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ON THE SUM $\sum_{k\equiv r \pmod m}\binom nk$ AND RELATED CONGRUENCES

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Abstract. In this paper we study $\left[{n\atop r}\right]_m=\sum_{k\equiv r(\mathrm{mod}\ m)}\binom nk$ where $m>0$, $n\geqslant 0$ and r are integers. We show that $\left[{n\atop r}\right]_m$ $(m>2)$ can be expressed in terms of some linearly recurrent sequences with orders not exceeding $\varphi(m)/2$. In particular we determine $\left[{n\atop r}\right]_{12}$ explicitly in terms of first order and second order recurrences. It follows that for any prime $p>3$ we have

$$\frac{2^{p-1}-1}{p}\equiv 2(-1)^{\frac{p-1}{2}}\sum_{1\leqslant k\leqslant \frac{p+1}{6}}\frac{(-1)^k}{2k-1}\pmod p$$

and

$$\sum_{0<k<p/2}\frac{3^k}{k}\equiv\sum_{0<k<p/6}\frac{(-1)^k}{k}\pmod p.$$

1. Introduction

Let $\mathbb{N}=\{0,1,2,\cdots\}$ and $\mathbb{Z}^+=\{1,2,3,\cdots\}$. For $m\in\mathbb{Z}^+$, $n\in\mathbb{N}$ and $r\in\mathbb{Z}$, we set

$$\text{(1.1)}\qquad \begin{bmatrix}n\\ r\end{bmatrix}_m=\sum_{\substack{k=0\\ k\equiv r\ (\mathrm{mod}\ m)}}^{n}\binom nk\quad\text{and}\quad \begin{Bmatrix}n\\ r\end{Bmatrix}_m=\sum_{\substack{k=0\\ k\equiv r\ (\mathrm{mod}\ m)}}^{n}(-1)^{\frac{k-r}{m}}\binom nk.$$

It is interesting to determine these two kinds of sums, which are closely related to various number-theoretic quotients (see [W], [SS], [S1-3] and [Su1]), values of Bernoulli and Euler polynomials at rational points (cf. [GS] and [Su3]), S.

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Jakubec's investigation ([J]) of divisibility of the class number of a real cyclotomic field of prime degree, and C. Helou's study of Terjanian's conjecture concerning Hilbert's norm residue symbol and cyclotomic units (see Proposition 2 and Lemma 3 of [H]). Observe that

$$\begin{bmatrix} n \\ r \end{bmatrix}_m + \begin{Bmatrix} n \\ r \end{Bmatrix}_m = 2\begin{bmatrix} n \\ r \end{bmatrix}_{2m}.$$

Also,

$$\begin{bmatrix} n \\ r \end{bmatrix}_m = \begin{bmatrix} n \\ n-r \end{bmatrix}_m \text{ and } \begin{bmatrix} n+1 \\ r \end{bmatrix}_m = \begin{bmatrix} n \\ r \end{bmatrix}_m + \begin{bmatrix} n \\ r-1 \end{bmatrix}_m. \tag{1.2}$$

So, it suffices to determine $\left[\begin{smallmatrix} n \\ r \end{smallmatrix}\right]_m$ with n odd. If $n > 0$ then

$$\begin{bmatrix} n \\ r \end{bmatrix}_m = |\{S \subseteq \{1, \cdots, n\}\colon |S| \equiv r \pmod m\}| \text{ and } \begin{bmatrix} n \\ r \end{bmatrix}_2 = \frac{1}{2}\begin{bmatrix} n \\ r \end{bmatrix}_1 = 2^{n-1}.$$

For explicit formulas of $\left[\begin{smallmatrix} n \\ r \end{smallmatrix}\right]_8$ and $\left[\begin{smallmatrix} n \\ r \end{smallmatrix}\right]_{10}$, the reader may consult [S2], [Su1] and [SS].

