

A few remarks on Euler and Bernoulli polynomials and their connections with binomial coefficients and modified Pascal matrices

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Abstract: We prove certain identities involving Euler and Bernoulli polynomials that can be treated as recurrences. We use these and also other known identities to indicate strong connection between Euler and Bernoulli numbers and entries of inverses of certain lower triangular matrices built of binomial coefficients. In other words we interpret Euler and Bernoulli numbers in terms of modified Pascal matrices.

Key Words: Euler polynomials, Bernoulli polynomials, Binomial Coefficients, Pascal matrices

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1. INTRODUCTION AND NOTATION

The aim of the paper is to indicate close relationship between Euler and Bernoulli polynomials and certain lower triangular matrices with entries depending on binomial coefficients and some other natural numbers. In this way we point out new interpretation of Euler and Bernoulli numbers.

In the series of papers [2], [3], [5], [4] Zhang, Kim and their associates have studied various generalizations of Pascal matrices and examined their properties. The results of this paper can be interpreted as the next step in studying properties of various modifications of Pascal matrices.

We do so by studying known and indicating new identities involving Euler and Bernoulli polynomials. One of them particularly simple involves these polynomials of either only odd or only even degrees.

More precisely we will express these numbers as entries of inverses of certain matrices build of almost entirely of binomial coefficients.

Throughout the paper we will use the following notation. Let a sequences of lower triangular matrices $\{A_n\}_{n \geq 0}$ be such that A_n is $(n+1) \times (n+1)$ matrix and matrix say A_k is a submatrix of every A_n , for $n \geq k$. Notice that the same property can be attributed to inverses of matrices A_n (of course if they exist) and to products of such matrices. Hence to simplify notation we will denote entire sequence of such matrices by one symbol. Thus e.g. sequence $\{A_n\}_{n \geq 0}$ will be denoted by \mathcal{A} or $[a_{ij}]$ if a_{ij} denotes (i, j) -th entry of matrices A_n , $n \geq i$. The sequence $\{A_n^{-1}\}_{n \geq 0}$ will be denoted by \mathcal{A}^{-1} or $[a_{ij}]^{-1}$. Analogously if we have two sequences say \mathcal{A} and \mathcal{B} then \mathcal{AB} would mean sequence $\{A_n B_n\}_{n \geq 0}$. It is easy to notice that all such lower

triangular matrices form a non-commutative ring with unity. Moreover this ring is also a linear space over reals as far as ring's addition is concerned. Diagonal matrix \mathcal{I} with 1 on the diagonal is this ring's unity.

Let us consider also $(n + 1)$ vectors $\mathbf{X}^{(n)} \stackrel{df}{=} (1, x, \dots, x^n)^T$, $\mathbf{f}(\mathbf{X})^{(n)} \stackrel{df}{=} (1, f(x), \dots, f(x)^n)$, By \mathbf{X} or by $[x^i]$ we will mean sequence of vectors $(\mathbf{X}^{(n)})_{n \geq 0}$ and by $\mathbf{f}(\mathbf{X})$ or by $[f(x)^i]$ the sequence of vectors $(\mathbf{f}(\mathbf{X})^{(n)})_{n \geq 0}$.

Let $\bar{E}_n(x)$ denote the n -th Euler polynomial and $B_n(x)$ the n -th Bernoulli polynomial. Let us introduce sequences of vectors $\mathbf{E}^{(n)}(x) = (1, 2E_1(x), \dots, 2^n E_n(x))^T$ and $\mathbf{B}^{(n)}(x) = (1, 2B_1(x), \dots, 2^n B_n(x))^T$. These sequences will be denoted briefly \mathbf{E} and \mathbf{B} respectively.

$[x]$ will denote so called 'floor' function of x that is the largest integer not exceeding x .

