A comparison of $(x_n + y_n\sqrt{2})(3 + 2\sqrt{2})$ and $\begin{pmatrix}3 & 4\\ 2 & 3\end{pmatrix}\begin{pmatrix}x_n\\ y_n\end{pmatrix}$ shows that all solutions of $t^2 - 2(2b)^2 = 1$ are obtained by

$$\begin{pmatrix}3 & 4\\ 2 & 3\end{pmatrix}\begin{pmatrix}t_n\\ 2b_n\end{pmatrix} = \begin{pmatrix}t_{n+1}\\ 2b_{n+1}\end{pmatrix}$$

and hence all solutions of $\frac{a(a+1)}{2} = b^2$ are obtained from $a_n = \frac{t_n - 1}{2}$, $b_n = \frac{2b_n}{2}$.

Note that t_n is odd for all n so a_n is an integer.

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CENTRAL FACTORIAL NUMBERS AND RELATED EXPANSIONS

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1. INTRODUCTION

The central factorials have been introduced and studied by Stephensen; properties and applications of these factorials have been discussed among others and by Jordan [3], Riordan [5], and recently by Roman and Rota [4].

For positive integer m,

$$x^{[m,b]} = x\left(x + \frac{1}{2}mb - b\right)\left(x + \frac{1}{2}mb - 2b\right) \cdots \left(x - \frac{1}{2}mb + b\right)$$

defines the generalized central factorial of degree m and increment b. This definition can be extended to any integer m as follows:

$$x^{[0,b]} = 1$$
$$x^{[-m,b]} = x^2/x^{[m+2,b]}, \ m \text{ a positive integer.}$$

The usual central factorial $(b = 1)$ will be denoted by $x^{[m]}$. Note that these factorials are called "Stephensen polynomials" by some authors.

Carlitz and Riordan [1] and Riordan [5, p. 213] studied the connection constants of the sequences $x^{[m]}$ and x^n, that is, the central factorial numbers $t(m, n)$ and $T(m, n)$:

$$x^{[m]} = \sum_{n=0}^{m} t(m, n)x^n, \quad x^m = \sum_{n=0}^{m} T(m, n)x^{[n]};$$

these numbers also appeared in the paper of Comtet [2]. In this paper we discuss some properties of the connection constants of the sequences $x^{[m,g]}$ and $x^{[n,h]}$, $h \neq g$, of generalized central factorials, that is, the numbers $K(m, n, s)$:

$$x^{[m,g]} = \sum_{n=0}^{m} g^m h^{-n} K(m, n, s)x^{[n,h]}, \quad s = h/g.$$

2. EXPANSIONS OF CENTRAL FACTORIALS

The central difference operator with increment a, denoted by δ_a, is defined by

$$\delta_a f(x) = f(x + a/2) - f(x - a/2)$$

Note that

$$\delta_a = E_a^{\frac{1}{2}} - E_a^{-\frac{1}{2}} = E_a^{-\frac{1}{2}}\Delta_a, \tag{2.1}$$

where E_α and Δ_α denote the displacement and difference operators with increment α, respectively. Therefore,

$$\delta_\alpha = \sum_{k=0}^{n} (-1)^k \binom{n}{k} E^{n/2-k} \tag{2.2}$$

When the increment $\alpha = 1$, we write $\delta_1 \equiv \delta$, $E_1 \equiv E$, and $\Delta_1 \equiv \Delta$.

The central factorial of degree m and increment b, denoted by $x^{[m,b]}$, is defined by

$$x^{[m,b]} = x\left(x + \frac{1}{2}mb - b\right)\left(x + \frac{1}{2}mb - 2b\right) \cdots \left(x - \frac{1}{2}mb + b\right).$$

Note that

$$x^{[m,b]} = x\left(x + \frac{1}{2}mb - b\right)_{m-1,b}, \tag{2.3}$$

where

$$(y)_{m,b} = y(y - b)(y - 2b) \cdots (y - mb + b)$$

is the falling factorial of degree m and increment b.

