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Extended Bernoulli and Stirling matrices and related combinatorial identities ☆

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ABSTRACT

In this paper we establish plenty of number theoretic and combinatoric identities involving generalized Bernoulli polynomials and Stirling numbers of both kinds, which generalize various known identities. These formulas are deduced from Pascal type matrix representations of Bernoulli and Stirling numbers. For this we define and factorize a modified Pascal matrix corresponding to Bernoulli and Stirling cases.

1. Introduction

Matrices and matrix theory are recently used in number theory and combinatorics. In particular Pascal type lower-triangular matrices are studied with Fibonacci, Bernoulli, Stirling and Pell numbers and other special numbers sequences. Cheon and Kim [13] factorized (generalized) Stirling matrices by Pascal matrices and obtained some combinatorial identities. Zhang and Wang [31] gave product

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formulas for the Bernoulli matrix and established several identities involving Fibonacci numbers, Bernoulli numbers and polynomials.

In this paper we employ matrices for degenerate Bernoulli polynomials and generalized Stirling numbers. We define degenerate Bernoulli and generalized Stirling matrices which generalize previous results and lead some new combinatorial identities. Some of these identities can hardly be obtained by classical ways such as by using generating functions or counting, however they are easily come up via matrix representations after elementary matrix multiplication.

The summary by sections is as follows: In Section 2, we define Pascal functional matrix which is a special case of Pascal functional matrices defined in [26,32] and factorize by the summation matrices. In Section 3, we generalize Bernoulli matrix and investigate some properties. In Section 4, we define two types generalized Stirling matrices and obtain relationships between Bernoulli matrices and Stirling matrices of the second type. Furthermore, degenerate Bernoulli and generalized Stirling matrices are factorized by Pascal matrices and several identities are developed as a result of matrix representations. In final section, we introduce some special cases of the results obtained in Section 4.

Throughout this paper we assume that i, j and n are nonnegative integers; μ, λ, w and x are real or complex numbers.

2. Pascal matrix

Let $g(t)$ be a formal power series of the form

$$g(t)=\sum_{m=0}^{\infty} g_m \frac{t^m}{m!}.$$

Define the multiplication matrix $M(g)$ as the lower triangular matrix whose (i,j) entry is $g_{i-j}/(i-j)!$. The map $g \to M(g)$ is an algebra isomorphism from the formal power series to the lower triangular Toeplitz matrices [18, Chapter 1]. Now define the diagonal matrix $F=\operatorname{diag}(0!,1!,2!,\ldots)$ and the Pascal matrix associated with $g(t)$ by $P(g)=FM(g)F^{-1}$. It is obvious that the set of all such Pascal matrices is isomorphic to the algebra of lower triangular Toeplitz matrices. These matrices satisfy

$$P(g)P(f)=P(gf), \qquad P(g)^k=P\left(g^k\right) \quad \text{and} \quad P(g)^{-1}=P(1/g) \quad \text{when } g_0 \neq 0. \tag{1}$$

The $n \times n$ section of an infinite matrix $P(g)$ is defined as the finite submatrix composed of the first n rows and columns of $P(g)$. Also, (1) is valid for the $n \times n$ sections.

Let $\mathcal{P}_n[\lambda,x]$ be the $n \times n$ section of the infinite Pascal matrix $P(g)$ associated with the generating function

$$g(t)=(1+\lambda t)^{x/\lambda}=\sum_{m=0}^{\infty}(x|\lambda)_m \frac{t^m}{m!},$$

i.e., let $\mathcal{P}_n[\lambda,x]$ be the $n \times n$ matrix defined by

$$\left(\mathcal{P}_n[\lambda,x]\right)_{i,j}=\begin{cases} \binom{i-1}{j-1}(x|\lambda)_{i-j}, & \text{if } i \geqslant j \geqslant 1, \\ 0, & \text{if } 1 \leqslant i < j, \end{cases}$$

where $(x|\lambda)_k=x(x-\lambda)(x-2\lambda)\cdots(x-(k-1)\lambda)$ with $(x|\lambda)_0=1$.

From (1) we have

$$\mathcal{P}_n[\lambda,x+y]=\mathcal{P}_n[\lambda,x]\mathcal{P}_n[\lambda,y], \qquad \left(\mathcal{P}_n[\lambda,x]\right)^h=\mathcal{P}_n[\lambda,hx] \quad \text{and} \quad \mathcal{P}_n^{-1}[\lambda,x]=\mathcal{P}_n[\lambda,-x].$$

The algebraic properties of $\mathcal{P}_n[\lambda,x]$ can be found in [3,7,16,26,29,30,32]. In fact, $\mathcal{P}_n[-\lambda,x]$ is the matrix $\mathcal{P}_{n,\lambda}[x]$ defined in [3] and this matrix is a special case of the generalized Pascal functional matrices defined in [26,32]. So we will not discuss the algebraic properties of this matrix. We will only focus on factorizing this matrix by the summation matrices. For this purpose, let us define the $n \times n$ matrices $\mathcal{R}_n[\lambda,x]$ and $G_k[\lambda,x]$ by

$$\big(\mathcal{R}_n[\lambda,x]\big)_{i,j}=\begin{cases}\frac{(x|\lambda)_{i-1}}{(x-\lambda|\lambda)_{j-1}}, & \text{if } i>j.\\ 1, & \text{if } i=j,\\ 0, & \text{if } i<j,\end{cases}$$

and

$$G_k[\lambda,x]=I_{n-k}\oplus\mathcal{R}_k[\lambda,x],\quad 1\leqslant k\leqslant n-1\quad\text{and}\quad G_n[\lambda,x]=\mathcal{R}_n[\lambda,x],$$

where the notation $\oplus$ denotes the direct sum of two matrices and I_n is the identity matrix of order n. Furthermore, we need the $(k+1)\times(k+1)$ matrices

$$\overline{\mathcal{P}}_k[\lambda,x]=[1]\oplus\mathcal{P}_k[\lambda,x],\quad k\geqslant 1.$$

Lemma 1. *For $k\geqslant 1$, we have*

$$\mathcal{R}_k[\lambda,x]\overline{\mathcal{P}}_{k-1}[\lambda,x]=\mathcal{P}_k[\lambda,x].$$

Proof. We must show that

$$\sum_{r=j}^{i-1}\frac{(x|\lambda)_{i-1}}{(x-\lambda|\lambda)_{r-1}}\binom{r-2}{j-2}(x|\lambda)_{r-j}+\binom{i-2}{j-2}(x|\lambda)_{i-j}=\binom{i-1}{j-1}(x|\lambda)_{i-j},\tag{2}$$

for $i\geqslant j$, since the left-hand side of (2) is the (i,j)-entry of the matrix $\mathcal{R}_k[\lambda,x]\overline{\mathcal{P}}_{k-1}[\lambda,x]$. We apply induction on i. For $i=j$, the assertion is clear. From the known property $\binom{i-1}{j-1}+\binom{i-1}{j}=\binom{i}{j}$, it is enough to show that

$$(x|\lambda)_{i-1}\sum_{r=j}^{i-1}\binom{r-2}{j-2}\frac{(x|\lambda)_{r-j}}{(x-\lambda|\lambda)_{r-1}}=\binom{i-2}{j-1}(x|\lambda)_{i-j}\tag{3}$$

for $i\geqslant j+1$. Suppose that (3) is true for $i=m>j$. For $i=m+1$, we have

$$\begin{aligned}&(x|\lambda)_m\sum_{r=j}^{m}\binom{r-2}{j-2}\frac{(x|\lambda)_{r-j}}{(x-\lambda|\lambda)_{r-1}}\\&=(x|\lambda)_m\sum_{r=j}^{m-1}\binom{r-2}{j-2}\frac{(x|\lambda)_{r-j}}{(x-\lambda|\lambda)_{r-1}}+\binom{m-2}{j-2}\frac{(x|\lambda)_m}{(x-\lambda|\lambda)_{m-1}}(x|\lambda)_{m-j}\\&=\big[x-(m-1)\lambda\big]\binom{m-2}{j-1}(x|\lambda)_{m-j}+x\binom{m-2}{j-2}(x|\lambda)_{m-j}\\&=\binom{m-1}{j-1}\big[x-(m-j)\lambda\big](x|\lambda)_{m-j}=\binom{m-1}{j-1}(x|\lambda)_{m+1-j}.\end{aligned}$$

This completes the proof. □

From the definition of the matrices $G_k[\lambda,x]$ and Lemma 1, we have the following factorization of $\mathcal{P}_n[\lambda,x]$, which generalizes the result of Zhang [29, Theorem 1].