Throughout this paper, for a real number x we use $\lfloor x \rfloor$ and $\{x\}$ to denote the integral and fractional parts of x respectively. For $a, b \in \mathbb{Z}$, as usual (a, b) stands for the greatest common divisor of a and b. When $a \in \mathbb{Z}$, $n \in \mathbb{Z}^+$ and $(a, n) = 1$, $(\frac{a}{n})$ denotes the Jacobi symbol if $2 \nmid n$; we write $q_n(a)$ for $(a^{n-1} - 1)/n$, which is often called a Fermat quotient if n is a prime p. For an assertion A we set

$$[A] = \begin{cases} 1 & \text{if } A \text{ holds,} \\ 0 & \text{otherwise.} \end{cases} \tag{1.3}$$

Our first aim is to express the sum $\left[\begin{smallmatrix} n \\ r \end{smallmatrix}\right]_m$ $(m > 2)$ in terms of some linearly recurrent sequences whose orders belong to $\{1\} \cup \{\varphi(d)/2\colon d \mid m \ \&\ d > 2\}$ where φ is Euler's totient function. Namely, we have

Theorem 1. *Let* $D_0(x) = 2$ *and*

$$D_n(x) = \sum_{i=0}^{\lfloor n/2 \rfloor} (-1)^i \frac{n}{n-i}\binom{n-i}{i} x^{\lfloor \frac{n}{2} \rfloor - i} \quad \textit{for } n \in \mathbb{Z}^+. \tag{1.4}$$

Let $k, m \in \mathbb{Z}$ *and* $m > 2$. *Write*

$$w_n(k, m) = \sum_{\substack{0<j<m/2 \\ (j,m)=1}} D_{|k|}\left(4\cos^2 \frac{j\pi}{m}\right)\left(4\cos^2 \frac{j\pi}{m}\right)^n \quad \textit{for } n \in \mathbb{Z}, \tag{1.5}$$

and

$$(1.6)\qquad \begin{aligned} A_m(x)=&\prod_{\substack{0<j<m/2\\(j,m)=1}}\left(x-4\cos^2\frac{j\pi}{m}\right)\\ =&x^{\varphi(m)/2}-a_1x^{\varphi(m)/2-1}-\cdots-a_{\varphi(m)/2-1}x-a_{\varphi(m)/2}.\end{aligned}$$

Then $(-1)^{s-1}a_s\in\mathbb{Z}^+$ *for* $s=1,\cdots,\varphi(m)/2,$ *and*

$$(1.7)\qquad w_n(k,m)=a_1w_{n-1}(k,m)+\cdots+a_{\varphi(m)/2}w_{n-\varphi(m)/2}(k,m)\quad \text{for } n\in\mathbb{Z}.$$

Whenever $n\in\mathbb{N}$ *and* $r\in\mathbb{Z}$, *we have*

$$(1.8)\qquad \begin{bmatrix}n\\r\end{bmatrix}_m=\frac{2^n+(-1)^r[2\mid m\ \&\ n=0]}{m}+\frac{1}{m}\sum_{\substack{d\mid m\\d>2}}w_{\lfloor\frac{n+1}{2}\rfloor}(n-2r,d).$$

Applying Theorem 1 with $m=4$ we find that

$$4\begin{bmatrix}n\\0\end{bmatrix}_4-2^n=w_{\frac{n+1}{2}}(n,4)=(-1)^{\frac{n^2-1}{8}}2^{\frac{n+1}{2}}\quad \text{for } n=1,3,5,\cdots,$$

consequently

$$(-1)^{\frac{p^2-1}{8}}\left(\frac{2}{p}\right)\equiv 2\begin{bmatrix}p\\0\end{bmatrix}_4-2^{p-1}\equiv 1 \pmod p\quad \text{for any odd prime } p.$$

This provides a new way to determine the quadratic character of 2 modulo an odd prime. (The author's brother Z.-H. Sun [S1] employed $\begin{bmatrix}p\\1\end{bmatrix}_4$ and $\begin{bmatrix}p\\2\end{bmatrix}_4$ to obtain $(\frac{2}{p})=(-1)^{(p^2-1)/8}$.)