It is not difficult to verify that

$$x^{[m,b]} = \left[x^2 - \left(\frac{1}{2}m - 1\right)^2 b^2\right] x^{[m-2,b]}. \tag{2.4}$$

Using the relation

$$(y)_{-m,b} = \frac{1}{(y + mb)_{m,b}}, \tag{2.5}$$

and, by (2.3), we get

$$x^{[-m,b]} = \frac{x^2}{x^{[m+2,b]}}. \tag{2.6}$$

When the increment $b = 1$, we write

$$x^{[m,1]} \equiv x^{[m]}, \quad (y)_{m,1} \equiv (y)_m.$$

Note also that

$$(bx)^{[m]} = b^m x^{[m,h]}, \quad h = 1/b. \tag{2.7}$$

From formula (2.8) (see Riordan [5, p. 147]),

$$u^\alpha = 1 + \sum_{n=1}^{\infty} \frac{\alpha}{n}\binom{\alpha + \beta n - 1}{n - 1} v^n = \sum_{n=0}^{\infty} \frac{\alpha}{\alpha + \beta n}\binom{\alpha + \beta n}{n} v^n, \quad v = (1 - u)u^{-\beta}, \tag{2.8}$$

with $\alpha = bx$, $\beta = 1/2$, $u = E$, $v = (E - 1)E^{-\frac{1}{2}} = \delta$, we get the symbolic formula

$$E^{bx} = \sum_{n=0}^{\infty} (bx)^{[n]} \frac{1}{n!}\delta^n$$

Since $[E^{bx}(sz)^{[m]}]_{z=0} = (ax)^{[m]}$, $s = a/b$, we obtain

$$(ax)^{[m]} = \sum_{n=0}^{m} \left[\frac{1}{n!}\delta^n (sx)^{[m]}\right]_{x=0} \cdot (bx)^{[n]}.$$

Denoting the number in brackets by

$$K(m, n, s) = \left[\frac{1}{n!}\delta^n (sx)^{[m]}\right]_{x=0}, \tag{2.9}$$

we have

$$(ax)^{[m]} = \sum_{n=0}^{m} K(m, n, s)(bx)^{[n]}, \quad s = a/b. \tag{2.10}$$

Using (2.7), (2.10) may be rewritten in the form

$$x^{[m,g]} = \sum_{n=0}^{m} g^m h^{-n} K(m, n, s) x^{[n,h]}, \quad s = h/g. \tag{2.11}$$

Note also that

$$K(m, n, s) = \left[\frac{1}{n!b^m}\delta_a^n x^{[m,b]}\right]_{x=0}, \quad s = a/b. \tag{2.12}$$

From the definition (2.9), we may deduce an explicit expression for the numbers $K(m, n, s)$. Indeed, from the symbolic formula (2.2) with $a = 1$, and since

$$\left[E^{n/2-k}(sx)^{[m]}\right]_{x=0} = \left(s\left[\frac{1}{2}n - k\right]\right)^{[m]},$$

we get

$$K(m, n, s) = \frac{1}{n!}\sum_{k=0}^{n}(-1)^k\binom{n}{k}\left(s\left[\frac{1}{2}n - k\right]\right)^{[m]}$$

$$= \frac{1}{n!}\sum_{k=0}^{n}(-1)^k\binom{n}{k}\left(\frac{1}{2}sn - sk\right)\left(\frac{1}{2}sn + \frac{1}{2}m - sk - 1\right)_{m-1} \tag{2.13}$$

A recurrence relation for the numbers $K(m, n, s)$, useful for tabulation purposes, may be obtained from (2.10) and (2.4) as follows:

$$(sk)^{[m+2]} = \sum_{n=0}^{m+2} K(m+2, n, s)x^{[n]} = \left(s^2x^2 - \frac{1}{4}m^2\right)\sum_{n=0}^{m} K(m, n, s)x^{[n]}$$

$$= \sum_{n=0}^{m} K(m, n, s)\left[s^2x^{[n+2]} + \frac{1}{4}(s^2n^2 - m^2)x^{[n]}\right].$$