Theorem 2.

$$\mathcal{P}_n[\lambda,x]=G_n[\lambda,x]G_{n-1}[\lambda,x]\cdots G_1[\lambda,x].$$

Example 3.

$$
\begin{aligned}
&G_4[\lambda,x]G_3[\lambda,x]G_2[\lambda,x]G_1[\lambda,x]\\
&=\begin{bmatrix}1&0&0&0\\x&1&0&0\\x^2-x\lambda&x&1&0\\x^3-3x^2\lambda+2x\lambda^2&x^2-2x\lambda&x&1\end{bmatrix}\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&x&1&0\\0&x^2-x\lambda&x&1\end{bmatrix}\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&x&1\end{bmatrix}\\
&=\begin{bmatrix}1&0&0&0\\x&1&0&0\\x^2-x\lambda&2x&1&0\\x^3-3x^2\lambda+2x\lambda^2&3x^2-3x\lambda&3x&1\end{bmatrix}=\mathcal{P}_4[\lambda,x].
\end{aligned}
$$

3. Degenerate Bernoulli matrices

3.1. Degenerate Bernoulli polynomials of the first kind

The higher order degenerate Bernoulli polynomials of the first kind $\beta_m^{(w)}(\lambda,x)$ are defined by means of the generating function [9]

$$\left(\frac{t}{(1+\lambda t)^{1/\lambda}-1}\right)^w(1+\lambda t)^{x/\lambda}=\sum_{m=0}^{\infty}\beta_m^{(w)}(\lambda,x)\frac{t^m}{m!} \tag{4}$$

for $\lambda\neq 0$. Clearly, $\beta_m^{(1)}(\lambda,x)=\beta_m(\lambda,x)$ and $\beta_m^{(1)}(\lambda,0)=\beta_m(\lambda)$ are the degenerate Bernoulli polynomials and the degenerate Bernoulli numbers, respectively. The first few of the degenerate Bernoulli polynomials are $\beta_0(\lambda,x)=1$, $\beta_1(\lambda,x)=x+\frac{1}{2}\lambda-\frac{1}{2}$, $\beta_2(\lambda,x)=x^2-x-\frac{1}{6}\lambda^2+\frac{1}{6}$, $\beta_3(\lambda,x)=x^3-\frac{3}{2}x^2+\frac{1}{2}x-\frac{3}{2}x^2\lambda+\frac{3}{2}x\lambda+\frac{1}{4}\lambda^3-\frac{1}{4}\lambda$, $\beta_4(\lambda,x)=x^4-2x^3+x^2-4x^3\lambda+4x^2\lambda^2+6x^2\lambda-4x\lambda^2-2x\lambda-\frac{19}{30}\lambda^4+\frac{2}{3}\lambda^2-\frac{1}{30}$. Explicit formulas and recurrence relations of (generalized) degenerate Bernoulli polynomials and numbers can be found in [1,8,9,11,19,28]. Divisibility properties [8,19,27,28] and symmetry relations [12, 22,28] are demonstrated as well.

Let $\mathcal{B}_n^{(w)}[\lambda,x]$ be the $n\times n$ matrix defined by

$$\left(\mathcal{B}_n^{(w)}[\lambda,x]\right)_{i,j}=\begin{cases}\binom{i-1}{j-1}\beta_{i-j}^{(w)}(\lambda,x), & \text{if } i\geqslant j\geqslant 1,\\ 0, & \text{if } 1\leqslant i<j,\end{cases}$$

with the notations $\mathcal{B}_n^{(1)}[\lambda,x]=\mathcal{B}_n[\lambda,x]$ and $\mathcal{B}_n^{(1)}[\lambda,0]=\mathcal{B}_n[\lambda]$.

Since $(1+\lambda t)^{1/\lambda}\to e^t$, as $\lambda\to 0$ it is evident that $\beta_m^{(w)}(0,x)=B_m^{(w)}(x)$ and $\beta_m^{(w)}(0,0)=B_m^{(w)}$, where $B_m^{(w)}(x)$ are the higher order Bernoulli polynomials defined by

$$\left(\frac{t}{e^t-1}\right)^w e^{xt}=\sum_{m=0}^{\infty}B_m^{(w)}(x)\frac{t^m}{m!}.$$

Hence, in the limiting case $\lambda=0$, $\mathcal{B}_n^{(w)}[0,x]$ is the generalized Bernoulli matrix $\mathcal{B}_{n-1}^{(w)}[x]$ defined in [31, p. 1623].

It is clear that $\mathcal{B}_n^{(0)}[\lambda,x]=\mathcal{P}_n[\lambda,x]$ and $\mathcal{B}_n^{(w)}[\lambda,x]$ is the $n\times n$ section of $P(g)$ where $g(t)$ is given by (4). We then have the following theorem which can be seen from (1).

Theorem 4.

$$
\begin{aligned}
&\mathcal{B}_n^{(w+z)}[\lambda,x+y]=\mathcal{B}_n^{(w)}[\lambda,x]\mathcal{B}_n^{(z)}[\lambda,y]=\mathcal{B}_n^{(z)}[\lambda,x]\mathcal{B}_n^{(w)}[\lambda,y],\\
&\mathcal{B}_n^{(w)}[\lambda,x+y]=\mathcal{P}_n[\lambda,x]\mathcal{B}_n^{(w)}[\lambda,y]=\mathcal{B}_n^{(w)}[\lambda,y]\mathcal{P}_n[\lambda,x],\\
&\left(\mathcal{B}_n^{(w)}[\lambda,x]\right)^k=\mathcal{B}_n^{(kw)}[\lambda,kx],\\
&\left(\mathcal{B}_n^{(w)}[\lambda,x]\right)^{-1}=\mathcal{B}_n^{(-w)}[\lambda,-x]=\mathcal{P}_n[\lambda,-x]\mathcal{B}_n^{(-w)}[\lambda].
\end{aligned}
$$

The following is a consequence of Theorems 2 and 4.