Let $m>2$ be an integer and $p>2$ be a prime not dividing m. From Theorem 1 we can deduce the following congruence:

$$(1.9)\qquad \frac{w_{\frac{p+1}{2}}(p,m)-(\varphi(m)+\mu(m))}{p}\equiv\varphi(m)\sum_{k=1}^{p-1}\frac{\mu(m/(k,m))}{\varphi(m/(k,m))}\cdot\frac{(-1)^{k-1}}{k}\pmod p$$

where μ denotes the well-known Möbius function.

Our second goal is to obtain an explicit formula for the sum $\begin{bmatrix}n\\r\end{bmatrix}_{12}$. This involves a special Lucas sequence $\{S_n\}_{n\in\mathbb{Z}}$ and its companion $\{T_n\}_{n\in\mathbb{Z}}$ defined as follows:

$$(1.10)\qquad \begin{aligned}&S_0=0,\ S_1=1 \text{ and } S_{n+1}+S_{n-1}=4S_n\quad \text{for } n=0,\pm1,\pm2,\cdots;\\ &T_0=2,\ T_1=4 \text{ and } T_{n+1}+T_{n-1}=4T_n\quad \text{for } n=0,\pm1,\pm2,\cdots.\end{aligned}$$

It is easy to check that $T_n=4S_n-2S_{n-1}$ and $6S_n=2T_n-T_{n-1}$ for all $n\in\mathbb{Z}$.

Theorem 2. *Let $n\in\mathbb{Z}^+$, $2\nmid n$ and $r\in\mathbb{Z}$. Then*

(1.11)
$$\begin{aligned}&12\begin{bmatrix}n\\ r\end{bmatrix}_{12}-2^n-1\\ &=\begin{cases}3^{\frac{n+1}{2}}+(-1)^{\frac{r(n-r)}{2}}(\frac{2}{n})(2^{\frac{n+1}{2}}+T_{\frac{n+1}{2}}) & \text{if } n-2r\equiv\pm1\pmod{12},\\ -3+(-1)^{\frac{r(n-r)}{2}}(\frac{2}{n})(2^{\frac{n+1}{2}}-T_{\frac{n+1}{2}}+T_{\frac{n-1}{2}}) & \text{if } n-2r\equiv\pm3\pmod{12},\\ -3^{\frac{n+1}{2}}+(-1)^{\frac{r(n-r)}{2}}(\frac{2}{n})(2^{\frac{n+1}{2}}-T_{\frac{n-1}{2}}) & \text{if } n-2r\equiv\pm5\pmod{12}.\end{cases}\end{aligned}$$

The author got Theorem 2 in 1988; it has the following application.

Theorem 3. *Let n be a positive integer with $(6,n)=1$. Set $\bar n=(n-(\frac{3}{n}))/2$. Then*

$$(1.12)\quad \left(\frac{2}{n}\right)\frac{S_{\bar n}}{n}=\frac{(-1)^{\frac{n-1}{2}}}{3}\sum_{k=1}^{\lfloor\frac{n+1}{6}\rfloor}\frac{(-1)^k}{2k-1}\binom{n-1}{6k-4}+\sum_{\substack{k=1\\ 6\mid k+n}}^{n-1}\frac{(-1)^{\frac{k+n}{6}}}{k}\binom{n-1}{k-1}.$$

For any prime $p>3$, we have the congruences

$$(1.13)\quad \sum_{k=1}^{\frac{p-1}{2}}\frac{3^k}{k}\equiv\sum_{0<k<p/6}\frac{(-1)^k}{k}\equiv-6\left(\frac{2}{p}\right)\frac{S_{\bar p}}{p}-q_p(2)\pmod p$$

and

$$(1.14)\quad q_p(2)\equiv2(-1)^{\frac{p-1}{2}}\sum_{k=1}^{\lfloor\frac{p+1}{6}\rfloor}\frac{(-1)^k}{2k-1}\pmod p.$$

Let $p>3$ be a prime. The first congruence in (1.13) was announced by the author [Su1] in 1995. (1.14) provides a quick way to compute $q_p(2)$ mod p. In Section 3 we will determine $\sum_{\substack{0<k<p\\ 12\mid k-r}}\frac{1}{k}$ mod p explicitly where $r\in\mathbb{Z}$.

