Combinatorial identities for the r-Lah numbers

Hacène Belbachir & Imad Eddine Bousbaa
USTHB, Faculty of Mathematics, RECITS Lab., DG-RSDT
BP 32, El Alia, 16111, Bab Ezzouar, Algiers, Algeria
hbelbachir@usthb.dz & ibousbaa@usthb.dz

Abstract

This paper is an orthogonal continuation of the work of Belbachir and Belkhir in sense where we establish, using bijective proofs, recurrence relations and convolution identities between lines of r-Lah triangle. It is also established a symmetric function form for the r-Lah numbers.

1 Introduction

The r-Lah numbers, denoted $\left\lfloor{n \atop k}\right\rfloor_r$, count the number of partitions of the set $\{1, 2, ..., n\}$ into k non empty ordered lists, such that the numbers $1, 2, ..., r$ are in distinct lists. They satisfy, see for instance [3, 1], the recurrence relation

$$\left\lfloor{n \atop k}\right\rfloor_r = \left\lfloor{n-1 \atop k-1}\right\rfloor_r + (n+k-1)\left\lfloor{n-1 \atop k}\right\rfloor_r, \tag{1}$$

with $\left\lfloor{n \atop k}\right\rfloor_r = \delta_{n,k}$ for $k = r$, where δ is the Kronecker delta, and $\left\lfloor{n \atop k}\right\rfloor_r = 0$ for $n < r$.

For $r = 0$ and $r = 1$, we get the classical Lah numbers.

The r-Lah numbers have the following explicit formula

$$\left\lfloor{n \atop k}\right\rfloor_r = \frac{(n+r-1)!}{(k+r-1)!}\binom{n-r}{k-r} = \frac{(n-r)!}{(k-r)!}\binom{n+r-1}{k+r-1}. \tag{2}$$

In a previous work, the first author and Belkhir [1], established a cross recurrence formula, a triangular recurrence with rational coefficient for the Lah numbers and a vertical recurrence relation using bijective proof.

Our aim is to give some new combinatorial identities for the r-Lah numbers. All the identities given in [1] deal with relations between columns of r-Lah triangle. Our work is a dual complement to [1] in sense that we give identities explaining relations between lines of r-Lah triangle. In section

ARS COMBINATORIA 115(2014), pp. 453-458

relation

$$\left\lfloor{n \atop k}\right\rfloor_r = (n+r-1)\left\lfloor{n-1 \atop k}\right\rfloor_r + \frac{(n+r-1)}{(k+r-1)}\left\lfloor{n-1 \atop k-1}\right\rfloor_r. \tag{7}$$

Under the restriction $r = 0$, we get relation (5) of [1].

Remark 3.2 *For $s = n - r$ in relation (6), we get the classical explicit form of r-Lah numbers given by (2).*

The following result improve the precedent one in sense that the coefficients are integers.

Theorem 3.3 *Let s, r, k and n nonnegative integers such that $r \leqslant k \leqslant n$ and $r \leqslant n - s$, we have*

$$\left\lfloor \begin{matrix} n \\ k \end{matrix} \right\rfloor_r = \sum_{j=0}^{s} \frac{(n+k-j-1)!}{(n+k-s-1)!} \binom{s}{j} \left\lfloor \begin{matrix} n-s \\ k-j \end{matrix} \right\rfloor_r . \quad (8)$$

Proof. We divide the n elements into two groups : a first one with s elements $\{1, \ldots, s\}$ and second one with $n - s$ elements . With the first group we can constitute j lists $(0 \leqslant j \leqslant s)$ and with the second group we can constitute $k - j$ lists such that $1, \ldots, r$ are in distinct lists (it is possible because $r \leqslant n - s$). The r fixed elements must be chosen from the elements of the second group. We have $\left\lfloor \begin{smallmatrix} n-s \\ k-j \end{smallmatrix} \right\rfloor_r$ possibilities to constitute the $k - j$ lists. It remains to count how to constitute the j remaining ones. We have $\binom{s}{j}$ possibilities to choose j elements from the first group with one element by list. Then, we order the remaining $s - j$ elements into the k lists, so the first one has $(n - s + k)$ choices ($n - s$ ways after each ordered element and k ways as head list), the second one has $(n - s + k + 1)$ choices (one possibility added by the previews insertion) and so on.... The last element $s - j$ has $(n - s + k + (s - j - 1)) = (n + k - j - 1)$ choices. It gives $\frac{(n+k-j-1)!}{(n+k-s-1)!} = (n - s + k)(n - s + k + 1) \cdots (n + k - j - 1)$ possibilities. We conclude by summing. □

