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ON SOME COMBINATORIAL IDENTITIES INVOLVING THE TERMS OF GENERALIZED FIBONACCI AND LUCAS SEQUENCES

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Abstract

In this paper, we consider the Horadam sequence and some summation formulas involving the terms of the Horadam sequence. We derive combinatorial identities by using the trace, the determinant, and the *nth* power of a special matrix.

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1. Preliminaries

Generalized Fibonacci sequence $W_n = W_n(a, b; p, q)$ is defined as follows;

(1.1) $$W_n = pW_{n-1} - qW_{n-2}\ ,\ W_0 = a,\ W_1 = b.$$

Where a, b, p, and q are arbitrary complex numbers, with $q \neq 0$. Since, these numbers have been studied firstly by Horadam(see, e.g., [1]) they are called as Horadam numbers. Some special cases of this sequence such as

(1.2) $$U_n = W_n(0, 1; p, q)\ ,\ V_n = W_n(2, p; p, q)$$

were investigated by Lucas[6]. Further and in detailed knowledge can be found in[1, 2, 3, 4, 5, 6]. If α, β assumed distinct, are the roots of

(1.3) $$\lambda^2 - p\lambda + q = 0$$

then the sequence W_n has the Binet representation

(1.4) $$W_n = \frac{A\alpha^n - B\beta^n}{\alpha - \beta},$$

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where $A = b - a\beta$ and $B = b - a\alpha$. For negative indices, this formula is given as

$$W_{-n} = \frac{pW_{-n+1} - W_{-n+2}}{q}.$$

So, for all integer numbers n, we can write

$$W_n = pW_{n-1} - qW_{n-2} \;;\; W_0 = a,\; W_1 = b. \tag{1.5}$$

In [16], the authors used the matrix in relation to the recurrence relation (1);

$$M = \begin{pmatrix} p & -q \\ 1 & 0 \end{pmatrix}. \tag{1.6}$$

Indeed, if $p = 1$ and $q = -1$, then the matrix M reduces to Fibonacci $Q-$ matrix. The matrix M is a special case of the general $k \times k$, $Q-$ matrix[11]. Now, we use the matrix M and its powers to prove and drive the some combinatorial identities involving the terms from the sequence $\{W_n\}$. Such identities are quite extensive on literature, but for this purpose we use only the trace and determinant of the matrix M^n. In [9], J. Mc. Laughlin gave a new formula for the nth power of a 2×2 matrix. The author proved that if $B = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is an arbitrary 2×2 matrix, then for $n \geq 1$, B^n is

$$B^n = \begin{pmatrix} y_n - dy_{n-1} & by_{n-1} \\ cy_{n-1} & y_n - ay_{n-1} \end{pmatrix}; y_n = \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n-i}{i} T^{n-2i}(-D)^i \tag{1.7}$$

where T and D are the trace and determinant of matrix B, respectively. In [10], K. S. Williams gave a formula for the nth power of any 2×2 matrix C with eigenvalues α and β as follows;

$$C^n = \begin{cases} \frac{\alpha^n(C-\beta I)-\beta^n(C-\alpha I)}{\alpha-\beta};\; \alpha \neq \beta \\ \alpha^{n-1}\left(nC - (n-1)\,\alpha I\right);\;\; \alpha = \beta \end{cases} \tag{1.8}$$

In [8], H. Belbachir extended this result to any matrix A of order m, $m \geq 2$. Also, he derived some identities concerning the Stirling numbers.

2. Some Combinatorial Identities involving the terms of Horadam Sequence

In this section, firstly we give a general formula for the generalized Lucas numbers. Then, we investigate the special cases of this sequence. And then, we give some formulae for generalized Fibonacci numbers.

2.1. Theorem. *For $n \geq 1$, we have the following identity;*

$$\sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \begin{pmatrix} n-k \\ k \end{pmatrix} \frac{n}{n-k} p^{n-2k}\,(-q)^{\,k} = \frac{1}{2^{n-1}} \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \begin{pmatrix} n \\ 2k \end{pmatrix} p^{n-2k}\left(p^2 - 4q\right)^{\,k}. \tag{2.1}$$

Where p and q are the trace and determinant of the matrix M, respectively.

Proof. Using the matrix M^n,

$$M^n = \begin{pmatrix} y_n & -qy_{n-1} \\ y_{n-1} & y_n - py_{n-1} \end{pmatrix},$$

we can write

$$tr(M^n) = \lambda_1^n + \lambda_2^n = 2y_n - py_{n-1} \tag{2.2}$$

$$2y_n - py_{n-1} = 2\sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n-k}{k} T^{n-2k}(-D)^k - p\sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n-1-k}{k} T^{n-1-2k}(-D)^k$$

$$2y_n - py_{n-1} = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n-k}{k} p^{n-2k}(-q)^k \left(2 - \frac{n-2k}{n-k}\right).$$

Thus, we have the right side of the equation (10).The left side of the equation (10) is follows;

$$\lambda_1^n + \lambda_2^n = \frac{1}{2^n}\left(\sum_{k=0}^{n} \binom{n}{k} p^{n-k}\left(\sqrt{p^2-4q}\right)^k + \sum_{k=0}^{n} \binom{n}{k} p^{n-k}\left(-\sqrt{p^2-4q}\right)^k\right)$$

$$\lambda_1^n + \lambda_2^n = \frac{1}{2^{n-1}}\left(\sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{2k} p^{n-2k}\left(p^2-4q\right)^k\right)$$

Thus, the proof is completed. □

In the following theorem, we give the nth term of the generalized Lucas sequence by using this method.