We will show Theorems 1 and 2 in the next section. Section 3 contains a proof of Theorem 3 and other applications of Theorems 1 and 2.

2. Proofs of Theorems 1 and 2

Let $m\in\mathbb{Z}^+$, $n\in\mathbb{N}$ and $a,r\in\mathbb{Z}$. Then

$$\sum_{\substack{0\leqslant k\leqslant n\\ k\equiv r\ (\mathrm{mod}\ m)}}\binom{n}{k}a^k=\sum_{k=0}^{n}\binom{n}{k}\frac{a^k}{m}\sum_{\gamma^m=1}\gamma^{k-r}=\frac{1}{m}\sum_{\gamma^m=1}\gamma^{-r}(1+a\gamma)^n.$$

This is (1.53) of H. W. Gould [G]. If p is a prime not dividing m, then we have

$$\sum_{\substack{0\leqslant k\leqslant pn\\ k\equiv pr \pmod m}}\binom{pn}{k}a^k \equiv \sum_{\substack{0\leqslant k\leqslant n\\ k\equiv r \pmod m}}\binom{n}{k}a^k \pmod p \tag{2.1}$$

(and in particular $\left[{pn \atop pr}\right]_m \equiv \left[{n \atop r}\right]_m \pmod p$ as observed by A. Granville) because

$$\sum_{\gamma^m=1}\gamma^{-pr}(1+a\gamma)^{pn} \equiv \sum_{\gamma^m=1}\gamma^{-pr}(1+a^p\gamma^p)^n \equiv \sum_{\gamma^m=1}\gamma^{-r}(1+a\gamma)^n \pmod p.$$

Lemma 2.1. *Let $k\in\mathbb{Z}$, $m\in\mathbb{Z}^+$ and $n\in\mathbb{N}$. Then*

$$\frac{1}{m}\sum_{\gamma^m=1}\gamma^k(2+\gamma+\gamma^{-1})^n = \begin{bmatrix}2n\\ k+n\end{bmatrix}_m \tag{2.2}$$

and

$$\frac{1}{m}\sum_{\gamma^m=-1}\gamma^k(2+\gamma+\gamma^{-1})^n = \begin{Bmatrix}2n\\ k+n\end{Bmatrix}_m \tag{2.3}$$

Proof. Let $\varepsilon\in\{1,-1\}$. Observe that

$$\begin{aligned}
&\sum_{\gamma^m=\varepsilon}\gamma^k(2+\gamma+\gamma^{-1})^n = \sum_{\gamma^m=\varepsilon}\gamma^{k+n}(1+2\gamma^{-1}+\gamma^{-2})^n\\
=&\sum_{\gamma^m=\varepsilon}\gamma^{k+n}(1+\gamma^{-1})^{2n} = \sum_{\gamma^m=\varepsilon}\gamma^{k+n}\sum_{s=0}^{2n}\binom{2n}{s}\gamma^{-s}\\
=&\sum_{s=0}^{2n}\binom{2n}{s}\sum_{\gamma^m=(-1)^{\frac{1-\varepsilon}{2}}}\gamma^{k+n-s} = \sum_{s=0}^{2n}\binom{2n}{s}\sum_{\gamma^m=1}\left(e^{\frac{\pi i}{m}\cdot\frac{1-\varepsilon}{2}}\gamma\right)^{k+n-s}\\
=&\sum_{\substack{0\leqslant s\leqslant 2n\\ m\mid k+n-s}}\binom{2n}{s}m(-1)^{\frac{1-\varepsilon}{2}\cdot\frac{k+n-s}{m}} = m\sum_{\substack{0\leqslant s\leqslant 2n\\ m\mid s-(k+n)}}\varepsilon^{\frac{s-k-n}{m}}\binom{2n}{s}.
\end{aligned}$$