Corollary 5.

$$\mathcal{B}_n[\lambda,x]=G_n[\lambda,x]G_{n-1}[\lambda,x]\cdots G_1[\lambda,x]\mathcal{B}_n[\lambda].$$

Consider the matrix

$$\left(\mathcal{B}_n[\lambda,x]-I_n\right)^h=\sum_{k=0}^{h}\binom{h}{k}(-1)^{h-k}\left(\mathcal{B}_n[\lambda,x]\right)^k=\sum_{k=0}^{h}\binom{h}{k}(-1)^{h-k}\mathcal{B}_n^{(k)}[\lambda,kx]$$

for positive integer h. Since $\operatorname{diag}(\mathcal{B}_n[\lambda,x]-I_n)=(0,0,\dots,0)$ and $(\mathcal{B}_n[\lambda,x]-I_n)$ is a lower-triangular matrix, it follows that $(\mathcal{B}_n[\lambda,x]-I_n)^h=[0]_{n\times n}$ for $n\leqslant h$. Then

$$\left(\mathcal{B}_n\left[\lambda,\frac{x}{h}\right]\right)^h=\mathcal{B}_n^{(h)}[\lambda,x]=\sum_{k=0}^{h-1}\binom{h}{k}(-1)^{h-1-k}\mathcal{B}_n^{(k)}\left[\lambda,\frac{k}{h}x\right],\quad \text{for } 1\leqslant n\leqslant h.$$

This yields

$$\sum_{k=0}^{h-1}\binom{h}{k}(-1)^{h-1-k}\beta_m^{(k)}\left(\lambda,\frac{k}{h}x\right)=\beta_m^{(h)}(\lambda,x),\quad \text{for } 0\leqslant m<h.$$

By the known identity $\beta_m^{(h)}(\lambda,1)=m\beta_{m-1}^{(h-1)}(\lambda)+\beta_m^{(h)}(\lambda)$ for $m\geqslant 1$, we have

$$\sum_{k=0}^{h}\binom{h}{k}(-1)^{h-k}\beta_m^{(k)}\left(\lambda,\frac{k}{h}\right)=-m\beta_{m-1}^{(h-1)}(\lambda),\quad \text{for } 1\leqslant m<h.$$

Similarly, we may get

$$\sum_{k=0}^{h-1}\binom{h}{k}(-1)^{h-1-k}(kx|\lambda)_m=(hx|\lambda)_m,\quad \text{for } 0\leqslant m<h.$$

3.2. Degenerate Bernoulli polynomials of the second kind

The higher order degenerate Bernoulli polynomials of the second kind $\alpha_m^{(w)}(\lambda,x)$ are defined by [1]

$$\left(\frac{\lambda t}{(1+t)^\lambda-1}\right)^w(1+t)^x=\sum_{m=0}^{\infty}\alpha_m^{(w)}(\lambda,x)\frac{t^m}{m!}. \tag{5}$$

For $x=0$, $\alpha_m^{(w)}(\lambda,0)=\alpha_m^{(w)}(\lambda)$ are called the higher order degenerate Bernoulli numbers of the second kind. In the limiting case $\lambda=0$ we have $\alpha_m^{(w)}(0,x)=m!b_m^{(w)}(x)$, where $b_m^{(w)}(x)$ are the higher order Bernoulli polynomials of the second kind defined by

$$\left(\frac{t}{\log(1+t)}\right)^w(1+t)^x=\sum_{m=0}^{\infty}b_m^{(w)}(x)t^m.$$

It is clear from (4) and (5) that

$$\beta_m^{(w)}\left(\frac{1}{\lambda},\frac{x}{\lambda}\right)=\left(\frac{1}{\lambda}\right)^m\alpha_m^{(w)}(\lambda,x). \tag{6}$$

If $\mathcal{L}_n^{(w)}[\lambda,x]$ denotes the $n\times n$ section of the Pascal matrix associated with the generating function given by (5), then

$$\left(\mathcal{L}_n^{(w)}[\lambda,x]\right)_{i,j} = \begin{cases} \binom{i-1}{j-1}\alpha_{i-j}^{(w)}(\lambda,x), & \text{if } i \geqslant j \geqslant 1, \\ 0, & \text{if } i < j. \end{cases}$$

It is obvious that $\mathcal{L}_n^{(0)}[\lambda,x] = \mathcal{P}_n[1,x]$. One can observe that $\mathcal{L}_n^{(w)}[\lambda,x]$ satisfies properties given for $\mathcal{B}_n^{(w)}[\lambda,x]$, however, we prefer not to list them here.

4. Generalized Stirling matrices

Recall the generalized Stirling numbers of the first and of the second kinds. For nonnegative integer m and real or complex parameters μ, λ and x, with $(\mu,\lambda,x) \neq (0,0,0)$, the generalized Stirling numbers of the first kind $S_1(m,k|\mu,\lambda,x)$ and of the second kind $S_2(m,k|\mu,\lambda,x)$ are defined by means of the generating functions (cf. [21, p. 372])

$$\left(\frac{(1+\mu t)^{\lambda/\mu}-1}{\lambda}\right)^k (1+\mu t)^{x/\mu} = k!\sum_{m=0}^{\infty} S_1(m,k|\mu,\lambda,x)\frac{t^m}{m!}, \tag{7}$$

$$\left(\frac{(1+\lambda t)^{\mu/\lambda}-1}{\mu}\right)^k (1+\lambda t)^{-x/\lambda} = k!\sum_{m=0}^{\infty} S_2(m,k|\mu,\lambda,x)\frac{t^m}{m!}, \tag{8}$$

with the notations

$$S_1(m,k|\mu,\lambda,x) = S^1(m,k) = S(m,k;\mu,\lambda,x),$$
$$S_2(m,k|\mu,\lambda,x) = S^2(m,k) = S(m,k;\lambda,\mu,-x)$$

and the convention $S_1(m,k|\mu,\lambda,x) = S_2(m,k|\mu,\lambda,x) = 0$ when $k > m$.

As Hsu and Shiue pointed out, the definitions or generating functions generalize various Stirling-type numbers studied previously, such as:

(i) $$\left\{S_1(m,k|1,0,0), S_2(m,k|1,0,0)\right\} = \left\{s(m,k), S(m,k)\right\} = \left\{(-1)^{m-k}\begin{bmatrix} m \\ k \end{bmatrix}, \begin{Bmatrix} m \\ k \end{Bmatrix}\right\} = \left\{S_1(m,k), S_2(m,k)\right\}$$

are the Stirling numbers of the first kind and of the second kind, respectively [17, Chapter 6].

(ii) $$\left\{S_1(m,k|1,\lambda,-x), S_2(m,k|1,\lambda,-x)\right\} = \left\{(-1)^{m-k}S_1(m,k,x+\lambda|\lambda), S_2(m,k,x|\lambda)\right\}$$

are the Howard degenerate weighted Stirling numbers of both kinds [20].

(iii) $$\left\{S_1(m,k|1,0,-x), S_2(m,k|1,0,-x)\right\} = \left\{(-1)^{m-k}R_1(m,k,x), R_2(m,k,x)\right\}$$

are Carlitz's weighted Stirling numbers of both kinds [10].

(iv) $$\left\{S_1(m,k|-1,0,r), S_2(m,k|-1,0,r)\right\} = \left\{\begin{bmatrix} m+r \\ k+r \end{bmatrix}_r, (-1)^{m-k}\begin{Bmatrix} m+r \\ k+r \end{Bmatrix}_r\right\}$$

are the r-Stirling numbers of both kinds [6].

(v) $$\left\{S_1(m,k|1,\lambda,0), S_2(m,k|1,\lambda,0)\right\} = \left\{(-1)^{m-k}S_1(m,k|\lambda), S_2(m,k|\lambda)\right\}$$

are Carlitz's degenerate Stirling numbers of both kinds [9].

(vi) $$\left\{S_1(m,k|-1,1,0), S_2(m,k|-1,1,0)\right\} = \left\{L(m,k), (-1)^{m-k}L(m,k)\right\},$$

where $L(m,k) = \frac{m!}{k!}\binom{m-1}{k-1}$ are the Lah numbers.

The list above may not be complete. The combinatorial interpretations of (i)–(iv) can be found in [6,10,17,20].