Remark 3.4 *For $s = 1$, we obtain the well known recurrence relation (1), and for $s = n - r$ we get again the explicit formula (2).*

4 Relation between r-Lah and Lah numbers

It is established [1], by combinatorial approach, that the r-Lah numbers can be expressed in terms of Lah numbers as follows

$$\left\lfloor \begin{matrix} n+r \\ k+r \end{matrix} \right\rfloor_r = \sum_{s=0}^{n-k} \sum_{i_1+\cdots+i_r=s} (i_1+1)! \cdots (i_r+1)! \binom{n}{i_1, \ldots, i_r, n-s} \left\lfloor \begin{matrix} n-s \\ k \end{matrix} \right\rfloor . \quad (9)$$

To prove the relation above, the authors consider the r first lists containing the r first elements and i_j $(1 \leqslant j \leqslant r)$ other elements. So the operation of counting the different situations was done in two steps : first we choose the i_j elements, then arrange the elements of each lists.

Now, we give an other formulation expressing r-Lah numbers in terms of Lah numbers without counting a multi-sum with a combinatorial argument.

Theorem 4.1 *Let r, k and n positive integers such that, $r \leqslant k \leqslant n$, we have*

$$\left\lfloor \begin{matrix} n \\ k \end{matrix} \right\rfloor_r = \sum_{s=0}^{n-k} \frac{(s+2r-1)!}{(2r-1)!} \binom{n-r}{s} \left\lfloor \begin{matrix} n-r-s \\ k-r \end{matrix} \right\rfloor. \tag{10}$$

Proof. The r first elements can be considered as representing of the r first lists. Because we have to constitute k lists, let us consider the s $(0 \leqslant s \leqslant n-k)$ elements that will belong to the r first lists. We have $\binom{n-r}{s}$ possibilities to choose them. Then, we insert the s elements to the r lists and we have $2r$ possibilities for the first one, $2r+1$ possibilities for the second and so on ..., until the last element s, it has $(s+2r-1)$ possibilities. This gives $2r\,(2r+1)\cdots(2r+s-1) = \frac{(s+2r-1)!}{(2r-1)!}$ possibilities. Finally, we constitute the remaining $k-r$ lists with the remaining $n-r-s$ elements and we have $\left\lfloor \begin{smallmatrix} n-r-s \\ k-r \end{smallmatrix} \right\rfloor$ possibilities. □

Corollary 4.1.1 *For $r = 1$, in the relations (9) and (10), we get the vertical recurrence relation for the Lah numbers*

$$\left\lfloor \begin{matrix} n \\ k \end{matrix} \right\rfloor = \sum_{i=0}^{n-k} (i+1)! \binom{n-1}{i} \left\lfloor \begin{matrix} n-i-1 \\ k-1 \end{matrix} \right\rfloor. \tag{11}$$

5 Expression of the r-Lah numbers in terms of the $(r \pm s)$-Lah numbers

The r-Lah numbers satisfy the following horizontal recurrence relations. They express an element $\left\lfloor \begin{smallmatrix} n \\ k \end{smallmatrix} \right\rfloor_r$ of r-Lah triangle in terms of the elements of the same line from the $(r+s)$-Lah triangle and $(r-s)$-Lah triangle.

