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SELF-INVERSE SEQUENCES RELATED TO A BINOMIAL INVERSE PAIR

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1. INTRODUCTION

Pairs of inverse relations are very useful in the study of combinatorial identities [2]. One of the classical inversion formulas is the binomial inverse pair:

$$a_n = \sum_{k=0}^{n} (-1)^k \binom{n}{k} b_k; \quad b_n = \sum_{k=0}^{n} (-1)^k \binom{n}{k} a_k. \tag{1}$$

We say that a sequence $\{a_n\}$ of complex numbers is self-inverse or invariant if

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} a_k = a_n \tag{2}$$

and denote by S^+ the set of self-inverse sequences. We also denote by S^- the set of sequences $\{a_n\}$ satisfying

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} a_k = -a_n.$$

By the definition it is not difficult to verify that $\{a_n\} \in S^-$ if and only if $a_0 = 0$ and $\left\{\frac{a_{n+1}}{n+1}\right\} \in S^+$ or $\{na_{n-1}\} \in S^+$.

Recently, Sun [5] studied self-inverse sequences by using their generating functions and gave many interesting examples and results of self-inverse sequences. In this paper, we explore self-inverse sequences by means of linear transformations, difference operators and the umbral calculus. We obtain various characterizations of self-inverse sequences from these different approaches. For S^+ we show that it is a vector space over the complex field and determine its dimension. We also give simpler proofs to certain results of Sun. It is worth noting that our results can give rise to many interesting identities.

2. LINEAR TRANSFORMATIONS

Let

$$P = \left((-1)^k \binom{n}{k}\right) = \begin{pmatrix} 1 & & & & \\ 1 & -1 & & O & \\ 1 & -2 & 1 & & \\ 1 & -3 & 3 & -1 & \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Then the inverse pair (1) can be written as

$$\alpha = P\beta\ ; \qquad \beta = P\alpha,$$

where $\alpha = (a_0, a_1, a_2, \dots)^T$ and $\beta = (b_0, b_1, b_2, \dots)^T$ are two column vectors. Note that the pair in (1) is self-inverse. Hence, P is an involutory matrix, i.e., $P^2 = I$, where I is the identity matrix. In what follows we identify a vector $\alpha = (a_0, a_1, a_2, \dots)^T$ with the sequence $\{a_n\}$. Then $\alpha \in S^+$ if and only if α is invariant under the linear transformation $y = Px$, i.e., $P\alpha = \alpha$. Thus, S^+ is precisely the null space of the matrix $P - I$. Similarly, we may show that S^- is the null space of the matrix $P + I$. The following proposition is therefore immediate.

Proposition 2.1: *Let α, β be two complex vectors.*
(a) *If $\alpha, \beta \in S^{\pm}$, then $a\alpha + b\beta \in S^{\pm}$ for arbitrary complex numbers a and b;*
(b) *if A is a matrix satisfying $AP = PA$, then $\alpha \in S^{\pm}$ implies that $A\alpha \in S^{\pm}$.*

Theorem 2.2: *$\alpha \in S^{\pm}$ if and only if $\alpha = (P \pm I)v$ for some vector v. In other words, $S^{\pm}$ is precisely the column space of the matrix $P \pm I$.*

Proof: Suppose that $\alpha \in S^+$. Then $P\alpha = \alpha$. Putting $v = \alpha/2$, we have

$$(P + I)v = (P + I)\alpha/2 = (P\alpha + \alpha)/2 = (\alpha + \alpha)/2 = \alpha.$$

Conversely, suppose that $\alpha = (P + I)v$ for some v. Then

$$P\alpha = P(P + I)v = (P^2 + P)v = (I + P)v = \alpha.$$

Thus, $\alpha \in S^+$.

The statement for S^- may be proved similarly. □

Remark 2.3: The above discussion for the binomial inverse pair is also suitable for general self-inverse pairs:

$$a_n = \sum_k A(n,k)b_k; \quad b_n = \sum_k A(n.k)a_k,$$

where the matrix $A = (A(n,k))$ satisfies $A^2 = I$.

An equivalent form of Theorem 2.2 is the following theorem, which has been observed by Sun [5, Remark 3.2].