From (7) and (8), it follows that

$$S_1(m,k|\mu,\lambda,x)=\mu^{m-k}S_1\left(m,k\middle|1,\frac{\lambda}{\mu},\frac{x}{\mu}\right)\quad\text{and}$$
$$S_2(m,k|\mu,\lambda,x)=\mu^{m-k}S_2\left(m,k\middle|1,\frac{\lambda}{\mu},\frac{x}{\mu}\right)\tag{9}$$

for $\mu\neq 0$. Letting $\lambda=0$ and $x=0$ in (9), we have

$$S_1(m,k|\mu,0,0)=\mu^{m-k}S_1(m,k)\quad\text{and}\quad S_2(m,k|\mu,0,0)=\mu^{m-k}S_2(m,k).\tag{10}$$

4.1. Stirling matrices of the first type

Let $s_n[\mu,\lambda,x]$ and $\mathcal{S}_n[\mu,\lambda,x]$ be the $n\times n$ matrices defined by

$$\big(s_n[\mu,\lambda,x]\big)_{i,j}=\begin{cases}S_1(i,j|\mu,\lambda,x), & \text{if } i\geqslant j\geqslant 1,\\ 0, & \text{if } i<j,\end{cases}$$

and

$$\big(\mathcal{S}_n[\mu,\lambda,x]\big)_{i,j}=\begin{cases}S_2(i,j|\mu,\lambda,x), & \text{if } i\geqslant j\geqslant 1,\\ 0, & \text{if } i<j\end{cases}$$

which we call generalized Stirling matrices of the first type. Then the relation (cf. [21, Eq. (3)])

$$\sum_{k=j}^{i}S_1(i,k|\mu,\lambda,x)S_2(k,j|\mu,\lambda,x)=\sum_{k=j}^{i}S_2(i,k|\mu,\lambda,x)S_1(k,j|\mu,\lambda,x)=\delta_{i,j}\tag{11}$$

yields $\mathcal{S}_n^{-1}[\mu,\lambda,x]=s_n[\mu,\lambda,x]$, where $\delta_{i,j}$ is the Kronecker symbol. From (10), it is seen that $s_n[\mu,0,0]$ and $\mathcal{S}_n[\mu,0,0]$ are the Stirling matrices $\mathcal{S}_n^{-1}[\mu]$ and $\mathcal{S}_n[\mu]$ defined in [13, p. 57].

Differentiate both sides of (8) with respect to t (with $x=0$) to get

$$\begin{aligned}\sum_{m=0}^{\infty}S_2(m+1,j|\mu,\lambda,0)\frac{t^m}{m!}&=\frac{1}{(j-1)!}(1+\lambda t)^{(\mu-\lambda)/\lambda}\left[\frac{(1+\lambda t)^{\mu/\lambda}-1}{\mu}\right]^{j-1}\\&=\sum_{m=0}^{\infty}(\mu-\lambda|\lambda)_m\frac{t^m}{m!}\sum_{m=0}^{\infty}S_2(m,j-1|\mu,\lambda,0)\frac{t^m}{m!}\\&=\sum_{m=0}^{\infty}\left[\sum_{k=j-1}^{m}\binom{m}{k}(\mu-\lambda|\lambda)_{m-k}S_2(k,j-1|\mu,\lambda,0)\right]\frac{t^m}{m!}.\end{aligned}$$

Thus,

$$S_2(m+1,j|\mu,\lambda,0)=\sum_{k=j-1}^{m}\binom{m}{k}(\mu-\lambda|\lambda)_{m-k}S_2(k,j-1|\mu,\lambda,0).$$

Putting $m=i-1$ gives

$$S_2(i,j|\mu,\lambda,0)=\sum_{k=j}^{i}\binom{i-1}{k-1}(\mu-\lambda|\lambda)_{i-k}S_2(k-1,j-1|\mu,\lambda,0)\tag{12}$$

which yields

$$\mathcal{S}_n[\mu,\lambda,0]=\mathcal{P}_n[\lambda,\mu-\lambda]\big([1]\oplus\mathcal{S}_{n-1}[\mu,\lambda,0]\big).\tag{13}$$

Note that (12) reduces to the well known *vertical* recurrence relation

$$S_2(i,j)=\sum_{k=j}^{i}\binom{i-1}{k-1}S_2(k-1,j-1)$$

by letting $\lambda=0$ and $\mu\neq 0$. The counterpart of (12) is

$$S_1(i,j|\mu,\lambda,0)=\sum_{k=j}^{i}\binom{k-1}{j-1}S_1(i-1,k-1|\mu,\lambda,0)(\lambda-\mu|\lambda)_{k-j}.$$

We also have

$$\binom{i-1}{j-1}(\mu-\lambda|\lambda)_{i-j}=\sum_{k=j}^{i}S_2(i,k|\mu,\lambda,0)S_1(k-1,j-1|\mu,\lambda,0),$$

$$\binom{i-1}{j-1}(\lambda-\mu|\lambda)_{i-j}=\sum_{k=j}^{i}S_2(i-1,k-1|\mu,\lambda,0)S_1(k,j|\mu,\lambda,0).$$

Furthermore, in consequence of (13) we have the following factorization of the matrix $\mathcal{S}_n[\mu,\lambda,0]$:

$$\mathcal{S}_n[\mu,\lambda,0]=Q_n[\lambda,\mu-\lambda]Q_{n-1}[\lambda,\mu-\lambda]\cdots Q_1[\lambda,\mu-\lambda],$$

where $Q_k[\lambda,x]=I_{n-k}\oplus\mathcal{P}_k[\lambda,x]$, $1\leqslant k\leqslant n-1$, and $Q_n[\lambda,x]=\mathcal{P}_n[\lambda,x]$.

4.2. Stirling matrices of the second type

Let us define the second type generalized Stirling matrices $\mathcal{G}_{n,h}[1,\lambda,x]$ and $g_{n,h}[1,\lambda,x]$ of order n by

$$\big(\mathcal{G}_{n,h}[1,\lambda,x]\big)_{i,j}=\begin{cases}\binom{i-1}{j-1}\binom{i-h}{j-h}^{-1}S_2(i-h,j-h|1,\lambda,x), & \text{if } i>j\geqslant 1 \text{ and } j\geqslant h,\\ 1, & \text{if } i=j,\\ 0, & \text{otherwise,}\end{cases}$$

and

$$\big(g_{n,h}[1,\lambda,x]\big)_{i,j}=\begin{cases}\binom{i-1}{j-1}\binom{i-h}{j-h}^{-1}S_1(i-h,j-h|1,\lambda,x), & \text{if } i>j\geqslant 1 \text{ and } j\geqslant h,\\ 1, & \text{if } i=j,\\ 0, & \text{otherwise.}\end{cases}$$

Remark 6. The lower triangular matrix whose (i,j) entry is in the form $c_{ij}S(i,j)=c_{i-j}S(i,j)$ is not in the form of lower triangular Toeplitz matrix since

$$S(i+1,j+1)=S(i,j)+\big((j+1)\lambda-i+x\big)S(i,j+1),$$

where $S(i,j)=S_k(i,j|1,\lambda,x)$, $k=1,2$ [21, Eq. (7)]. Therefore, we do not have the matrices $g_{n,h}[1,\lambda,x]$ and $G_{n,h}[1,\lambda,x]$ as an $n\times n$ section of $P(g)$ associated with the generating function $g(t)$ given by (7) and (8), respectively.

