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COMBINATORIAL SUMS AND SERIES INVOLVING INVERSES OF BINOMIAL COEFFICIENTS

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0. INTRODUCTION

In this note we deal with several combinatorial sums and series involving inverses of binomial coefficients. Some of them have already been considered by other authors (see, e.g., [3], [4]), but it should be noted that our approach is different. It is based on Euler's well-known Beta function defined by

$$B(m,n)=\int_0^1 t^{m-1}(1-t)^{n-1}dt$$

for all positive integers m and n. Since

$$B(m,n)=\frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}=\frac{(m-1)!(n-1)!}{(m+n-1)!},$$

we get

$$\binom{n}{k}^{-1}=(n+1)\int_0^1 t^k(1-t)^{n-k}dt \tag{1}$$

for all nonnegative integers n and k with $n\geq k$.

1. SUMS INVOLVING INVERSES OF BINOMIAL COEFFICIENTS

Theorem 1.1 ([4], Theorem 1): If n is a nonnegative integer, then

$$\sum_{k=0}^{n}\binom{n}{k}^{-1}=\frac{n+1}{2^{n+1}}\sum_{k=1}^{n+1}\frac{2^k}{k}.$$

Proof: Let S_n be the sum of inverses of binomial coefficients. From (1) we get

$$\begin{aligned}S_n&=\sum_{k=0}^{n}(n+1)\int_0^1 t^k(1-t)^{n-k}dt\\&=(n+1)\int_0^1\left\{(1-t)^n\sum_{k=0}^{n}\left(\frac{t}{1-t}\right)^k\right\}dt=(n+1)\int_0^1\frac{(1-t)^{n+1}-t^{n+1}}{1-2t}dt.\end{aligned}$$

Making the substitution $1-2t=x$, we obtain

$$\begin{aligned}S_n&=\frac{n+1}{2^{n+2}}\int_{-1}^{1}\frac{(1+x)^{n+1}-(1-x)^{n+1}}{x}dx=\frac{n+1}{2^{n+2}}\left\{\int_{-1}^{1}\frac{(1+x)^{n+1}-1}{x}dx+\int_{-1}^{1}\frac{1-(1-x)^{n+1}}{x}dx\right\}\\&=\frac{n+1}{2^{n+2}}\sum_{k=0}^{n}\left\{\int_{-1}^{1}(1+x)^k dx+\int_{-1}^{1}(1-x)^k dx\right\}=\frac{n+1}{2^{n+1}}\sum_{k=1}^{n+1}\frac{2^k}{k}.\end{aligned}$$

Theorem 1.2: If n is a positive integer, then

$$\sum_{k=0}^{2n}(-1)^k\binom{2n}{k}\binom{4n}{2k}^{-1}=\frac{4n+1}{2n+1}.$$

Proof: Formula (1) yields

$$\begin{aligned}\sum_{k=0}^{2n}(-1)^k\binom{2n}{k}\binom{4n}{2k}^{-1}&=\sum_{k=0}^{2n}(-1)^k\binom{2n}{k}(4n+1)\int_0^1 t^{2k}(1-t)^{4n-2k}dt\\&=(4n+1)\int_0^1\left\{(1-t)^{4n}\sum_{k=0}^{2n}\binom{2n}{k}\left(\frac{-t^2}{(1-t)^2}\right)^k\right\}dt\\&=(4n+1)\int_0^1(1-t)^{4n}\left(1-\frac{t^2}{(1-t)^2}\right)^{2n}dt\\&=(4n+1)\int_0^1(1-2t)^{2n}dt=\frac{4n+1}{2n+1}.\end{aligned}$$