Since we will use in the sequel often Euler and Bernoulli polynomials we will recall now briefly the definition of these polynomials. Their characteristic functions are given e.g. by the formulae (23.1.1) of [7] and respectively:

$$\sum_{j \geq 0} \frac{t^j}{j!} E_j(x) = \frac{2 \exp(xt)}{\exp(t) + 1}, \quad (1)$$

$$\sum_{j \geq 0} \frac{t^j}{j!} B_j(x) = \frac{t \exp(xt)}{\exp(t) - 1}. \quad (2)$$

Numbers $E_n = 2^n E_n(1/2)$ and $B_n = B_n(0)$ are called respectively Euler and Bernoulli numbers.

By standard manipulation on characteristic functions we obtain for example the following identities some of which are well known $\forall k \geq 0$:

$$2^k E_k(x) = \sum_{j=0}^k \binom{k}{j} E_{k-j} \times (2x - 1)^j, \quad (3)$$

$$B_k(x) = \sum_{j=0}^k \binom{k}{j} B_{k-j} \times x^j, \quad x^k = \sum_{j=0}^k \binom{k}{j} \frac{1}{k-j+1} B_j(x), \quad (4)$$

$$E_k(x) = \sum_{j=0}^k \binom{k}{j} 2^j B_j(x) \frac{(1-x)^{k-j+1} - (-x)^{k-j+1}}{(n-j+1)}, \quad (5)$$

$$E_k(x) = \sum_{j=0}^k \binom{k}{j} 2^j B_j\left(\frac{x}{2}\right) \frac{1}{k-j+1}, \quad (6)$$

$$B_k(x) = \sum_{j=0}^k \binom{k}{j} 2^j B_j(x) \frac{(1-x)^{k-j} + (-x)^{k-j}}{2}, \quad (7)$$

$$B_k(x) = 2^k B_k\left(\frac{x}{2}\right) + \sum_{j=1}^k \binom{k}{j} 2^{k-j-1} B_{k-j}\left(\frac{x}{2}\right), \quad (8)$$

which are obtained almost directly from the following trivial identities respectively:

$$\begin{aligned} \frac{2 \exp(xt)}{\exp(t) + 1} &= \frac{1}{\cosh(t/2)} \times \exp\left(\frac{t}{2}(2x - 1)\right), \\ \frac{t \exp(xt)}{\exp(t) - 1} &= \frac{t}{\exp(t) - 1} \times \exp(xt), \quad \exp(xt) = \frac{t \exp(xt)}{\exp(t) - 1} \times \frac{\exp(t) - 1}{t}, \\ \frac{2 \exp(xt)}{\exp(t) + 1} &= \frac{2t \exp(2xt)}{\exp(2t) - 1} \times (\exp((1-x)t) - \exp(-xt))/t, \\ \frac{2 \exp(xt)}{\exp(t) + 1} &= \frac{2t \exp(\frac{x}{2}(2t))}{\exp(2t) - 1} \times (\exp(t) - 1)/t, \\ \frac{t \exp(xt)}{\exp(t) - 1} &= \frac{2t \exp(2xt)}{\exp(2t) - 1} \times (\exp((1-x)t) + \exp(-xt))/2, \\ \frac{t \exp(xt)}{\exp(t) - 1} &= \frac{2t \exp(\frac{x}{2}(2t))}{\exp(2t) - 1} \times (\exp(t) + 1)/2. \end{aligned}$$

By direct calculation one can easily check that:

$$\left[\binom{i}{j}\right]^{-1} = [(-1)^{i-j} \binom{i}{j}], \quad [\lambda^{i-j} \binom{i}{j}]^{-1} = [(-\lambda)^{i-j} \binom{i}{j}], \quad (9)$$

for any λ . Identities (9) are well known. They expose properties of Pascal matrices discussed in [2]. Similarly by direct application of (4) we have:

$$\left[\binom{i}{j} \frac{1}{i-j+1}\right]^{-1} = \left[\binom{i}{j} B_{i-j}\right], \quad (10)$$

giving new interpretation of Bernoulli numbers. Now notice that we can multiply both sides of (4) by say λ^k and define new vectors $[(\lambda x)^i]$ and $[\lambda^i B_i(x)]$. Thus (10) can be trivially generalized to

$$\left[\binom{i}{j} \frac{\lambda^{i-j}}{i-j+1}\right]^{-1} = \left[\binom{i}{j} \lambda^{i-j} B_{i-j}\right],$$

for all $\lambda \in \mathbb{R}$, presenting first of the series of modifications of Pascal matrices and their properties that we will present in the sequel.