It is obvious from (11) that $g_{n,h}[1,\lambda,x]=(\mathcal{G}_{n,h}[1,\lambda,x])^{-1}$. We have

Theorem 7.

$$\left(\mathcal{G}_{n,h}[1,\lambda,-x]\right)_{i,j} = \left(\mathcal{B}_n^{(h)}[\lambda,x-y]\mathcal{G}_{n,0}[1,\lambda,-y]\right)_{i,j} = \left(\mathcal{G}_{n,0}[1,\lambda,-y]\mathcal{L}_n^{(-h)}[\lambda,x-y]\right)_{i,j},$$
$$\left(g_{n,h}[1,\lambda,-x]\right)_{i,j} = \left(g_{n,0}[1,\lambda,-y]\mathcal{B}_n^{(-h)}[\lambda,y-x]\right)_{i,j} = \left(\mathcal{L}_n^{(h)}[\lambda,y-x]g_{n,0}[1,\lambda,-y]\right)_{i,j},$$

for $j \geqslant h$. In particular,

$$\mathcal{P}_n[\lambda,x-y] = \mathcal{G}_{n,0}[1,\lambda,-x]g_{n,0}[1,\lambda,-y],$$
$$\mathcal{P}_n[1,y-x] = g_{n,0}[1,\lambda,-x]\mathcal{G}_{n,0}[1,\lambda,-y].$$

Proof. By (4) and (8), we have

$$\begin{aligned}
&\left(\frac{t}{(1+\lambda t)^{1/\lambda}-1}\right)^h \frac{(1+\lambda t)^{x/\lambda}}{j!}\left[(1+\lambda t)^{1/\lambda}-1\right]^j \\
&= \sum_{m=j}^{\infty}\left(\sum_{k=j}^{m}\binom{m}{k}\frac{1}{m!}\beta_{m-k}^{(h)}(\lambda,x-y)S_2(k,j|1,\lambda,-y)\right)t^m \\
&= \sum_{m=j}^{\infty}\left(\frac{(j-h)!}{j!(m-h)!}S_2(m-h,j-h|1,\lambda,-x)\right)t^m \\
&= \frac{t^h}{j!}(1+\lambda t)^{x/\lambda}\left[(1+\lambda t)^{1/\lambda}-1\right]^{j-h}
\end{aligned}$$

for an integer h and $j \geqslant h$. Then

$$\frac{\binom{i}{j}}{\binom{i-h}{j-h}}S_2(i-h,j-h|1,\lambda,-x) = \sum_{k=j}^{i}\binom{i}{k}\beta_{i-k}^{(h)}(\lambda,x-y)S_2(k,j|1,\lambda,-y) \tag{14}$$

or

$$\frac{\binom{i-1}{j-1}}{\binom{i-h}{j-h}}S_2(i-h,j-h|1,\lambda,-x) = \sum_{k=j}^{i}\binom{i-1}{k-1}\beta_{i-k}^{(h)}(\lambda,x-y)\frac{j}{k}S_2(k,j|1,\lambda,-y),$$

which gives $(\mathcal{G}_{n,h}[1,\lambda,-x])_{i,j} = (\mathcal{B}_n^{(h)}[\lambda,x-y]\mathcal{G}_{n,0}[1,\lambda,-y])_{i,j}$.

It can be seen from (9) and generating functions that

$$S_2\left(i,j\,\middle|\,1,\frac{1}{\lambda},\frac{x}{\lambda}\right) = \left(\frac{1}{\lambda}\right)^{i-j}S_1(i,j|1,\lambda,-x). \tag{15}$$

Thus, taking into account (6) and (15), replace (λ,x,y) by $(\frac{1}{\lambda},-\frac{x}{\lambda},-\frac{y}{\lambda})$ in (14) to get

$$\frac{\binom{i}{j}}{\binom{i-h}{j-h}}S_1(i-h,j-h|1,\lambda,-x) = \sum_{k=j}^{i}\binom{i}{k}\alpha_{i-k}^{(h)}(\lambda,y-x)S_1(k,j|1,\lambda,-y) \tag{16}$$

or

$$\frac{\binom{i-1}{j-1}}{\binom{i-h}{j-h}}S_1(i-h,j-h|1,\lambda,-x) = \sum_{k=j}^{i}\binom{i-1}{k-1}\alpha_{i-k}^{(h)}(\lambda,y-x)\frac{j}{k}S_1(k,j|1,\lambda,-y)$$

which gives $(g_{n,h}[1,\lambda,-x])_{i,j} = (\mathcal{L}_n^{(h)}[\lambda,y-x]g_{n,0}[1,\lambda,-y])_{i,j}$. □

The theorem above shows that the Bernoulli polynomials can be expressed in terms of the Stirling numbers.

Corollary 8. *For $i \geqslant j \geqslant 0$ and $j \geqslant h$, we have*

$$\binom{i}{j}\beta_{i-j}^{(h)}(\lambda, x-y) = \sum_{k=j}^{i} \frac{\binom{i}{k}}{\binom{i-h}{k-h}} S_2(i-h, k-h|1, \lambda, -x) S_1(k, j|1, \lambda, -y), \tag{17}$$

$$\binom{i}{j}\alpha_{i-j}^{(h)}(\lambda, y-x) = \sum_{k=j}^{i} \frac{\binom{i}{k}}{\binom{i-h}{k-h}} S_1(i-h, k-h|1, \lambda, -x) S_2(k, j|1, \lambda, -y). \tag{18}$$

In particular,

$$\beta_i(\lambda, x) = \sum_{k=0}^{i} \frac{1}{k+1} S_2(i, k|1, \lambda, -x)(\lambda-1)_k,$$

$$\alpha_i(\lambda, x) = \sum_{k=0}^{i} \frac{1}{k+1} S_1(i, k|1, \lambda, x)(1|\lambda)_{k+1}.$$

Note that we can equally well write the result (17) in the form

$$\binom{i-m}{j-m}\beta_{i-j}^{(h-m)}(\lambda, x-y) = \sum_{k=j}^{i} \frac{\binom{i-m}{k-m}}{\binom{i-h}{k-h}} S_2(i-h, k-h|1, \lambda, -x) S_1(k-m, j-m|1, \lambda, -y) \tag{19}$$

for $i \geqslant j \geqslant 0$ and $j \geqslant \max\{h, m\}$ because of

$$\begin{aligned}\mathcal{G}_{n,h}[1, \lambda, -x] g_{n,m}[1, \lambda, -y] &= \mathcal{G}_{n,h}[1, \lambda, -x] g_{n,0}[1, \lambda, -z] \mathcal{G}_{n,0}[1, \lambda, -z] g_{n,m}[1, \lambda, -y] \\ &= \mathcal{B}_n^{(h)}[\lambda, x-z]\mathcal{B}_n^{(-m)}[\lambda, z-y] = \mathcal{B}_n^{(h-m)}[\lambda, x-y].\end{aligned}$$

Set $h = y = \lambda = 0$ and $-m = l \geqslant 0$ to compute any positive (integer) order of Bernoulli polynomials

$$\binom{i}{j}\binom{j+l}{j}^{-1} B_{i-j}^{(l)}(x) = \sum_{k=j}^{i} \binom{l+k}{l}^{-1} R_2(i, k, x) S_1(l+k, l+j)$$

and in particular

$$B_i^{(i)} = \sum_{k=0}^{i} \binom{i+k}{i}^{-1} S_2(i, k) S_1(i+k, i).$$

By (9) and Corollary 8, we have the generalized orthogonality relations

$$\binom{i}{j}(x-y|\lambda)_{i-j} = \sum_{k=j}^{i} S_2(i, k|\mu, \lambda, -x) S_1(k, j|\mu, \lambda, -y),$$

$$\binom{i}{j}(y-x|\mu)_{i-j} = \sum_{k=j}^{i} S_1(i, k|\mu, \lambda, -x) S_2(k, j|\mu, \lambda, -y).$$

Theorem 7 also entails the following.