Theorem 2.4: *The sequence $\{a_n\} \in S^{\pm}$ if and only if there exists a sequence $\lambda_0, \lambda_1, \lambda_2, \dots$ of complex numbers such that*

$$a_n = \sum_{k=0}^{n} (-1)^k \binom{n}{k} \lambda_k \pm \lambda_n, \quad n = 0, 1, 2, \dots.$$

Taking $\lambda_n = \lambda^n$ in Theorem 2.4 and noting

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} \lambda^k = (1 - \lambda)^n,$$

we obtain

Corollary 2.5: *Suppose that λ is a complex number. Then $\{(1-\lambda)^n \pm \lambda^n\} \in S^{\pm}$.*

Remark 2.6: Let $\{a_n\}$ is a sequence defined by

$$\begin{cases} a_{n+2} = a_{n+1} + ta_n, \ n \geq 0, \\ a_0 = 2, a_1 = 1. \end{cases}$$

Then, the Binet form of $\{a_n\}$ is $a_n = (1-\lambda)^n + \lambda^n$, where λ is a root of the equation $\lambda^2 - \lambda - t = 0$. Thus, $\{a_n\} \in S^+$. In particular, we have $\{L_n\} \in S^+$, where the L_n are the Lucas numbers defined by $L_0 = 2, L_1 = 1$ and $L_n = L_{n-1} + L_{n-2}$ for $n \geq 2$.

Similarly, if a sequence $\{a_n\}$ satisfies the recursive relation $a_{n+2} = a_{n+1} + ta_n (n \geq 0)$ with $a_0 = 0$, then $\{a_n\} \in S^-$. In particular, we have $\{F_n\} \in S^-$, where the F_n are the Fibonacci numbers defined by $F_0 = 0, F_1 = 1$ and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$.

Now let m be a positive integer and P_m the $m \times m$ matrix $\left((-1)^k \binom{n}{k}\right)$, $n, k = 0, 1, \ldots, m-1$. Denote $S_m^{\pm} = \{\alpha \in \boldsymbol{C}^m : P_m \alpha = \pm\alpha\}$. Then $S_m^{\pm}$ is the column space of the matrices $P_m \pm I$. Moreover, we have the following result.

Theorem 2.7: *With the above notation, we have*
(a) $\boldsymbol{C}^m = S_m^+ \oplus S_m^-$, *and*
(b) $\dim S_m^+ = \lceil m/2 \rceil$ *and* $\dim S_m^- = \lfloor m/2 \rfloor$,
where $\lceil x \rceil$ *and* $\lfloor x \rfloor$ *denote the least integer greater than or equal to* x *and the greatest integer less than or equal to* x *respectively.*

Proof: (a) Any vector $\alpha \in \boldsymbol{C}^m$ can be written in the form

$$\alpha = \beta + \gamma,$$

where $\beta = (P_m + I)\alpha/2$ and $\gamma = -(P_m - I)\alpha/2 \in S_m^-$. By Theorem 2.2, $\beta \in S_m^+$ and $\gamma \in S_m^-$. So $\boldsymbol{C}^m = S_m^+ + S_m^-$. On the other hand, it is clear that $S_m^+ \cap S_m^- = \{0\}$. Thus, we conclude that $\boldsymbol{C}^m = S_m^+ \oplus S_m^-$.

(b) For $k = 0, 1, \ldots, m-1$, let u_k be the kth column of the matrix

$$P_m + I = \begin{pmatrix} 2 & & & & & \\ 1 & 0 & & & & \\ 1 & -2 & 2 & & O & \\ 1 & -3 & 3 & 0 & & \\ \vdots & \vdots & \vdots & \vdots & \ddots & \\ 1 & -\binom{m-1}{1} & \binom{m-1}{2} & -\binom{m-1}{3} & \cdots & (-1)^{m-1}+1 \end{pmatrix}.$$

Then, $u_0, u_2, u_4, \ldots$ are linearly independent since the position of the first non-zero element of these vectors is different. The rank of matrix $P_m + I$ is therefore not less than $\lceil m/2 \rceil$. It follows that $\dim S_m^+ \geq \lceil m/2 \rceil$ from Theorem 2.2. Similarly, we have $\dim S_m^- \geq \lfloor m/2 \rfloor$. However, $\dim S_m^+ + \dim S_m^- = m$ by (a). Hence, $\dim S_m^+ = \lceil m/2 \rceil$ and $\dim S_m^- = \lfloor m/2 \rfloor$, as claimed. □

Remark 2.8: From the proof of Theorem 2.7(b) we see that $u_0, u_2, u_4, \ldots$ form a basis of the vector space S_m^+. Similarly we may determine a basis of the vector space S_m^-.