Hence

$$K(m+2, n, s) = \frac{1}{4}(s^2n^2 - m^2)K(m, n, s) + s^2K(m, n-2, s). \tag{2.14}$$

The initial conditions are

$$K(0, 0, s) = 1,\ K(0, n, s) = 0,\ n > 0,\ K(m, 0, s) = 0,\ m > 0.$$

Moreover,

$$K(2m, 2n+1, s) = 0,\ K(2m+1, 2n, s) = 0.$$

From the recurrence relation and the initial conditions, it follows that:

If s is an integer, the numbers

$$s^{-2n}K(2m, 2n, s) \quad \text{and} \quad 4^{m-n}s^{-2n-1}K(2m+1, 2n+1, s)$$

are positive integers and, moreover,

If s is a negative integer, the numbers

$$K(2m, 2n, s) = 0,\ m < n,\ m > n|s|,$$

$$K(2m+1, 2n+1, s) = 0,\ m < n,\ 2m+1 > (2n+1)|s|.$$

Other properties of these numbers will be discussed in the next section.

We now proceed to determine the coefficients $A(n, m, s)$ in the expansion

$$x^{[-m]} = \sum_{n=m}^{\infty} A(n, m, s)(sx)^{[-n]}.$$

Since $x^{[-m+2]} = \left(x^2 - \frac{1}{4}m^2\right)x^{[-m]}$, we get

$$\sum_{n=n-2}^{\infty} A(n, m-2, s)(sx)^{[-n]} = \left(x^2 - \frac{1}{4}m^2\right)\sum_{n=m}^{\infty} A(n, m, s)(sx)^{[-n]}$$

$$= \sum_{n=m}^{\infty} A(n, m, s)\left[s^{-2}(sx)^{[-n+2]} + \frac{1}{4}(s^{-2}n^2 - m^2)(sx)^{[-n]}\right].$$

Hence

$$A(n+2, m, s) = \frac{1}{4}(s^2m^2 - n^2)A(n, m, s) + s^2A(n, m-2, s)$$

with

$$A(0, 0, s) = 1, \; A(0, m, s) = 0, \quad > 0.$$

Comparing this recurrence with (2.14), we conclude that

$$x^{[-m]} = \sum_{n=m}^{\infty} K(n, m, s)(sx)^{[-n]}, \tag{2.15}$$

which may be written in the form

$$(bx)^{[-m]} = \sum_{n=m}^{\infty} K(n, m, s)(ax)^{[-n]} \tag{2.16}$$

or

$$x^{[-m, g]} = \sum_{n=m}^{\infty} g^n h^{-m} K(n, m, s) x^{[-n, h]}, \quad s = h/g. \tag{2.17}$$

3. SOME PROPERTIES OF THE CENTRAL FACTORIAL NUMBERS

Some other properties of the numbers $K(m, n, s)$, defined by (2.9) or, equivalently, by (2.12), will be discussed in this section.

From (2.10) we may easily get the relation

$$\sum_{k=n}^{m} K(m, k, a/b)K(k, n, b/a) = \delta_{mn}, \tag{3.1}$$

where δ_{mn} denotes the Kronecker delta. This relation implies the pairs of inverse relation

$$a_m = \sum_{n=0}^{m} K(m, n, a/b)b_n, \qquad b_m = \sum_{n=0}^{m} K(m, n, b/a)a_n,$$

$$c_n = \sum_{m=n}^{\infty} K(m, n, a/b)d_m, \qquad d_n = \sum_{m=n}^{\infty} K(m, n, b/a)c_m.$$