So (2.2) and (2.3) hold. $\square$

Remark 2.1. Let $k\in\mathbb{Z}$, $m\in\mathbb{Z}^+$, $n\in\mathbb{N}$ and $\varepsilon\in\{1,-1\}$. By Lemma 2.1,

$$\begin{aligned}
\sum_{\gamma^m=\varepsilon}\gamma^k(2-\gamma-\gamma^{-1})^n &= \sum_{\gamma^m=(-1)^m\varepsilon}(-\gamma)^k(2+\gamma+\gamma^{-1})^n\\
&=(-1)^k m\times\begin{cases}\left[{2n \atop k+n}\right]_m & \text{if } \varepsilon=(-1)^m,\\ \left\{{2n \atop k+n}\right\}_m & \text{otherwise.}\end{cases}
\end{aligned}$$

For $n=0,1,2,3,\cdots$ the nth Chebyshev polynomial $T_n(x)$ of the first kind is defined by

$$\cos(n\theta)=T_n(\cos\theta).$$

It is known that if $n\in\mathbb{Z}^+$ then

$$T_{n+1}(x)=2xT_n(x)-T_{n-1}(x)\text{ and }2T_n(x)=\sum_{i=0}^{\lfloor n/2\rfloor}(-1)^i\frac{n}{n-i}\binom{n-i}{i}(2x)^{n-2i}.$$

Thus $2T_n(x)=D_n(4x^2)(2x)^{[2\nmid n]}$ for any $n\in\mathbb{N}$.

Proof of Theorem 1. Let $y_j=\cos\frac{j\pi}{m}$ and $x_j=4y_j^2$ for $j\in\mathbb{Z}$. As $x_j-2=2\cos(2\pi\frac{j}{m})=e^{2\pi i\frac{j}{m}}+e^{-2\pi i\frac{j}{m}}$, the coefficients of $A_m(x+2)$ are symmetric polynomials in those primitive mth roots of unity with integer coefficients. Since

$$\Phi_m(x)=\prod_{\substack{1\leqslant j\leqslant m\\(j,m)=1}}\left(x-e^{2\pi i\frac{j}{m}}\right)\in\mathbb{Z}[x],$$

we have $A_m(x+2)\in\mathbb{Z}[x]$ by the Fundamental Theorem on Symmetric Polynomials, therefore $A_m(x)\in\mathbb{Z}[x]$.

Let $1\leqslant s\leqslant\varphi(m)/2$. By Viéte's theorem

$$-a_s=\sum_{\substack{0<j_1<\cdots<j_s<m/2\\(j_1\cdots j_s,m)=1}}\prod_{t=1}^{s}(-x_{j_t}),$$

therefore

$$0<(-1)^{s-1}a_s<\binom{\varphi(m)/2}{s}4^s.$$

For any integer n we clearly have

$$\begin{aligned}\sum_{i=1}^{\varphi(m)/2}a_i\sum_{\substack{0<j<m/2\\(j,m)=1}}D_{|k|}(x_j)\,x_j^{n-i}&=\sum_{\substack{0<j<m/2\\(j,m)=1}}D_{|k|}(x_j)\sum_{i=1}^{\varphi(m)/2}a_ix_j^{n-i}\\&=\sum_{\substack{0<j<m/2\\(j,m)=1}}D_{|k|}(x_j)\,x_j^{n-\varphi(m)/2}\left(x_j^{\varphi(m)/2}-A_m(x_j)\right)=\sum_{\substack{0<j<m/2\\(j,m)=1}}D_{|k|}(x_j)\,x_j^{n}.\end{aligned}$$

So (1.7) follows.