Theorem 1.3 ([5]): If n is a positive integer, then

$$\sum_{k=0}^{2n}(-1)^k\binom{4n}{2k}\binom{2n}{k}^{-1}=-\frac{1}{2n-1}.$$

Proof: Let S_n be the sum to evaluate. From (1) we get

$$\begin{aligned}S_n&=\sum_{k=0}^{2n}(-1)^k\binom{4n}{2k}(2n+1)\int_0^1 t^k(1-t)^{2n-k}dt\\&=(2n+1)\int_0^1\left\{\sum_{k=0}^{2n}\binom{4n}{2k}(-1)^k t^k(1-t)^{2n-k}\right\}dt\\&=\frac{2n+1}{2}\int_0^1\left\{\left(\sqrt{1-t}+i\sqrt{t}\right)^{4n}+\left(\sqrt{1-t}-i\sqrt{t}\right)^{4n}\right\}dt.\end{aligned}$$

Since

$$\sqrt{1-t}\pm i\sqrt{t}=\cos\left(\arctan\sqrt{\frac{t}{1-t}}\right)\pm i\sin\left(\arctan\sqrt{\frac{t}{1-t}}\right),$$

it follows that

$$S_n=(2n+1)\int_0^1\cos\left(4n\arctan\sqrt{\frac{t}{1-t}}\right)dt.$$

Making the substitution $\arctan\sqrt{\frac{t}{1-t}}=x$, we obtain

$$\begin{aligned}S_n&=(2n+1)\int_0^{\pi/2}\cos(4nx)\sin(2x)dx\\&=\frac{2n+1}{2}\int_0^{\pi/2}\{\sin(4n+2)x-\sin(4n-2)x\}dx=-\frac{1}{2n-1}.\end{aligned}$$

Theorem 1.4 ([2]): If m, n, and p are nonnegative integers with $p\le n$, then

$$\sum_{k=0}^{m}\binom{m}{k}\binom{n+m}{p+k}^{-1}=\frac{n+m+1}{n+1}\binom{n}{p}^{-1}.$$

Proof: Formula (1) yields

$$\sum_{k=0}^{m}\binom{m}{k}\binom{n+m}{p+k}^{-1}=\sum_{k=0}^{m}\binom{m}{k}(n+m+1)\int_0^1 t^{p+k}(1-t)^{n+m-p-k}dt$$
$$=(n+m+1)\int_0^1\left\{t^p(1-t)^{n+m-p}\sum_{k=0}^{m}\binom{m}{k}\left(\frac{t}{1-t}\right)^k\right\}dt$$
$$=(n+m+1)\int_0^1 t^p(1-t)^{n+m-p}\left(1+\frac{t}{1-t}\right)^m dt$$
$$=(n+m+1)\int_0^1 t^p(1-t)^{n-p}dt=\frac{n+m+1}{n+1}\binom{n}{p}^{-1}.$$

Remark: In the special case $p=n$, from the above theorem we get

$$\sum_{k=0}^{m}\binom{m}{k}\binom{n+m}{n+k}^{-1}=\frac{n+m+1}{n+1}.$$

Theorem 1.5: If m and n are nonnegative integers, then

$$\sum_{k=0}^{n}(-1)^k\binom{m+n}{m+k}^{-1}=\frac{m+n+1}{m+n+2}\left(\binom{m+n+1}{m}^{-1}+(-1)^n\right).$$

Proof: We have

$$\sum_{k=0}^{n}(-1)^k\binom{m+n}{m+k}^{-1}=\sum_{k=0}^{n}(-1)^k(m+n+1)\int_0^1 t^{m+k}(1-t)^{n-k}dt$$
$$=(m+n+1)\int_0^1\left\{t^m(1-t)^n\sum_{k=0}^{n}\left(\frac{-t}{1-t}\right)^k\right\}dt$$
$$=(m+n+1)\left(\int_0^1 t^m(1-t)^{n+1}dt+(-1)^n\int_0^1 t^{m+n+1}dt\right)$$
$$=\frac{m+n+1}{m+n+2}\left(\binom{m+n+1}{m}^{-1}+(-1)^n\right).$$

Remark: In the special case $m=n$ we get

$$\sum_{k=0}^{n}(-1)^k\binom{2n}{n+k}^{-1}=\frac{2n+1}{2n+2}\left(\binom{2n+1}{n}^{-1}+(-1)^n\right),$$

while in the special case $m=0$ we obtain

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}^{-1}=\frac{n+1}{n+2}(1+(-1)^n).$$