For the central factorial numbers

$$t(m, n) = \left[\frac{1}{n!}D^n x^m\right]_{x=0} \quad \text{and} \quad T(m, n) = \left[\frac{1}{n!}\delta^n x^m\right]_{x=0}$$

we have (see Riordan [5, p. 213])

$$x^{[m]} = \sum_{n=0}^{m} t(m, n)x^n \tag{3.2}$$

$$x^m = \sum_{n=0}^{m} T(m, n)x^{[n]}. \tag{3.3}$$

Expanding $(sx)^{[m]}$ into powers of x by means of (3.2) and then the powers into central factorials by means of (3.3), we obtain

$$(sx)^{[m]} = \sum_{k=0}^{m} s^k t(m, k)x^k = \sum_{k=0}^{m}\sum_{n=0}^{k} s^k t(m, k)T(k, n)x^{[n]}$$

or

$$(sx)^{[m]} = \sum_{n=0}^{m}\sum_{k=n}^{m} s^k t(m, k)T(k, n)x^{[n]},$$

which, in virtue of (2.10) with $b = 1$, $a = s$, gives

$$K(m, n, s) = \sum_{k=n}^{m} s^k t(m, k)T(k, n); \tag{3.4}$$

similarly, it can be shown that

$$t(m, n) = s^{-n}\sum_{k=n}^{m} K(m, k, s)t(k, n) \tag{3.5}$$

and

$$T(m, n) = s^{-m}\sum_{k=n}^{m} T(m, k)K(k, n, s). \tag{3.6}$$

Since $\lim_{s\to\pm\infty} s^{-m}(sx)^{[m]} = x^m$, we get, from (2.9),

$$\lim_{s\to\pm\infty} s^{-m}K(m, n, s) = \left[\frac{1}{n!}\delta^n x^m\right]_{x=0} = T(m, n). \tag{3.7}$$

From (2.12) with $b = 1$, $a = s$, and noting that $\lim_{s\to 0} s^{-1}\delta_s = D$, we deduce

$$\lim_{s\to 0} s^{-n}K(m, n, s) = \left[\frac{1}{n!}D^n x^m\right]_{x=0} = t(m, n). \tag{3.8}$$

Turning to the generating function, we find, on using (2.13) and (2.8), with

$$\alpha = \frac{1}{2}sn - sk,\ \beta = \frac{1}{2},\ v = y,\ (u - 1)u^{-\frac{1}{2}} = y,$$

that

$$\begin{aligned} g_n(y; s) &= \sum_{m=0}^{\infty} K(m, n, s)\frac{y^m}{m!} \\ &= \frac{1}{n!}\sum_{k=0}^{n}(-1)^k\binom{n}{k}\left[1 + \sum_{m=1}^{\infty}\frac{\frac{1}{2}sn - sk}{m}\binom{\frac{1}{2}sn - sk + \frac{1}{2}m - 1}{m - 1}y^m\right] \\ &= \frac{1}{n!}(u^{s/2} - u^{-s/2}),\quad (u - 1)u^{-1/2} = y. \end{aligned}$$

Putting $u = e^w$ and $s = r$ to avoid mistakes in the hyperbolic formulas, we get

$$g_n(y; r) = \frac{1}{n!}\left[2 \sinh\left(\frac{1}{2}rw\right)\right]^n$$

and

$$y = 2 \sinh\left(\frac{1}{2}w\right).$$

Therefore,

$$\begin{aligned} g_n(y; r) &= \frac{1}{n!}\left[2 \sinh\left\{r \sinh^{-1}\left(\frac{1}{2}y\right)\right\}\right]^n \\ &= \frac{1}{n!}\left[2 \sinh\left\{r \log\left(\frac{1}{2}y + \frac{1}{2}\sqrt{y^2 + 4}\right)\right\}\right]^n. \end{aligned} \tag{3.9}$$