Theorem 5.1 *The r-Lah numbers satisfy*

$$\left\lfloor \begin{matrix} n \\ k \end{matrix} \right\rfloor_r = \frac{(n+r-1)!}{(k+r-1)!} \sum_{i=0}^{s} \frac{(k+i+(r+s)-1)!}{(n+(r+s)-1)!} \binom{s}{i} \left\lfloor \begin{matrix} n \\ k+i \end{matrix} \right\rfloor_{r+s}, \tag{12}$$

$$\left\lfloor \begin{matrix} n \\ k \end{matrix} \right\rfloor_r = \frac{(n-r)!}{(k-r)!} \sum_{i=0}^{s} \binom{s}{i} \frac{(k+i-r+s)!}{(n-r+s)!} \left\lfloor \begin{matrix} n \\ k+i \end{matrix} \right\rfloor_{r-s}, \ (r \geq s). \tag{13}$$

Proof. From (2), $\left\lfloor \begin{smallmatrix} n \\ k \end{smallmatrix} \right\rfloor_r = \frac{(n+r-1)!}{(k+r-1)!}\binom{n-r}{k-r}$, Vandermonde's formula gives $\left\lfloor \begin{smallmatrix} n \\ k \end{smallmatrix} \right\rfloor_r = \frac{(n+r-1)!}{(k+r-1)!}\sum_{i=0}^{s}\binom{s}{i}\binom{n-r-s}{k+i-r-s}$, thus we get the result. The same approach gives the second relation. □

An expression of the Lah numbers in terms of the s-Lah numbers can be deduced from (12) for $r = 1$.

Corollary 5.1.1 *For $s \geq 1$, we get*

$$\left\lfloor \begin{matrix} n \\ k \end{matrix} \right\rfloor = \frac{n!}{k!} \sum_{i=0}^{s-1} \binom{s-1}{i} \frac{(k+i+s-1)!}{(n+s-1)!} \left\lfloor \begin{matrix} n \\ k+i \end{matrix} \right\rfloor_s, \qquad (14)$$

And for $s = 1$, in relations (12) and (13), we get

Corollary 5.1.2 *Triangular recurrence relations*

$$\left\lfloor \begin{matrix} n \\ k \end{matrix} \right\rfloor_r = (k+r+1) \frac{(k+r)}{(n+r)} \left\lfloor \begin{matrix} n \\ k+1 \end{matrix} \right\rfloor_{r+1} + \frac{(k+r)}{(n+r)} \left\lfloor \begin{matrix} n \\ k \end{matrix} \right\rfloor_{r+1}, \qquad (15)$$

$$\left\lfloor \begin{matrix} n \\ k \end{matrix} \right\rfloor_{r+1} = (k-r+1) \frac{(k-r)}{(n-r)} \left\lfloor \begin{matrix} n \\ k+1 \end{matrix} \right\rfloor_r + \frac{(k-r)}{(n-r)} \left\lfloor \begin{matrix} n \\ k \end{matrix} \right\rfloor_r. \qquad (16)$$

Using (7) in (15), we get a recurrence relation of order 3 with integer coefficients which improve the quality of the recurrence relation.

Corollary 5.1.3 *The following recurrence of order three holds*

$$\left\lfloor \begin{matrix} n \\ k \end{matrix} \right\rfloor_r = \left\lfloor \begin{matrix} n-1 \\ k-1 \end{matrix} \right\rfloor_{r+1} + 2(k+r) \left\lfloor \begin{matrix} n-1 \\ k \end{matrix} \right\rfloor_{r+1} + (k+r+1)(k+r) \left\lfloor \begin{matrix} n-1 \\ k+1 \end{matrix} \right\rfloor_{r+1}.$$

As a special case of (13), for $s = r$, we get

Corollary 5.1.4 *Expression of the r-Lah numbers in terms of the Lah numbers*

$$\left\lfloor \begin{matrix} n \\ k \end{matrix} \right\rfloor_r = \frac{(n-r)!}{n!\,(k-r)!} \sum_{i=0}^{r} (k+i)! \binom{r}{i} \left\lfloor \begin{matrix} n \\ k+i \end{matrix} \right\rfloor.$$

Acknowledgement 1 *The authors would like to thank the anonymous referee for careful reading and suggestions that improved the clarity of this manuscript.*

References

[1] H. Belbachir and A. Belkhir. Cross recurrence relations for r-Lah numbers. *Ars Combin.*, 110:199–203, 2013.

[2] A. Z. Broder. The r-Stirling numbers. *Discrete Math.*, 49(3):241–259, 1984.

[3] C. A. Charalambides. *Enumerative combinatorics.* CRC Press Series on Discrete Mathematics and its Applications. Chapman & Hall/CRC, Boca Raton, FL, 2002.