Corollary 9. *For $i \geqslant j \geqslant 0$ and $j \geqslant h$, we have*

$$\frac{\binom{i}{j}}{\binom{i-h}{j-h}} S_2(i-h, j-h|1, \lambda, -x) = \sum_{k=j}^{i} \binom{k}{j} S_2(i, k|1, \lambda, -y) \alpha_{k-j}^{(-h)}(\lambda, x-y), \tag{20}$$

$$\frac{\binom{i}{j}}{\binom{i-h}{j-h}} S_1(i-h, j-h|1, \lambda, -x) = \sum_{k=j}^{i} \binom{k}{j} S_1(i, k|1, \lambda, -y) \beta_{k-j}^{(-h)}(\lambda, y-x). \tag{21}$$

In particular,

$$\frac{j+1}{i+1} S_2(i+1, j+1|1, \lambda, -x) = \sum_{k=j}^{i} \binom{k}{j} S_2(i, k|\lambda) \alpha_{k-j}(\lambda, x),$$

$$\frac{j+1}{i+1} S_1(i+1, j+1|1, \lambda, x) = \sum_{k=j}^{i} \binom{k}{j} (-1)^{i-k} S_1(i, k|\lambda) \beta_{k-j}(\lambda, x).$$

In consequence of Theorem 2 and Theorem 7 we have

Corollary 10.

$$\begin{aligned}\mathcal{G}_{n,0}[1, \lambda, -x] &= G_n[\lambda, x] G_{n-1}[\lambda, x] \cdots G_1[\lambda, x] \mathcal{G}_{n,0}[1, \lambda, 0] \\ &= \mathcal{G}_{n,0}[1, \lambda, 0] G_n[1, x] G_{n-1}[1, x] \cdots G_1[1, x].\end{aligned}$$

It is evident from definitions that

$$\beta_m^{(-h)}(\lambda, x) = \binom{m+h}{h}^{-1} S_2(m+h, h|1, \lambda, -x), \tag{22}$$

$$\alpha_m^{(-h)}(\lambda, x) = \binom{m+h}{h}^{-1} S_1(m+h, h|1, \lambda, x) \tag{23}$$

for an integer $h \geqslant 0$. Therefore, (14), (16), (20) and (21) may be specified according to h is negative or positive:

$$\binom{h+j}{j} S_2\big(m, j+h\big|1, \lambda, -(x+y)\big) = \sum_{k=j}^{m-h} \binom{m}{k} S_2(m-k, h|1, \lambda, -x) S_2(k, j|1, \lambda, -y), \quad m = i+h, \tag{24}$$

$$\binom{h+j}{j} S_1\big(m, j+h\big|1, \lambda, -(x+y)\big) = \sum_{k=j}^{m-h} \binom{m}{k} S_1(m-k, h|1, \lambda, -x) S_1(k, j|1, \lambda, -y), \quad m = i+h, \tag{25}$$

$$\frac{\binom{i}{j}}{\binom{i+h}{j+h}} S_2\big(i+h, j+h\big|1, \lambda, -(x+y)\big) = \sum_{k=j}^{i} \binom{k}{j} \alpha_{k-j}^{(h)}(\lambda, x) S_2(i, k|1, \lambda, -y), \tag{26}$$

$$\frac{\binom{i}{j}}{\binom{i+h}{j+h}} S_1\big(i+h, j+h\big|1, \lambda, -(y-x)\big) = \sum_{k=j}^{i} \binom{k}{j} \beta_{k-j}^{(h)}(\lambda, x) S_1(i, k|1, \lambda, -y) \tag{27}$$

for $i \geqslant j \geqslant 0$ and arbitrary integer $h \geqslant 0$, and

$$\frac{\binom{i}{j}}{\binom{i-h}{j-h}} S_2\big(i-h, j-h \big| 1, \lambda, -(x+y)\big) = \sum_{k=j}^{i} \binom{i}{k} \beta_{i-k}^{(h)}(\lambda, x) S_2(k, j|1, \lambda, -y), \tag{28}$$

$$\frac{\binom{i}{j}}{\binom{i-h}{j-h}} S_1\big(i-h, j-h \big| 1, \lambda, -(y-x)\big) = \sum_{k=j}^{i} \binom{i}{k} \alpha_{i-k}^{(h)}(\lambda, x) S_1(k, j|1, \lambda, -y), \tag{29}$$

$$\binom{i}{h} S_2\big(i-h, j-h \big| 1, \lambda, -(y-x)\big) = \sum_{k=j}^{i} \binom{k}{j-h} S_2(i, k|1, \lambda, -y) S_1(k-j+h, h|1, \lambda, -x), \tag{30}$$

$$\binom{i}{h} S_1\big(i-h, j-h \big| 1, \lambda, -(y-x)\big) = \sum_{k=j}^{i} \binom{k}{j-h} S_1(i, k|1, \lambda, -y) S_2(k-j+h, h|1, \lambda, -x) \tag{31}$$

for $i \geqslant j \geqslant h \geqslant 0$.

By (9), identities (24) and (25) coincide the "addition theorems" (cf. [21, Corollary 2]), and Eqs. (30) and (31) are still valid for $S_1(n, m|\mu, \lambda, x)$ and $S_2(n, m|\mu, \lambda, x)$.

Remark 11. From (24)–(27) we have identities given in [24, Part 5.3.2].

Remark 12. (28) and (29) reduce to [10, Eqs. (3.2) and (5.3)] and [6, Theorems 12 and 14] taking account of (33) and (34) given later. We also have identities that appear in [2, Theorem 5.1] and [25, Theorem 4.4 and their results].

Remark 13. From Corollary 8 we have identities given by [6, Theorem 25], [10, Eqs. (6.3) and (6.5)] and [17, Eq. (6.99)] (and [14, Corollary 3.3]). Additionally, by $(s+r)_r(s)_{k-r} = (s+r)_k$, (19) gives [5, Eq. (2.89)] for $x = r = j = h = m$, $y = -s$ and $\lambda = 0$.

Remark 14. As a special case of Corollary 9, we have Carlitz's results appearing in [10, Eqs. (3.3), (3.4), (3.25), (5.2) and (5.8)]. Furthermore, [23, Theorem 5] is a special case of (27) for $\lambda = y = j = 0$ and $h = 1$.

Remark 15. (28) gives [11, Theorem 4.1] for $h = j$, $x = y = 0$ and $i = m + j$.

Remark 16. The identities given by [17, Eqs. (6.15), (6.17), (6.21), (6.24), (6.25), (6.28) and (6.29)] are the special cases of (24) for $h = y = 0$ and $x = 1$, (28) for $h = 0$, $y = 1$ and $x = -1$, (29) for $h = y = 0$ and $x = -1$, (17) for $h = y = 0$ and $x = 1$, (18) for $h = y = 0$ and $x = 1$, (24) for $x = y = 0$, (25) for $x = y = 0$ and in all cases $\lambda = 0$, respectively.

Remark 17. Broder noted that Nielsen in "Traite Elementaire des Nombres de Bernoulli, Gauthier-Villars, Paris, 1923, Chapter 12" developed a large number of formulas relating $R_2(n, m, x) = S_2(n, m|1, 0, -x)$ to the Bernoulli and Euler polynomials. (Nielsen's notation is $A_m^n(x) = m!R_2(n, m, x)$.) This note reveals that identities following from our results for $\lambda = 0$ may be studied by Nielsen with probably different notations.

5. Applications of the results in Section 4.2

By using the results presented in (24)–(31) and Corollary 8 several identities for some related number sequences can be deduced for the special values of h, μ, λ, x and y, and from the identities $S_1(i,j|1,\lambda,-x)=(-1)^{i-j}S_1(i+1,j+1|\lambda)$ and $S_2(i,j|1,\lambda,-x)=S_2(i+1,j+1|\lambda)$ for $x=1-\lambda$. In previous section we mentioned some of them. In this section we partly specify identities given by (24)–(31) and Corollary 8.