To find inverses of other matrices built of binomial coefficients we will have to refer to the results of the next section.

2. MAIN RESULTS

THEOREM 1. $\forall n \geq 1$:

$$\sum_{j=0}^{\lfloor n/2 \rfloor} \binom{n}{2 \lfloor n/2 \rfloor - 2j} 2^{2j+n-2 \lfloor n/2 \rfloor} E_{2j+n-2 \lfloor n/2 \rfloor}(x) = (2x-1)^n, \quad (11)$$

$$\sum_{j=0}^{\lfloor n/2 \rfloor} \binom{n}{2 \lfloor n/2 \rfloor - 2j} 2^{2j+n-2 \lfloor n/2 \rfloor} \frac{B_{2j+n-2 \lfloor n/2 \rfloor}(x)}{2 \lfloor n/2 \rfloor - 2j + 1} = (2x-1)^n. \quad (12)$$

Proof. We start with the following identities:

$$\begin{aligned} \cosh(t/2) \frac{2 \exp(xt)}{\exp(t) + 1} &= \exp(t(x - 1/2)), \\ \frac{t \exp(xt)}{\exp(t) - 1} \frac{2 \sinh(t/2)}{t} &= \exp(t(x - 1/2)). \end{aligned}$$

Recall that we also have:

$$\cosh(t/2) = \sum_{j \geq 0} \frac{t^{2j}}{2^{2j} (2j)!}, \quad \frac{2 \sinh(t/2)}{t} = \sum_{j \geq 0} \frac{t^{2j}}{2^{2j} (2j)! (2j + 1)}.$$

So applying the standard Cauchy multiplication of two series we get respectively:

$$\sum_{n \geq 0} \frac{t^n}{n! 2^n} (2x - 1)^n = \sum_{j \geq 0} \frac{t^{2j}}{2^{2j} (2j)!} \sum_{j \geq 0} \frac{t^j}{j!} E_j(x) \quad (13)$$

$$= \sum_{n \geq 0} \frac{t^n}{n!} \sum_{j=0}^n \binom{n}{j} c_j E_{n-j}(x), \quad (14)$$

$$\sum_{n \geq 0} \frac{t^n}{n! 2^n} (2x - 1)^n = \sum_{j \geq 0} \frac{t^{2j}}{2^{2j} (2j)! (2j + 1)} \sum_{j \geq 0} \frac{t^j}{j!} B_j(x) \quad (15)$$

$$= \sum_{n \geq 0} \frac{t^n}{n!} \sum_{j=0}^n \binom{n}{j} c'_j B_{n-j}(x), \quad (16)$$

where we denoted by c_n and c'_n the following numbers:

$$c_n = \begin{cases} \frac{1}{2^n} & \text{if } n = 2 \lfloor n/2 \rfloor \\ 0 & \text{if otherwise} \end{cases}, \quad c'_n = \begin{cases} \frac{1}{2^n (n+1)} & \text{if } n = 2 \lfloor n/2 \rfloor \\ 0 & \text{if otherwise} \end{cases}.$$

Making use of the uniqueness of characteristic functions we can equate functions of x standing by t^n . Finally let us multiply both sides so obtained identities by 2^n . We have obtained (11) and (12). ■

We have the following other result:

THEOREM 2. Let $e(i) = \begin{cases} 0 & \text{if } i \text{ is odd,} \\ 1 & \text{if } i \text{ is even,} \end{cases}$ then

$$[e(i-j) \binom{i}{j}]^{-1} = \left[\binom{i}{j} E_{i-j} \right], \quad (17)$$

$$\left[e(i-j) \binom{i}{j} \frac{1}{i-j+1} \right]^{-1} = \left[\binom{i}{j} \sum_{k=0}^{i-j} \binom{i-j}{k} 2^k B_k \right]. \quad (18)$$

Proof. Let us define by $W_n(x) = 2^n E_n((x+1)/2)$ and $V_n(x) = 2^n B_n((x+1)/2)$.