For each $k\in\mathbb{N}$, if $2\mid k$ then $D_k(4x^2)=2T_k(x)$; if $2\nmid k$ then

$$D_k(4x^2)=\frac{2T_k(x)}{2x}=\frac{T_{k-1}(x)+T_{k+1}(x)}{2x^2}.$$

Let $n\in\mathbb{N}$ and $r\in\mathbb{Z}$. Then

$$\begin{aligned}\sum_{\substack{d\mid m\\ d>2}} w_{\lfloor\frac{n+1}{2}\rfloor}(n-2r,d) =&\sum_{d\mid m}\sum_{\substack{0<c<d/2\\ (c,d)=1}} D_{|n-2r|}\left(4\cos^2\frac{c\pi}{d}\right)\left(4\cos^2\frac{c\pi}{d}\right)^{\lfloor\frac{n+1}{2}\rfloor}\\ =&\sum_{0<j<m/2} D_{|n-2r|}(x_j)x_j^{\lfloor\frac{n+1}{2}\rfloor}.\end{aligned}$$

If $2\mid n$, then

$$\begin{aligned}&\sum_{\substack{d\mid m\\ d>2}} w_{\lfloor\frac{n+1}{2}\rfloor}(n-2r,d) = \sum_{0<j<m/2} 2T_{|n-2r|}(y_j)x_j^{n/2}\\ =&\sum_{0<j<m/2}\left(e^{\pi i\frac{j}{m}(n-2r)}+e^{-\pi i\frac{j}{m}(n-2r)}\right)\left(2+e^{2\pi i\frac{j}{m}}+e^{-2\pi i\frac{j}{m}}\right)^{n/2}\\ =&\sum_{\gamma^m=1}\gamma^{n/2-r}(2+\gamma+\gamma^{-1})^{n/2}-4^{n/2}-(-1)^{n/2-r}[2\mid m\ \&\ n/2=0]\\ =&m\begin{bmatrix} n\\ n-r\end{bmatrix}_m-2^n-(-1)^r[2\mid m\ \&\ n=0]\\ =&m\begin{bmatrix} n\\ r\end{bmatrix}_m-2^n-(-1)^r[2\mid m\ \&\ n=0].\end{aligned}$$

When $2\nmid n$, we have

$$\begin{aligned}&\sum_{\substack{d\mid m\\ d>2}} w_{\lfloor\frac{n+1}{2}\rfloor}(n-2r,d) = \sum_{0<j<m/2}\frac{T_{|n-2r|-1}(y_j)+T_{|n-2r|+1}(y_j)}{2y_j^2}x_j^{\frac{n+1}{2}}\\ =&\sum_{0<j<m/2}\left(2\cos(n-2r-1)\frac{j\pi}{m}+2\cos(n-2r+1)\frac{j\pi}{m}\right)x_j^{\frac{n-1}{2}}\\ =&\sum_{\gamma^m=1}\left(\gamma^{\frac{n-1}{2}-r}+\gamma^{\frac{n+1}{2}-r}\right)(2+\gamma+\gamma^{-1})^{\frac{n-1}{2}}-(1+1)4^{\frac{n-1}{2}}\\ =&m\begin{bmatrix} n-1\\ n-1-r\end{bmatrix}_m+m\begin{bmatrix} n-1\\ n-r\end{bmatrix}_m-2^n=m\begin{bmatrix} n\\ n-r\end{bmatrix}_m-2^n=m\begin{bmatrix} n\\ r\end{bmatrix}_m-2^n.\end{aligned}$$

