots)$.*

Proof. Since $R(\alpha) = (1+\alpha)S(\alpha) - \alpha$, it follows from (11) that

$$zR(\alpha) = z((1+\alpha)S(\alpha) - \alpha) = z\frac{zS(\alpha)((1+\alpha)S(\alpha) - \alpha)}{zS(\alpha)} = \frac{S(\alpha) - 1}{S(\alpha)}.$$

Hence $(S(\alpha), zR(\alpha))$ is consistent. In addition, $zR(\alpha) = z\frac{1+\alpha z R(\alpha)}{1-zR(\alpha)}$ in (11) implies that $A(z) = \frac{1+\alpha z}{1-z} = 1 + (\alpha+1)z + (\alpha+1)z^2 + \cdots$. □

Let $(S(\alpha), zR(\alpha)) = [s_{n,k}^{(\alpha)}]_{n,k\geq 0}$. By Theorem 2.1, $s_{n,k}^{(\alpha)}$ can be viewed as the sum of weights of L-paths from the origin to the point (n, k) using the weights

$$\omega(S_{-r}) = \begin{cases} 1 & \text{if } r = 1 \text{ or if the flat step } S_0 \text{ is on the } x\text{-axis}, \\ \alpha + 1 & \text{otherwise}. \end{cases} \tag{12}$$

A weighted Schröder path is said to be *α-Schröder path* if its flat step has weight α, which can be thought of as α different colors. We denote the flat step with color ℓ by F_ℓ (also see [10]).

Theorem 3.4. *Let $\alpha \geq 1$. There is a bijection between the set of weighted L-paths ending at the point (n, k) using the weights given by (12) and the collection of sets of partial α-Schöder paths ending at the point $(2n-k, k)$ with no flat steps on the x-axis.*

Proof. We proceed by a similar method to Theorem 3.1. Let $\mathcal{L}$ be the set of weighted L-paths from the origin to (n, k) using the weights (12) with $|\mathcal{L}| = \eta$. We begin with the rule Ω_2 changing a weighted L-path to a partial α-Schröder path.

(i) For an $L \in \mathcal{L}$, if S_0 meets the x-axis then we replace S_0 with UD and if S_0 does not meet the x-axis then we replace S_0 with one of $\{UD, F_1, \ldots, F_\alpha\}$.
(ii) We replace S_1 with U and replace S_{-r} $(r \geq 1)$ with one of $\{UD^{r+1}, F_1D^r, \ldots, F_\alpha D^r\}$.

It can be shown that every partial α-Schröder path obtained from L by the rule Ω_2 has the same end point $(2n-k, k)$. The set of such α-Schröder paths is denoted by $A(L)$. Obviously, $|A(L)| = \omega(L)$. Let $\mathcal{S}$ be the set of all partial α-Schröder paths ending at $(2n-k, k)$.

Let us now show that $\mathcal{S}$ is completely partitioned into the sets $A(L)$ where $L \in \mathcal{L}$ and $|\mathcal{L}| = \eta$. First we consider the inverse rule of Ω_2.

Given a partial α-Schröder path $s \in \mathcal{S}$ of length $2n-k$, let $s = s_1s_2\cdots s_{2n-k}$ where $s_\ell \in \{U, D, F_1, \ldots, F_\alpha\}$. Then divide s into disjoint subpaths $P_1, P_2, \ldots$ starting with a step which is not D and containing exactly one of $\{U, F_1, \ldots, F_\alpha\}$. Replace

P_ℓ with S_{1-r} (resp. S_{-r}) if P_ℓ begins with U (resp. F_i), where r is the number of D's in P_ℓ. Finally, assign a weight on each step S_{-r} by $\omega(S_{-r})$ in (12). Then we obtain the weighted L-path $L^{(s)}$ ending at (n, k) corresponding to the α-Schröder path $s \in \mathcal{S}$ ending at $(2n-k, k)$.

Let us now define the relation $\sim$ on the set $\mathcal{S}$ by $s \sim s'$ if and only if $L^{(s)} = L^{(s')}$. It is easy to show that $\sim$ is an equivalence relation on $\mathcal{S}$. Thus $A(L)$ coincides with the equivalence class containing a α-Schröder path s such that $L = L^{(s)}$, and so $\mathcal{S} = \bigcup_{L\in\mathcal{L}} A(L)$. Hence a map $\phi : \mathcal{L} \to \{A(L)|L \in \mathcal{L}\}$ defined by $\phi(L) = A(L)$ is indeed a bijection. Thus the proof is completed. $\square$

By Theorems 2.1 and 3.4, we have the following corollary.