2.2. Theorem. *For $n \geq 0$ we have the following identities;*

$$\text{(2.3)} \qquad i)\ V_n = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n-k}{k} \frac{n}{n-k} p^{n-2k}(-q)^k$$

and

$$\text{(2.4)} \qquad ii)\ V_n = \frac{1}{2^{n-1}} \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{2k} p^{n-2k}\left(p^2-4q\right)^k$$

where p and q are the trace and determinant of the matrix M, respectively.

Note that if we take $p = 1$ and $q = -1$ in the Theorem 2.1 and Theorem 2.2, then we obtain that

$$L_n = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n-k}{k} \frac{n}{n-k} = \frac{1}{2^{n-1}} \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{2k} 5^k.$$

The each sides of the last equation can be found in [13]. In addition to this, in the Theorem 2.1 and Theorem 2.2 if we take $p = 2$, $q = -1$ and $p = 1$, $q = -2$ then we obtain the identities for the Pell-Lucas and Jacobsthal-Lucas sequences, respectively, as follows;

$$Q_n = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n-k}{k} \frac{n}{n-k} 2^{n-2k} = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{2k} 2^{k+1},$$

and

$$j_n = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n-k}{k} \frac{n}{n-k} 2^k = \frac{1}{2^{n-1}} \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{2k} 9^k.$$

Similarly, for the certain values of p and q we can get the bivariate Lucas, Pell-Lucas and Jacobsthal-Lucas polynomials . The equation is given in the following theorem can be seen many studies, but we give this identity by using a different method.

2.3. Theorem. *For $n \geq 0$ we have the following identity;*

$$U_n = \sum_{i=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n-1-i}{i} p^{n-1-2i} (-q)^i, \tag{2.5}$$

where $U_n = W_n(0,1;p,q)$.

By using the Theorem 2.3 we can write the following identities;

$$F_n = \sum_{i=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n-1-i}{i}, \quad p=1,\ q=-1,$$

$$P_n = \sum_{i=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n-1-i}{i} 2^{n-1-2i}, \quad p=2,\ q=-1,$$

and

$$J_n = \sum_{i=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n-1-i}{i} 2^i, \quad p=1,\ q=-2.$$

Now, we give a formula for the generalized Fibonacci numbers by using the studies of Shannon and Laughlin.

2.4. Theorem. *For $k \geq 1$, we have*

$$U_{nk} = U_n \sum_{i=0}^{\lfloor \frac{k-1}{2} \rfloor} \binom{k-1-i}{i} {V_n}^{k-1-2i} (-q^n)^i, \tag{2.6}$$

where $U_n = W_n(0,1;p,q)$ and $V_n = W_n(2,p;p,q)$.

Finally, we give the most general formula for generalized Fibonacci numbers in the following theorem.

2.5. Theorem. *For $n, k \geq 1$ and $r \neq 0$, we have*

$$U_{nk+r} = \sum_{i=0}^{\lfloor \frac{k}{2} \rfloor} \binom{k-i}{i} {V_n}^{k-2i} (-q^n)^i \delta, \tag{2.7}$$

where $\delta = q^r \frac{U_{n-r}}{V_n} \frac{k-2i}{k-i} + U_r$.

Proof. The proof can be seen by the powers of the matrix M as follows; Note that

$$M^n = \begin{pmatrix} U_{n+1} & -qU_n \\ U_n & -qU_{n-1} \end{pmatrix}$$

and

$$M^{nk+r} = \begin{pmatrix} U_{nk+r+1} & -qU_{nk+r} \\ U_{nk+r} & -qU_{nk+r-1} \end{pmatrix}.$$

Then, we write

$$U_{nk+r} = y_{k-1}(U_n U_{r+1} - U_r U_{n+1}) + U_r y_k \tag{2.8}$$

where $y_k = \sum_{i=0}^{\lfloor \frac{k}{2} \rfloor} \binom{k-i}{i} T^{k-2i}(-D)^i$, $T = U_{n+1} - qU_{n-1} = V_n$, $D = q^n$ and since

$$U_n U_{r+1} - U_r U_{n+1} = q^r U_{n-r},$$

we obtain

$$U_{nk+r} = q^r U_{n-r} \sum_{i=0}^{\lfloor \frac{k-1}{2} \rfloor} \binom{k-1-i}{i} T^{k-1-2i}(-D)^i + U_r \sum_{i=0}^{\lfloor \frac{k}{2} \rfloor} \binom{k-i}{i} T^{k-2i}(-D)^i,$$

$$U_{nk+r} = \sum_{i=0}^{\lfloor \frac{k}{2} \rfloor} \binom{k-i}{i} {V_n}^{k-2i}(-q^n)^i \left(q^r \frac{U_{n-r}}{V_n} \frac{k-2i}{k-i} + U_r \right).$$

Thus, the proof is completed. □

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