Consequently (see [3], p. 343),

$$\sum_{k=0}^{2n}(-1)^k\binom{2n}{k}^{-1}=\frac{2n+1}{n+1}.$$

Theorem 1.6: If n is a positive integer, then

$$\frac{1}{3n+1}\sum_{k=0}^{n}\binom{3n}{3k}^{-1}-\frac{1}{3n+2}\sum_{k=0}^{n}\binom{3n+1}{3k+1}^{-1}+\frac{1}{3n+3}\sum_{k=0}^{n}\binom{3n+2}{3k+2}^{-1}$$
$$=\frac{1}{3n+3}\sum_{k=0}^{3n+2}\binom{3n+2}{k}^{-1}.$$

Proof: We have

$$\frac{1}{3n+1}\sum_{k=0}^{n}\binom{3n}{3k}^{-1}-\frac{1}{3n+2}\sum_{k=0}^{n}\binom{3n+1}{3k+1}^{-1}+\frac{1}{3n+3}\sum_{k=0}^{n}\binom{3n+2}{3k+2}^{-1}$$
$$=\sum_{k=0}^{n}\int_0^1\{t^{3k}(1-t)^{3n-3k}-t^{3k+1}(1-t)^{3n-3k}+t^{3k+2}(1-t)^{3n-3k}\}\,dt$$
$$=\int_0^1\left\{(1-t)^{3n}(1-t+t^2)\sum_{k=0}^{n}\left(\frac{t^3}{(1-t)^3}\right)^k\right\}dt=\int_0^1\frac{(1-t)^{3n+3}-t^{3n+3}}{1-2t}\,dt.$$

On the other hand,

$$\frac{1}{3n+3}\sum_{k=0}^{3n+2}\binom{3n+2}{k}^{-1}=\sum_{k=0}^{3n+2}\int_0^1 t^k(1-t)^{3n+2-k}\,dt$$
$$=\int_0^1\left\{(1-t)^{3n+2}\sum_{k=0}^{3n+2}\left(\frac{t}{1-t}\right)^k\right\}dt=\int_0^1\frac{(1-t)^{3n+3}-t^{3n+3}}{1-2t}\,dt,$$

completing the proof.

2. SERIES INVOLVING INVERSES OF BINOMIAL COEFFICIENTS

Theorem 2.1: If m and n are positive integers with $m>n$, then

$$\sum_{k=0}^{\infty}\binom{mk}{nk}^{-1}=\int_0^1\frac{1+(m-1)t^n(1-t)^{m-n}}{(1-t^n(1-t)^{m-n})^2}\,dt.$$

Proof: From (1) we get

$$\sum_{k=0}^{\infty}\binom{mk}{nk}^{-1}=\sum_{k=0}^{\infty}(mk+1)\int_0^1 t^{nk}(1-t)^{(m-n)k}\,dt$$
$$=m\sum_{k=1}^{\infty}\int_0^1 k(t^n(1-t)^{m-n})^k\,dt+\sum_{k=0}^{\infty}\int_0^1(t^n(1-t)^{m-n})^k\,dt.$$

Let $f:[0,1]\to\mathbf{R}$ be the function defined by $f(t)=t^n(1-t)^{m-n}$. It is immediately seen that f attains its maximum at the point $t_0=n/m$. Since $f(t_0)<1$, it follows that

$$\sum_{k=1}^{\infty}k(t^n(1-t)^{m-n})^k=\frac{t^n(1-t)^{m-n}}{(1-t^n(1-t)^{m-n})^2}$$

and

$$\sum_{k=1}^{\infty}(t^n(1-t)^{m-n})^k=\frac{1}{1-t^n(1-t)^{m-n}}$$

uniformly on [0, 1]. Therefore, we obtain

$$\sum_{k=0}^{\infty}\binom{mk}{nk}^{-1}=m\int_0^1\frac{t^n(1-t)^{m-n}}{(1-t^n(1-t)^{m-n})^2}dt+\int_0^1\frac{dt}{1-t^n(1-t)^{m-n}},$$

completing the proof.