Corollary 3.5. *$s^{(\alpha)}_{n,k}$ counts the number of α-Schröder paths from $(0, 0)$ to $(2n-k, k)$ with no flat steps on the x-axis.*

3.3. Colored Motzkin path interpretation

A *Motzkin path* is a lattice path that starts at the origin and ends on the x-axis, cannot go below the x-axis and has as its possible steps $U = (1, 1), D = (1, -1)$ and the flat step $F = (1, 0)$. The number of Motzkin paths ending at the point $(n, 0)$ is denoted M_n and is called the *nth Motzkin number*. A Motzkin path with no flat steps on the x-axis is called a *Riordan path*. The number of Riordan paths ending at $(n, 0)$ is called the *nth Riordan number* R_n.

The Motzkin numbers have been generalized in several ways, for example see [9,12]. The Motzkin paths whose steps are all colored have been studied in [7] in connection with Riordan arrays. In this section, we consider so called *α-Motzkin path* which is the (partial) Motzkin path with flat steps $F_1, \dots, F_\alpha$ of α colors. Let $M_n(\alpha)$ and $R_n(\alpha)$ be the number of α-Motzkin paths and the number of α-Riordan paths ending at $(n, 0)$. The corresponding generating functions will be denoted by $M(\alpha)$ and $R(\alpha)$, respectively. It is known [12] that

$$M(\alpha) = \frac{1-\alpha z - \sqrt{(1-\alpha z)^2 - 4z^2}}{2z^2} = 1 + \alpha z M(\alpha) + z^2(M(\alpha))^2 \tag{13}$$

and

$$R(\alpha) = \frac{1+\alpha z - \sqrt{(1-\alpha z)^2 - 4z^2}}{2z(\alpha+z)} = 1 + z^2 M(\alpha)R(\alpha). \tag{14}$$

We note that $M(1)$, $R(1)$ and $R(2)$ give the Motzkin numbers, the Riordan numbers (A005043, [13]) and the Fine numbers (A000957, [13]), respectively. Also $R(3)$ is known as the generating function for (A117641, [13]), which counts the number of 3-Motzkin paths with no flat steps on the x-axis.

We are now interested in combinatorial interpretations for the two consistent Riordan arrays

$$\mathcal{M}_\alpha = \left(\frac{1}{1-zM(\alpha)}, zM(\alpha)\right) \quad \text{and} \quad \mathcal{R}_\alpha = \left(\frac{1}{1-\alpha z R(\alpha)}, \alpha z R(\alpha)\right).$$

Theorem 3.6. *For $\alpha \geq 0$, let $\mathcal{M}_\alpha = [m^{(\alpha)}_{n,k}]_{n,k\geq 0}$. Then $m^{(\alpha)}_{n,k}$ counts the number of α-Motzkin paths of length $n-1$ ending at level $\geq k-1$.*

Proof. Since the 0th column and the 1st column of $\mathcal{M}_\alpha$ are the same, it suffices to give a combinatorial interpretation for $\overline{\mathcal{M}}_\alpha := (M(\alpha)/(1-zM(\alpha)), zM(\alpha))$. First we count the α-Motzkin paths of length n ending at any level. The generating function for the numbers of those paths is given by

$$M(\alpha) + z(M(\alpha))^2 + z^2(M(\alpha))^3 + \cdots = \frac{M(\alpha)}{1-zM(\alpha)},$$

which is the generating function for the 0th column of $\overline{\mathcal{M}}_\alpha$.

Similarly, the α-Motzkin paths of length n ending at level k or higher are counted by the generating function:

$$z^k(M(\alpha))^{k+1} + z^{k+1}(M(\alpha))^{k+2} + \cdots = \frac{z^k(M(\alpha))^{k+1}}{1-zM(\alpha)} = \frac{M(\alpha)}{1-zM(\alpha)}(zM(\alpha))^k,$$

which is the generating function for the k-th column of $\overline{\mathcal{M}}_\alpha$. Since $\mathcal{M}_\alpha$ is consistent, $m^{(\alpha)}_{n,k}$ counts the number of α-Motzkin paths of length $n-1$ ending at level $\geq k-1$. $\square$