Notice that characteristic function of polynomials W_n and V_n are given by

$$\begin{aligned} \sum_{j \geq 0} \frac{t^j}{j!} W_j(x) &= \sum_{j \geq 0} \frac{(2t)^j}{j!} E_j((x+1)/2) \\ &= \frac{2 \exp(2t(x+1)/2)}{\exp(2t) + 1} = \frac{\exp(tx)}{\cosh(t)}, \\ \sum_{j \geq 0} \frac{t^j}{j!} V_j(x) &= \sum_{j \geq 0} \frac{(2t)^j}{j!} B_j((x+1)/2) \\ &= \frac{\exp(2t(x+1)/2) 2t}{\exp(2t) - 1} = \frac{t \exp(tx)}{\sinh(t)}. \end{aligned}$$

Now recall that $\frac{1}{\cosh(t)}$ is the characteristic function of Euler numbers while $\frac{t}{\sinh t}$ equals to the characteristic function of numbers $\left\{ \sum_{j=0}^n \binom{n}{j} 2^j B_j \right\}_{n \geq 0}$. Hence on one hand we see that

$$\begin{aligned} W_n(x) &= \sum_{j=0}^n \binom{n}{j} x^{n-j} E_j, \\ V_n(x) &= \sum_{j=0}^n \binom{n}{j} x^{n-j} \sum_{k=0}^j \binom{j}{k} 2^k B_k. \end{aligned}$$

On the other substituting x by $(x+1)/2$ in (11) and (12) we see that

$$\begin{aligned} x^n &= \sum_{j=0}^n e(n-j) \binom{n}{j} W_j(x), \\ x^n &= \sum_{j=0}^n e(n-j) \binom{n}{j} \frac{1}{n-j+1} V_j. \end{aligned}$$

By uniqueness of the polynomial expansion we deduce (17) and (18). ■

Remark 1. Notice that $\forall j > 0 : \sum_{k=0}^j \binom{j}{k} 2^k B_k = 2^j B_j(\frac{1}{2})$.

As a corollary, using also well known properties of lower triangular matrices (see e.g. : [6]), we get the following result.

COROLLARY 1.

$$\begin{aligned} \left[\binom{2i}{2j} \right]^{-1} &= \left[\binom{2i}{2j} E_{2(i-j)} \right], \\ \left[\binom{2i}{2j} \frac{1}{2(i-j)+1} \right]^{-1} &= \left[\binom{2i}{2j} \sum_{k=0}^{2i-2j} \binom{2i-2j}{k} 2^k B_k \right]. \end{aligned}$$

As in Section 1. we can multiply both sides of (11) and (12) by λ^n and redefine appropriate vectors and rephrase out results in terms of modified Pascal matrices.

COROLLARY 2. For all $\lambda \in \mathbb{R}$:

$$[e(i-j) \binom{i}{j} \lambda^{i-j}]^{-1} = \left[\binom{i}{j} \lambda^{i-j} E_{i-j} \right], \quad (19)$$

$$[e(i-j) \binom{i}{j} \frac{\lambda^{i-j}}{i-j+1}]^{-1} = \left[\binom{i}{j} \lambda^{i-j} \sum_{k=0}^{i-j} \binom{i-j}{k} 2^k B_k \right], \quad (20)$$

$$\left[\binom{2i}{2j} \lambda^{i-j} \right]^{-1} = \left[\binom{2i}{2j} \lambda^{i-j} E_{2(i-j)} \right], \quad (21)$$

$$\left[\binom{2i}{2j} \frac{\lambda^{i-j}}{2(i-j)+1} \right]^{-1} = \left[\binom{2i}{2j} \lambda^{i-j} \sum_{k=0}^{2i-2j} \binom{2i-2j}{k} 2^k B_k \right]. \quad (22)$$

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