The corresponding generating functions for the Carlitz-Riordan central factorial numbers may be obtained as

$$\sum_{m=0}^{\infty} t(m, n)\frac{y^m}{m!} = \frac{1}{n!}\left[2 \sinh^{-1}\left(\frac{1}{2}y\right)\right]^n \tag{3.10}$$

$$\sum_{m=0}^{\infty} T(m, n)\frac{y^m}{m!} = \frac{1}{n!}\left[2 \sinh\left(\frac{1}{2}y\right)\right]^n. \tag{3.11}$$

Using formulas (3.10), (3.11), and (3.9), and since

$$\delta_a^n = \left[2 \sinh\left(\frac{1}{2}aD\right)\right]^n,\quad a^n D^n = \left[2 \sinh^{-1}\left(\frac{1}{2}\delta_a\right)\right]^n,\quad \delta_a^n = \left[2 \sinh\left\{r \sinh^{-1}\left(\frac{1}{2}\delta_b\right)\right\}\right]^n,$$

we get

$$\delta_a^n = \sum_{m=0}^{\infty} \frac{n!}{m!} T(m, n) a^m D^m, \qquad a^n D^n = \sum_{m=0}^{\infty} \frac{n!}{m!} t(m, n) \delta_a^m,$$

$$\delta_a^n = \sum_{m=0}^{\infty} \frac{n!}{m!} K(m, n, r) \delta_b^m, \quad r = a/b.$$

Finally, let

$$Q_m(z; s) = \sum_{x=0}^{z} (sx)^{[m]}$$

and put

$$Q_{2m}(z; s) = \frac{2z + 1}{2} \sum_{n=0}^{m} \frac{Q_{m,n,s}}{2n + 1} \frac{(z + n)!}{(z - n)!}.$$

Then

$$(sx)^{[2m]} = \sum_{n=0}^{m} Q_{m,n,s} \frac{x(x + n - 1)!}{(x - n)!} = \sum_{n=0}^{m} Q_{m,n,s} x^{[2m]},$$

and by (2.10),

$$Q_{m,n,s} = K(2m, 2n, s).$$

A similar expression may be obtained for $Q_{2m+1}(z; s)$.

REFERENCES

1. L. Carlitz & J. Riordan. "The Divided Central Differences of Zero." *Canad. J. Math.* 15 (1963):94-100.
2. L. Comtet. "Nombres de Stirling generaux et fonctions symetriques." *C. R. Acad. Sc. Paris*, Serie A 275 (1972):747-50.
3. C. Jordan. *Calculus of Finite Differences*. New York: Chelsea, 1960.
4. S. M. Roman & G.-C. Rota. "The Umbral Calculus." *Advances in Mathematics* 27 (1978):95-188.
5. J. Riordan. *Combinatorial Identities*. New York: Wiley, 1968.

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ON THE FIBONACCI NUMBERS MINUS ONE

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Let A be the $n \times n$ matrix with elements defined by

$$a_{ij} = -1 \text{ if } i = j - 1;\ 1 + \mu \text{ if } i = j;\ -\mu \text{ if } i = j + 2;$$

and 0 otherwise. If $n \geq 3$ and μ is a positive number, then A is a special case of a matrix that was shown in [1] to be useful in the design of two-up, one-down ideal cascades for uranium enrichment. The purpose of this paper is to derive certain properties of the determinant D_n of A and to point out its relation to the Fibonacci numbers.

Expansion of the determinant of A according to its first column leads to the recurrence relation

(1) $\quad D_1 = 1 + \mu,\ D_2 = (1 - \mu)^2, \text{ and } D_n = (1 + \mu)D_{n-1} - \mu D_{n-3} \text{ for } n \geq 3.$

For convenience, set $D_0 = 1$.

By using standard techniques for generating functions, it can be shown that the generating function $D(x)$ for $\{D_n\}$ (with positive radius of convergence) is

(2) $$D(x) = [1 - (1 + \mu)x + \mu x^3]^{-1} = \sum_{i=0}^{\infty} \sum_{j=0}^{i} (-1)^j \binom{i}{j} \mu^j (1 + \mu)^{i-j} x^{i+2j}.$$