A simple computation shows that $\overline{\mathcal{M}}_\alpha$ can be expressed as the matrix decomposition:

$$\overline{\mathcal{M}}_\alpha = \left(\frac{1}{1-z}, \frac{z}{1-z}\right)\left(\frac{M(\alpha-1)}{1-zM(\alpha-1)}, zM(\alpha-1)\right) := P\overline{\mathcal{M}}_{\alpha-1}.$$

Here $P = [\binom{n}{k}]_{n,k\ge 0}$ is the Pascal matrix. By induction we have $\overline{\mathcal{M}}_\alpha = P^\alpha \overline{\mathcal{M}}_0$ where $P^\alpha = [\alpha^{n-k}\binom{n}{k}]_{n,k\ge 0}$ and $\overline{\mathcal{M}}_0 = [\binom{n}{\lceil (n+k)/2\rceil}]_{n,k\ge 0}$. Hence we obtain the explicit form for $m^{(\alpha)}_{n,k}$ as

$$m^{(\alpha)}_{n+1,k+1} = \sum_{j=0}^{n} \alpha^{n-j}\binom{n}{j}\binom{j}{\lceil \frac{j+k}{2}\rceil}.$$

Theorem 3.7. *For $\alpha \ge 1$, let $\mathcal{R}_\alpha = [r^{(\alpha)}_{n,k}]_{n,k\ge 0}$. Then $\frac{1}{\alpha} r^{(\alpha)}_{n+1,k+1}$ counts the number of α-Motzkin paths ending at $(n, 0)$ that have at least k flat steps on the x-axis.*

Proof. Since $\frac{1}{1-\alpha z R(\alpha)} = 1 + \alpha z M(\alpha)$ from (13) and (14), we have $\mathcal{R}_\alpha = (1 + \alpha z M(\alpha), \alpha z M(\alpha))$. Deleting the first row and the first column of $\mathcal{R}_\alpha$ yields

$$(\alpha M(\alpha), \alpha z R(\alpha)) = (M(\alpha), \alpha z R(\alpha))(\alpha, z).$$

Thus $(M(\alpha), \alpha z R(\alpha)) = [\frac{1}{\alpha} r^{(\alpha)}_{n+1,k+1}]_{n,k\ge 0}$. Since $M(\alpha)(\alpha z R(\alpha))^k$ is the generating function for the number of α-Motzkin paths ending at $(n, 0)$ that have at least k flat steps on the x-axis, we obtain the desired one. □

3.4. Ordered tree interpretation

Let us consider the consistent Riordan array $\mathcal{T}_k = [t^{(k)}_{i,j}]_{i,j\ge 0}$ with $A(z) = 1 + z + \cdots + z^{k-1} (k \ge 2)$. In particular, it is known [1] that $(t^{(3)}_{i,0})_{i\ge 0} = (1, 1, 2, 5, 13, 35, 96, \ldots)$ (A005773, [13]) counts the number of ordered trees with i edges and having nonroot vertices of outdegree at most 2. It also counts the number of directed animals in the first quadrant using unit North and East steps.

Deutsch [1] introduced a simple method to obtain the generating function $T(z)$ for ordered trees with the prescribed sets R, N, and L of root degrees, node degrees, and branch lengths, respectively. In fact,

$$T(z) = 1 + \sum_{\ell\in R} P^\ell(z) H^\ell(z) \tag{15}$$

where $P(z)$ and $H(z)$ are the generating functions of all paths and of all trees except the planted trees (ordered trees with the root degree 1) with the prescribed sets R, N, and L respectively, i.e.,

$$P(z) = \sum_{\ell\in L} z^\ell \quad \text{and} \quad H(z) = 1 + \sum_{\ell\in N, \ell\ne 1} P^\ell(z) H^\ell(z).$$

For regular ordered trees, $H(z) = 1 + z^2C^2(z) + z^3C^3(z) + z^4C^4(z) + \cdots = 1 + \frac{z^2C^2(z)}{1-zC(z)}$.

Enumeration of ordered trees with certain prescribed sets R, N, and L gives a new combinatorial interpretation for $t^{(k)}_{i,j}$.