This ends the proof. $\square$

Remark 2.2. For any integer $m>2$, clearly

$$\begin{aligned}&A_m\left((1+x)(1+x^{-1})\right)=A_m(2+x+x^{-1})\\ =&\prod_{\substack{0<j<m/2\\ (j,m)=1}}\left(x+x^{-1}-e^{2\pi i\frac{j}{m}}-e^{-2\pi i\frac{j}{m}}\right)\\ =&\prod_{\substack{0<j<m/2\\ (j,m)=1}}\frac{1}{x}\left(x-e^{2\pi i\frac{j}{m}}\right)\left(x-e^{-2\pi i\frac{j}{m}}\right)=\frac{\Phi_m(x)}{x^{\varphi(m)/2}}.\end{aligned}$$

Now we list $A_m(x)$ for $2 < m \leqslant 12$:

$$\begin{aligned}&A_3(x)=x-1,\ A_4(x)=x-2,\ A_5(x)=x^2-3x+1,\\&A_6(x)=x-3,\ A_7(x)=x^3-5x^2+6x-1,\ A_8(x)=x^2-4x+2,\\&A_9(x)=x^3-6x^2+9x-1,\ A_{10}(x)=x^2-5x+5,\\&A_{11}(x)=x^5-9x^4+28x^3-35x^2+15x-1,\ A_{12}(x)=x^2-4x+1.\end{aligned}$$

Let $m,n\in\mathbb{Z}$ and $m>2$. Clearly $w_n(0,m)=2w_n(1,m)$ since $D_0(x)=2D_1(x)=2$. For $k,l\in\mathbb{Z}$ we have

$$w_n(k,m)=w_n(l,m)\ \text{ if } k\equiv\pm l \pmod{2m}, \tag{2.4}$$

and

$$w_n(m-k,m)=-w_n(k,m)\ \text{ if } m\equiv 0 \pmod 2. \tag{2.5}$$

(Thus $w_n(m/2,m)=0$ when m is even.) This is because

$$D_{|k|}\left(4\cos^2\frac{j\pi}{m}\right)\left(2\cos\frac{j\pi}{m}\right)^{[2\nmid k]}=2T_{|k|}\left(\cos\frac{j\pi}{m}\right)=2\cos\left(\frac{jk}{m}\pi\right).$$

When $m\in\{5,8,10,12\}$ (i.e. $\varphi(m)/2=2$) we will express $w_n(k,m)$ $(k,n\in\mathbb{Z})$ in terms of several second order recurrences of integers, namely the Fibonacci sequence $\{F_n\}_{n\in\mathbb{Z}}$ and its companion $\{L_n\}_{n\in\mathbb{Z}}$, the Pell sequence $\{P_n\}_{n\in\mathbb{Z}}$ and its companion $\{Q_n\}_{n\in\mathbb{Z}}$, and the sequence $\{S_n\}_{n\in\mathbb{Z}}$ and its companion $\{T_n\}_{n\in\mathbb{Z}}$ given by (1.10). The sequences $\{F_n\}_{n\in\mathbb{Z}},\{L_n\}_{n\in\mathbb{Z}},\{P_n\}_{n\in\mathbb{Z}},\{Q_n\}_{n\in\mathbb{Z}}$ are defined as follows:

$$\begin{aligned}&F_0=0,\ F_1=1,\ F_{n+1}=F_n+F_{n-1}\ (n=0,\pm1,\pm2,\cdots);\\&L_0=2,\ L_1=1,\ L_{n+1}=L_n+L_{n-1}\ (n=0,\pm1,\pm2,\cdots);\\&P_0=0,\ P_1=1,\ P_{n+1}=2P_n+P_{n-1}\ (n=0,\pm1,\pm2,\cdots);\\&Q_0=2,\ Q_1=2,\ Q_{n+1}=2Q_n+Q_{n-1}\ (n=0,\pm1,\pm2,\cdots).\end{aligned}\tag{2.6}$$