Remark: As special cases of Theorem 2.1 we get

$$\sum_{k=0}^{\infty}\binom{2k}{k}^{-1}=\frac{4}{3}+\frac{2\pi\sqrt{3}}{27}, \tag{2}$$

$$\sum_{k=0}^{\infty}\binom{4k}{2k}^{-1}=\frac{16}{15}+\frac{\pi\sqrt{3}}{27}-\frac{2\sqrt{5}}{25}\ln\frac{1+\sqrt{5}}{2}. \tag{3}$$

Indeed, according to Theorem 2.1, we have

$$\sum_{k=0}^{\infty}\binom{2k}{k}^{-1}=\int_0^1\frac{1+t-t^2}{(1-t+t^2)^2}dt=2\int_0^1\frac{dt}{(1-t+t^2)^2}-\int_0^1\frac{dt}{1-t+t^2} \tag{4}$$

and

$$\sum_{k=0}^{\infty}\binom{4k}{2k}^{-1}=\int_0^1\frac{1+3t^2(1-t)^2}{(1-t^2(1-t)^2)^2}dt,$$

respectively. Since

$$\frac{1+3x^2}{(1-x^2)^2}=\frac{1+x}{2(1-x)^2}+\frac{1-x}{2(1+x)^2},$$

we obtain

$$\begin{aligned}\sum_{k=0}^{\infty}\binom{4k}{2k}^{-1}&=\int_0^1\frac{1+t-t^2}{2(1-t+t^2)^2}dt+\int_0^1\frac{1-t+t^2}{2(1+t-t^2)^2}dt\\&=\int_0^1\frac{dt}{(1-t+t^2)^2}-\frac{1}{2}\int_0^1\frac{dt}{1-t+t^2}+\int_0^1\frac{dt}{(1+t-t^2)^2}-\frac{1}{2}\int_0^1\frac{dt}{1+t-t^2}.\end{aligned} \tag{5}$$

Taking into account that

$$\int_0^1\frac{dt}{1-t+t^2}=\frac{2\pi\sqrt{3}}{9}\quad\text{and}\quad\int_0^1\frac{dt}{1+t-t^2}=\frac{4\sqrt{5}}{5}\ln\frac{1+\sqrt{5}}{2},$$

and that (see, e.g., [1])

$$\int\frac{dt}{(a+bt+ct^2)^2}=\frac{b+2ct}{(4ac-b^2)(a+bt+ct^2)}+\frac{2c}{4ac-b^2}\int\frac{dt}{a+bt+ct^2},$$

from (4) and (5) one can easily obtain (2) and (3).

Theorem 2.2 ([4], Theorem 2): If $n\geq 2$ is an integer, then

$$\sum_{k=0}^{\infty}\binom{n+k}{k}^{-1}=\frac{n}{n-1}.$$

Proof: For each positive integer p, we have

$$\begin{aligned} s_p &:= \sum_{k=0}^{p} \binom{n+k}{k}^{-1} = \sum_{k=0}^{p} (n+k+1)\int_0^1 t^k (1-t)^n dt \\ &= \int_0^1 \left\{ (n+1)(1-t)^n \sum_{k=0}^{p} t^k + (1-t)^n \sum_{k=0}^{p} kt^k \right\} dt \\ &= (n+1)\int_0^1 (1-t)^n dt - (n+1)\int_0^1 t^{p+1}(1-t)^{n-1} dt + \int_0^1 t(1-t)^{n-2} dt \\ &\qquad - (p+1)\int_0^1 t^{p+1}(1-t)^{n-2} dt + p\int_0^1 t^{p+2}(1-t)^{n-2} dt. \end{aligned}$$

Formula (1) yields

$$s_p = \frac{n}{n-1} - (n-2)!\frac{(np+p+1)(p+1)!}{(p+n+1)!} - (n+1)(n-1)!\frac{(p+1)!}{(p+n+1)!}.$$

Taking into account that $n \geq 2$, we conclude that $s_p \to \frac{n}{n-1}$ when $p \to \infty$.

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