5.1. Carlitz's weighted Stirling numbers

In this part we give results involving Carlitz's weighted Stirling numbers and Bernoulli polynomials. For $\lambda=0$, we have from (17)

$$\binom{i}{j}B_{i-j}(x)=\sum_{k=j}^{i}\frac{i}{k}R_2(i-1,k-1,x)S_1(k,j),\quad \text{for } h=1,\ y=0,$$

$$B_i(x)=\sum_{k=0}^{i}(-1)^k\frac{k!}{k+1}R_2(i,k,x),\quad \text{for } h=j=1,\ y=0,$$

from (18)

$$i!b_i(-x)=\sum_{k=0}^{i}\frac{(-1)^{i-k}}{k+1}R_1(i,k,x),\quad \text{for } h=j=1,\ y=0,$$

$$(-1)^{i-1}(i-1)!=\sum_{k=0}^{i}(k+1)S_1(i+1,k+1),\quad \text{for } h=-1,\ y=1,\ x=j=0,$$

by (23) and

$$S_1(m,1|1,0,1)=S_1(m,1)+mS_1(m-1,1)=\begin{cases}1, & m=1,\\ (-1)^m(m-2)!, & m>1,\end{cases}\tag{32}$$

which can be held from (29) for $\lambda=h=y=0$, $j=1$ and $x=1$.

From (26)

$$\frac{i!}{(i+h)!}R_2(i+h,j+h,x)=\frac{1}{(j+h)!}\sum_{k=j}^{i}S_2(i,k)b_{k-j}^{(h)}(x)k!,\quad \text{for } y=0,$$

$$\frac{(x+1)^{i+1}-x^{i+1}}{i+1}=\sum_{k=0}^{i}S_2(i,k)b_k(x)k!,\quad \text{for } h=1,\ j=y=0,$$

and from (27)

$$(-1)^{i-j}\frac{\binom{i}{j}}{\binom{i+h}{j+h}}R_1(i+h,j+h,-x)=\sum_{k=j}^{i}\binom{k}{j}S_1(i,k)B_{k-j}^{(h)}(x),\quad \text{for } y=0,$$

$$\frac{\binom{i}{j}}{\binom{i+h}{j+h}}S_1(i+h+1,j+h+1)=\sum_{k=j}^{i}\binom{k}{j}S_1(i+1,k+1)B_{k-j}^{(h)},\quad \text{for } y=1,\ x=0.$$

5.2. r-Stirling numbers

Setting $\mu = -1$ in (9) we have

$$S_1(m,k|1,-\lambda,-x) = (-1)^{m-k} S_1(m,k|-1,\lambda,x), \tag{33}$$

$$S_2(m,k|1,-\lambda,-x) = (-1)^{m-k} S_2(m,k|-1,\lambda,x). \tag{34}$$

So that

$$S_1(m,k|1,0,-r) = (-1)^{m-k} \begin{bmatrix} m+r \\ k+r \end{bmatrix}_r \quad \text{and} \quad S_2(m,k|1,0,-r) = \begin{Bmatrix} m+r \\ k+r \end{Bmatrix}_r$$

for $\lambda = 0$ and nonnegative integer $x = r$. Then the results presented in Section 4.2 can be specialized in terms of r-Stirling numbers for $\lambda = 0$ and integers x and y.

In the first place note that

$$\begin{bmatrix} m \\ k \end{bmatrix}_0 = \begin{bmatrix} m \\ k \end{bmatrix}_1 = \begin{bmatrix} m \\ k \end{bmatrix}, \qquad \begin{Bmatrix} m \\ k \end{Bmatrix}_0 = \begin{Bmatrix} m \\ k \end{Bmatrix}_1 = \begin{Bmatrix} m \\ k \end{Bmatrix}.$$

The following are special cases of (31) for $\lambda = 0$, nonnegative integers $y = p$, $y - x = r$ and $j \geqslant h$:

$$\begin{bmatrix} i+r \\ j+r \end{bmatrix}_r = \sum_{k=j}^{i} \binom{k}{j} \begin{bmatrix} i+p \\ k+p \end{bmatrix}_p (r-p)^{k-j}, \quad \text{for } h = 0, \tag{35}$$

$$\begin{bmatrix} i+r \\ j+r \end{bmatrix}_r = \frac{1}{i+1} \sum_{k=j}^{i} \binom{k+1}{j} \begin{bmatrix} i+r \\ k+r \end{bmatrix}_{r-1}, \quad \text{for } h = 1,\ p = r-1,$$

$$\binom{i}{j} \langle p+j \rangle_{i-j} = \sum_{k=j}^{i} \begin{bmatrix} i+p \\ k+p \end{bmatrix}_p \begin{Bmatrix} k \\ j \end{Bmatrix}, \quad \text{for } h = j,\ r = p+j,$$

where $\langle m \rangle_k = \begin{cases} m(m+1)\cdots(m+k-1), & k > 0, \\ 1, & k = 0 \end{cases}$ and we use that

$$S_2(k,m|1,0,m) = (-1)^{k-m} S_2(k,m) = (-1)^{k-m} \begin{Bmatrix} k \\ m \end{Bmatrix}.$$

Additionally, for $p = r-1$ and $p = r+1$ in (35) we get recurrences

$$\begin{bmatrix} i+r \\ j+r \end{bmatrix}_r = \sum_{k=j}^{i} \binom{k}{j} \begin{bmatrix} i+r-1 \\ k+r-1 \end{bmatrix}_{r-1},$$

$$\begin{bmatrix} i+r \\ j+r \end{bmatrix}_r = \sum_{k=j}^{i} (-1)^{k-j} \binom{k}{j} \begin{bmatrix} i+r+1 \\ k+r+1 \end{bmatrix}_{r+1},$$

respectively. The identities above reduce to [17, Eq. (6.16) and (6.18)] for $r = 1$ and $r = 0$, respectively.

Some identities deduced from (30), for $\lambda = 0$, nonnegative integers $y = p$, $y - x = r$ and $j \geqslant h$, are

$$\begin{Bmatrix} i+r \\ j+r \end{Bmatrix}_r = \sum_{k=j}^{i} \binom{k}{j} \begin{Bmatrix} i+p \\ k+p \end{Bmatrix}_p (r-p)_{k-j}, \quad \text{for } h = 0,$$

$$\begin{Bmatrix} i+p-1 \\ j+p-1 \end{Bmatrix}_{p-1} = \frac{1}{j!} \sum_{k=j}^{i} (-1)^{k-j} \begin{Bmatrix} i+p \\ k+p \end{Bmatrix}_p k!, \quad \text{for } h = 0,\ r = p-1, \tag{36}$$

$$\left\{ {i-1+r \atop j-1+r} \right\}_r = \frac{1}{i(j-1)!} \sum_{k=j}^{i} (-1)^{k-j} \frac{k!}{k-j+1} \left\{ {i+r \atop k+r} \right\}_r, \quad \text{for } h=1,\ p=r, \tag{37}$$

$$\left\{ {i-1+r \atop j-1+r} \right\}_r = \frac{1}{i(j-1)!} \sum_{k=j}^{i} (-1)^{k-j} \left\{ {i+r+1 \atop k+r+1} \right\}_{r+1} k! H_{k-j+1}, \quad \text{for } h=1,\ p=r+1,$$