3. DIFFERENCE OPERATORS

Given a function $f : \boldsymbol{Z} \to \boldsymbol{C}$, define a new function $\Delta f : \boldsymbol{Z} \to \boldsymbol{C}$ by

$$\Delta f(n) = f(n+1) - f(n).$$

Δ is called the forward difference operator. For $k \geq 1$, we may iterate Δ k times to obtain the kth forward difference operator,

$$\Delta^k f = \Delta(\Delta^{k-1} f),$$

where Δ^0 is the identity operator. It is well-known (see, e.g., [4, p.37]) that

$$\Delta^n f(0) = \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} f(k).$$

Thus, Theorem 2.4 can be restated as follows.

Theorem 3.1: *The sequence $\{a_n\} \in S^+$ if and only if there exists a function $f : \boldsymbol{Z} \to \boldsymbol{C}$ such that*

$$a_n = (-1)^n \Delta^n f(0) + f(n), \quad n = 0, 1, 2, \ldots .$$

It is also well-known that if f is a polynomial of degree d, then $\Delta^k f(0) = 0$ for $k > d$ (see, e.g., [4, Proposition 1.4.2(a)]). Hence, we have

Theorem 3.2: *If f is a polynomial of degree d, then there exists an self-inverse sequence $\{a_n\}$ such that $a_n = f(n)$ when $n > d$.*

Example 3.3: Let d be a positive integer.

1. Suppose that $f(n) = n^d$. Then $\Delta^n f(0) = n!S(d.n)$ where $S(d, n)$ are the Stirling numbers of the second kind (see, e.g., [4, Proposition 1.4.2(c)]). Thus, we obtain a self-inverse sequence $a_n = (-1)^n n! S(d, n) + n^d$ satisfying $a_n = n^d$ for $n > d$.

2. Suppose that $f(n) = \binom{n}{d}$. Then $\Delta^n f(0) = \delta_{nd}$ by induction, where $\delta_{nd} = 1$ if $n = d$ and $\delta_{nd} = 0$ otherwise. Thus, we obtain a self-inverse sequence $a_n = (-1)^n \delta_{nd} + \binom{n}{d}$ satisfying $a_n = \binom{n}{d}$ for $n > d$.

Theorem 3.4: *Suppose that $\{a_n\} \in S^+$. Then $\{\Delta^k a_{n+k}\} \in S^+$ for $k \geq 0$.*

Proof: We use induction on k. The case $k = 0$ is trivial. Now assume that $\Delta^k a_{n+k} \in S^+$. Then there exists a function f such that

$$\Delta^k a_{n+k} = (-1)^n \Delta^n f(0) + f(n), \quad n = 0, 1, 2, \ldots .$$

It follows that

$$\begin{aligned}
\Delta^{k+1} a_{n+(k+1)} &= \Delta(\Delta^k a_{(n+1)+k}) \\
&= \Delta[(-1)^{n+1} \Delta^{n+1} f(0) + f(n+1)] \\
&= (-1)^{n+2} \Delta^{n+2} f(0) - (-1)^{n+1} \Delta^{n+1} f(0) + \Delta f(n+1) \\
&= (-1)^n \Delta^{n+1} f(1) + \Delta f(n+1) \\
&= (-1)^n \Delta^n F(0) + F(n),
\end{aligned}$$

where $F(n) = \Delta f(n+1)$. Hence $\{\Delta^{k+1} a_{n+(k+1)}\} \in S^+$ and the proof is complete by induction. □

In the case $k = 1$, Theorem 3.4 states that $\{a_n\} \in S^+$ yields $\{a_{n+2} - a_{n+1}\} \in S^+$, which is precisely the result of Corollary 3.1(c) in [5].

Remark 3.5: The discussion for S^+ in this section is also suitable for S^-.

4. UMBRAL CALCULUS

Many results in combinatorics are often easily read, verified and expanded by means of the umbral method. In this section we apply the umbral method to explore self-inverse sequences. But here, we do not want to describe the umbral method (for a rigorous description, see [1] or [3]). The basic idea of the umbral method relies on the use of a notation where certain exponents can be interchanged with suffixes. For example, (2) can be written symbolically as

$$(1-a)^n \equiv a^n,$$

with the understanding that the expression on the left be expanded in powers of a, and then each term a^k be replaced by a_k. The symbol a is referred to as an "umbra", and the symbol $\equiv$ is used to denote symbolic or umbral equivalences, in which we put $a^k \equiv a_k$. Thus, a sequence $\{a_n\} \in S^+$ if and only if $(1-a)^n \equiv a^n$ for all n. Similarly, $\{a_n\} \in S^-$ if and only if $(1-a)^n \equiv -a^n$ for all n.