Theorem 3.8. *Let $t^{(k)}_{i,j}$ be the number of ordered trees with i edges having root degree $\ge j$ and nonroot outdegrees $\le k - 1$. Then $\mathcal{T}_k = [t^{(k)}_{i,j}]_{i,j\ge 0}$ is the consistent Riordan array with $A = (1, 1, \ldots, 1, 0, \ldots)$ where 1's appear k times.*

Proof. Since $t^{(k)}_{i,j} = 0$ for $i, j \ge 0$ such that $i < j$, $\mathcal{T}_k$ is a lower triangular matrix. Let us consider the generating function $T(z)$ for the number of ordered trees with $i \ge 0$ edges having root degree at least 0 and nonroot outdegrees at most $k - 1$. Thus we have $R = \{1, 2, 3, \ldots\}$, $N = \{1, 2, \ldots, k - 1\}$, and $L = \{1, 2, 3, \ldots\}$. By (15) we obtain $T(z) = \frac{1}{1-P(z)H(z)}$, $P(z) = \frac{z}{1-z}$ and

$$H(z) = 1 + (P(z)H(z))^2 + (P(z)H(z))^3 + \cdots + (P(z)H(z))^{k-1}. \tag{16}$$

For $j \ge 1$, let $T_j(z) = \sum_{i\ge j} t^{(k)}_{i,j} z^i$. In this case, we have $R = \{j, j + 1, j + 2, \ldots\}$, $N = \{1, 2, \ldots, k - 1\}$ and $L = \{1, 2, 3, \ldots\}$. Since $i \ge j \ge 1$, there is no empty tree. Hence by (15) we obtain

$$T_j(z) = \sum_{\ell\ge j} (P(z)H(z))^\ell = \frac{1}{1 - P(z)H(z)}(P(z)H(z))^j = T(z)\cdot\left(1 - \frac{1}{T(z)}\right)^j.$$

It follows $\mathcal{T}_k$ is the consistent Riordan array given by $\left(T(z), 1 - \frac{1}{T(z)}\right)$. In addition, by a simple computation the A-sequence follows from (2) and (16) that

$$1 - \frac{1}{T(z)} = z\sum_{\ell=0}^{k-1}\left(1 - \frac{1}{T(z)}\right)^\ell = zA\left(1 - \frac{1}{T(z)}\right) \tag{17}$$

where $A(z) = 1 + z + \cdots + z^{k-1}$. □

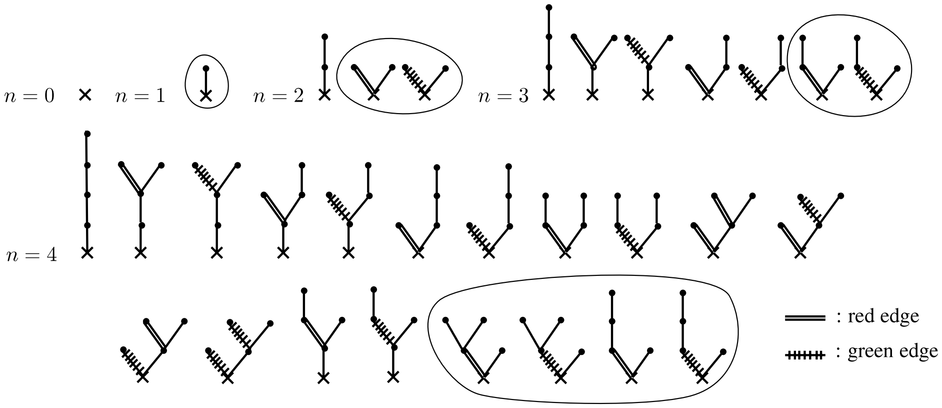

Fig. 6. $0 \cdot 1 \cdot 2$ trees counted by $1/\sqrt{1-2z-3z^2}$ for $n = 0, 1, 2, 3, 4$.

We end this section with a consistent Riordan array $(1/\sqrt{1-2z-3z^2}, z(1+2zm(z)))$ observed in (6) and a combinatorial interpretation to accompany it. We note that

$$f(z) := z(1+2zm(z)) = 1 - \sqrt{1-2z-3z^2} = z + 2z^2 + 2z^3 + 4z^4 + 8z^5 + \cdots$$

where m is the generating function for the Motzkin numbers.

In terms of $0 \cdot 1 \cdot 2$ trees, the generating function $1/\sqrt{1-2z-3z^2}$ counts such trees where, along the right most branch any vertex of updegree 2 has the left edge red or green. The first few terms are illustrated in Fig. 6.

If the right most branch has length 1 we get the f trees as circled in g Fig. 6. We note that the (n, k)-entry counts the number of such trees with n edges which have at least k edges on the right most branch at the root.

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