where $\left[{m+1 \atop 2} \right] = m! H_m$ and

$$H_m = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{m}.$$

By (32), we also have

$$\frac{i}{j} \left\{ {i-1+r \atop j-1+r} \right\}_r - \left\{ {i+r-1 \atop j+r-1} \right\}_{r-1} = -\frac{1}{j!} \sum_{k=j+1}^{i} \frac{(-1)^{k-j} k!}{(k+1-j)(k-j)} \left\{ {i+r-1 \atop k+r-1} \right\}_{r-1} \tag{38}$$

for $h=1$ and $p=r-1$. Additionally, regular Stirling numbers of the second kind satisfy

$$\begin{aligned} \left\{ {i \atop j} \right\} &= \frac{1}{j!} \sum_{k=j}^{i} (-1)^{k-j} \left\{ {i+1 \atop k+1} \right\} k!, \quad \text{for } j \geqslant 0,\ p=1 \text{ in (36)} \\ &= \frac{1}{i(j-1)!} \sum_{k=j}^{i} (-1)^{k-j} \frac{k!}{k-j+1} \left\{ {i+1 \atop k+1} \right\}, \quad \text{for } j \geqslant 1,\ r=1 \text{ in (37)} \\ &= \frac{1}{(j-i)(j-1)!} \sum_{k=j+1}^{i} \frac{(-1)^{k-j} k!}{(k-j+1)(k-j)} \left\{ {i \atop k} \right\}, \quad \text{for } j \geqslant 1,\ r=1 \text{ in (38)}. \end{aligned}$$

The following are special cases of (17) for $\lambda = 0$, nonnegative integers $y = p$, $x = r$ and $j \geqslant h$:

$$\binom{i}{j} B_{i-j}(p) = \sum_{k=j}^{i} (-1)^{i-k} \frac{i}{k} \left\{ {i \atop k} \right\} \left[{k+p \atop j+p} \right]_p, \quad \text{for } h=1,\ r=1,$$

$$B_i(r) = \sum_{k=0}^{i} (-1)^k \frac{k!}{k+1} \left\{ {i+r \atop k+r} \right\}_r, \quad \text{for } h=j=1,\ p=0,$$

$$\binom{i+1}{j} = \sum_{k=j}^{i} (-1)^{i-k} (k+1) \left\{ {i+p \atop k+p} \right\}_{p-1} \left[{k+p \atop j+p} \right]_p, \quad \text{for } h=-1,\ r=p-1,$$

$$\binom{i+1}{j} = \sum_{k=j}^{i} (-1)^{k-j} (k+1) \left\{ {i+1+p \atop k+1+p} \right\}_p \left[{k+p \atop j+p} \right]_p, \quad \text{for } h=-1,\ r=p.$$

Moreover, from (17) for $\lambda = y = 0$, $h = j = 1$ and $x = 1-i$, we have

$$B_i(i+1) = \sum_{k=0}^{i} \frac{(i-k)!}{i-k+1} \left\{ {i+k \atop i} \right\}_k$$

by making use of $R_2(m,k,-x) = (-1)^{m-k} R_2(m,k,x-k)$ and $B_m(1-x) = (-1)^m B_m(x)$.

5.3. Hyperharmonic numbers

The hyperharmonic number of order r denoted by H_m^r is defined by

$$H_m^r = \sum_{k=1}^{m} H_k^{r-1}$$

for $r, m \geqslant 1$, $H_m^0 = \frac{1}{m}$ for $m \geqslant 1$, and $H_m^r = 0$ for $r < 0$ or $m \leqslant 0$ [4].

A generating function for the hyperharmonic numbers is

$$-(1-x)^{-r}\ln(1-x) = \sum_{n=1}^{\infty} H_n^r x^n.$$

It follows from (7) and (33) that

$$m!H_m^r = \begin{bmatrix} m+r \\ 1+r \end{bmatrix}_r = S_1(m,1|-1,0,r) = (-1)^{m-1}S_1(m,1|1,0,-r).$$

A combinatorial proof of this fact can be found in [4, Theorem 2]. Thus, we have from (27)

$$i!H_{i+1}^r = \sum_{k=0}^{i} (-1)^k B_k \begin{bmatrix} i+r \\ k+r \end{bmatrix}_r, \quad \text{for } h=1,\ y=r,\ x=j=\lambda=0,$$

$$i!H_{i+1}^{r-1} = \sum_{k=0}^{i} B_k \begin{bmatrix} i+r \\ k+r \end{bmatrix}_r, \quad \text{for } h=x=1,\ y=r,\ j=\lambda=0,$$

$$i!H_i^{r+p} = \sum_{k=1}^{i} k \begin{bmatrix} i+r \\ k+r \end{bmatrix}_r p^{k-1}, \quad \text{for } j=1,\ y=r,\ y-x=r+p \geqslant 0,\ h=\lambda=0,$$

$$i!H_i^p = \sum_{k=1}^{i} k \begin{bmatrix} i \\ k \end{bmatrix} p^{k-1}$$

and from (29)

$$H_i^{r-m} = \sum_{k=1}^{i} \frac{(-1)^{i-k}}{(i-k)!}(m)_{i-k} H_k^r$$

for $j=1$, $y=r$, $y-x=r-m \geqslant 0$ and $h=\lambda=0$. This gives

$$\sum_{k=1}^{i} \binom{i-k+p-1}{p-1} H_k^r = H_i^{p+r}, \quad \text{for } m=-p<0, \tag{39}$$

$$\sum_{k=1}^{i} kH_k^r = (i+1)!H_i^{r+1} - i!H_i^{r+2}, \quad \text{for } p=2.$$

(39) is Eq. (7) of [4] and thereby [4, Theorem 1] (see also [15, Theorem 5]) for $r=0$.

5.4. Lah numbers

For $\lambda=1$ and $x=0$ in (33) and (34), we have

$$S_1(m,k|1,-1,0) = (-1)^{m-k}L(m,k) \quad \text{and} \quad S_2(m,k|1,-1,0) = L(m,k).$$

Then, it follows from (18), (26) and (30) that

$$\binom{i-j+m-1}{i-j} = \sum_{k=j}^{i} (-1)^{k-j} \binom{i+m-1}{k+m-1} \binom{k-1}{j-1}, \quad \text{for } x = y = 0,\ h = -m < 0,$$

$$\binom{i+h}{j+h} = \sum_{k=j}^{i} \binom{i}{k} \binom{h}{k-j}, \quad \text{for } x = y = 0,\ h \geqslant 0,$$

$$\binom{i-h+1}{j-h+1} = \sum_{k=j}^{i} (-1)^{k-j} \binom{i+1}{k+1} \binom{k-j+h-1}{h-1}, \quad \text{for } x = 0,\ y = 1-\lambda,\ j \geqslant h \geqslant 1,$$

and $\lambda = -1$, respectively.

In general, for arbitrary x and y, we have

$$S_2(m, j|1, -1, -x) = (-1)^{m-j} S_1(m, j|1, -1, -x) = \binom{m}{j} \langle x + j \rangle_{m-j},$$

$$\beta_m^{(h)}(-1, x) = (-1)^m \alpha_m^{(h)}(-1, -x) = \langle x - h \rangle_m.$$

Then (18), (28) and (30) reduce to

$$\langle x - h - y \rangle_{i-j} = \sum_{k=0}^{i-j} \binom{i-j}{k} (-1)^k \langle y + j \rangle_k \langle x - h + j + k \rangle_{i-j-k},$$

$$\langle y + j + x - h \rangle_{i-j} = \sum_{k=0}^{i-j} \binom{i-j}{k} \langle y + j \rangle_k \langle x - h \rangle_{i-j-k},$$

$$\langle y + j - x - h \rangle_{i-j} = \sum_{k=0}^{i-j} \binom{i-j}{k} (-1)^k \langle x + h \rangle_k \langle y + j + k \rangle_{i-j-k},$$

respectively.

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