Proposition 4.1: *Let $\{a_n\}$ and $\{b_n\}$ be two sequences of complex numbers.*
(a) *Suppose that both $\{a_n\}$ and $\{b_n\}$ are in S^+ or in S^-. Then,*

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}a_{n-k}b_k = 0$$

holds for odd n.
(b) *Suppose that $\{a_n\} \in S^-$ and $\{b_n\} \in S^+$. Then, (3) holds for even n.*

Proof: Suppose that both $\{a_n\}$ and $\{b_n\}$ are self-inverse. Then,

$$a^n \equiv (1-a)^n, \quad b^n \equiv (1-b)^n.$$

Thus,

$$(a-b)^n \equiv [(1-a)-(1-b)]^n \equiv (b-a)^n.$$

When n is odd, we have $(a-b)^n \equiv 0$. Expanding the binomial expression yields (3).

Other results may be obtained similarly. □

Corollary 4.2: *Suppose that $\{a_n\} \in S^{\pm}$. Then for odd n,*

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}a_k a_{n-k} = 0.$$

It is easy to see that the sequence $\{1/2^n\}$ is self-inverse. By Proposition 4.1, if $\{a_n\} \in S^+$, then

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}2^k a_{n-k} = 0 \tag{4}$$

holds for odd n, and if $\{a_n\} \in S^-$, then (4) holds for even n. The converse is also true.

Theorem 4.3: *Let $\{a_n\}$ be a sequence of complex numbers.*
(a) *$\{a_n\} \in S^+$ if and only if (4) holds for odd n.*
(b) *$\{a_n\} \in S^-$ if and only if (4) holds for even n.*

Proof: (a) It suffices to show sufficiency. Suppose that (4) holds for odd n. Then $(a-1/2)^n \equiv 0$ for odd n. Thus, $(a-1/2)^n \equiv (1/2-a)^n$ for all n. It follows that

$$(1-a)^n \equiv [1/2+(1/2-a)]^n \equiv [1/2+(a-1/2)]^n \equiv a^n.$$

Hence, $\{a_n\} \in S^+$.

(b) The proof is similar to that of part (a). □

Finally, we apply the umbral method to give an elegant proof of the following proposition due to Sun [5, Theorem 4.1].

Proposition 4.4: *Suppose that* $\{f_n\}, \{a_n\}$ *are two sequences of complex numbers.*

(a) *If* $\{a_n\} \in S^+$, *then*

$$\sum_{k=0}^{n} \binom{n}{k} \left(f_k - (-1)^{n-k} \sum_{s=0}^{k} \binom{k}{s} f_s \right) a_{n-k} = 0. \tag{5}$$

(b) *If* $\{a_n\} \in S^-$, *then*

$$\sum_{k=0}^{n} \binom{n}{k} \left(f_k + (-1)^{n-k} \sum_{s=0}^{k} \binom{k}{s} f_s \right) a_{n-k} = 0.$$

Proof: We only prove (a). The proof of (b) is similar. Suppose that $\{a_n\} \in S^+$. Then,

$$(f+a)^n \equiv [f+(1-a)]^n \equiv [(f+1)-a]^n.$$

Expanding and noting that $(f+1)^k \equiv \sum_{s=0}^{k} \binom{k}{s} f_s$, we obtain

$$\sum_{k=0}^{n} \binom{n}{k} f_k a_{n-k} = \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} \sum_{s=0}^{k} \binom{k}{s} f_s a_{n-k},$$

which yields (5). □

Remark 4.5: We have seen that the Lucas sequence $\{L_n\} \in S^+$ and the Fibonacci sequence $\{F_n\} \in S^-$. It is also easy to see that $\{(-1)^n B_n\} \in S^+$ where B_n are the Bernoulli numbers defined by $B_0 = 1, B_1 = -1/2$ and $\sum_{k=0}^{n} \binom{n}{k} B_k = B_n$ for $n > 1$. So, the results of this section can produce many identities about these well-known numbers.

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