It is easy to check that for each $n\in\mathbb{Z}$ we have

$$\begin{aligned}&F_n=\frac{1}{\sqrt5}\left(\left(\frac{1+\sqrt5}{2}\right)^n-\left(\frac{1-\sqrt5}{2}\right)^n\right),\quad L_n=\left(\frac{1+\sqrt5}{2}\right)^n+\left(\frac{1-\sqrt5}{2}\right)^n;\\&P_n=\frac{1}{2\sqrt2}\left((1+\sqrt2)^n-(1-\sqrt2)^n\right),\qquad Q_n=(1+\sqrt2)^n+(1-\sqrt2)^n;\\&S_n=\frac{1}{2\sqrt3}\left((2+\sqrt3)^n-(2-\sqrt3)^n\right),\qquad T_n=(2+\sqrt3)^n+(2-\sqrt3)^n.\end{aligned}$$

For those $m\in\mathbb{Z}^+$ with $\varphi(m)=2$ or 4, we give below values of $w_n(k,m)$ $(n\in\mathbb{Z})$ where $1\leqslant k\leqslant m$ if $2\nmid m$, and $0<k<m/2$ if $2\mid m$. They can be obtained through trivial computations.

$$w_n(1,3)=1,\ w_n(2,3)=-1,\ w_n(3,3)=-2.$$
$$w_n(1,4)=2^n;\ w_n(1,6)=w_n(2,6)=3^n.$$
$$w_n(1,5)=L_{2n},\ w_n(2,5)=L_{2n-1},\ w_n(3,5)=-L_{2n-2},$$
$$w_n(4,5)=-L_{2n+1},\ w_n(5,5)=-2L_{2n-1}.$$

	$w_n(1,8)$	$w_n(2,8)$	$w_n(3,8)$	$w_n(1,10)$	$w_n(2,10)$
$2\nmid n$	$2^{(n+3)/2}P_n$	$2^{(n+1)/2}Q_n$	$2^{(n+3)/2}P_{n-1}$	$5^{(n+1)/2}F_n$	$5^{(n+1)/2}F_{n+1}$
$2\mid n$	$2^{n/2}Q_n$	$2^{(n+4)/2}P_n$	$2^{n/2}Q_{n-1}$	$5^{n/2}L_n$	$5^{n/2}L_{n+1}$

$$w_n(3,10)=w_n(4,10)=\begin{cases}5^{(n+1)/2}F_{n-1} & \text{if } 2\nmid n,\\ 5^{n/2}L_{n-1} & \text{if } 2\mid n.\end{cases}$$
$$w_n(1,12)=w_n(4,12)=T_n,\ w_n(2,12)=6S_n,$$
$$w_n(3,12)=6S_n-T_n=2(S_n+S_{n-1})=T_n-T_{n-1},\ w_n(5,12)=T_{n-1}.$$

Proof of Theorem 2. Let $k=n-2r$. By Theorem 1,

$$12\begin{bmatrix}n\\ r\end{bmatrix}_{12}-2^n=\sum_{\substack{d\mid 12\\ d>2}}w_{\frac{n+1}{2}}(k,d)=b_k+c_k$$

where

$$b_k=w_{\frac{n+1}{2}}(k,3)+w_{\frac{n+1}{2}}(k,6)\text{ and }c_k=w_{\frac{n+1}{2}}(k,4)+w_{\frac{n+1}{2}}(k,12).$$

Observe that

$$b_1=1+3^{\frac{n+1}{2}},\ b_3=-2,\ b_5=w_{\frac{n+1}{2}}(1,3)-w_{\frac{n+1}{2}}(1,6)=1-3^{\frac{n+1}{2}}.$$

Also, $c_1=2^{\frac{n+1}{2}}+T_{\frac{n+1}{2}}$,

$$c_3=-w_{\frac{n+1}{2}}(1,4)+w_{\frac{n+1}{2}}(3,12)=-2^{\frac{n+1}{2}}+T_{\frac{n+1}{2}}-T_{\frac{n-1}{2}},$$
$$c_5=-w_{\frac{n+1}{2}}(1,4)+w_{\frac{n+1}{2}}(5,12)=-2^{\frac{n+1}{2}}+T_{\frac{n-1}{2}}.$$

Let l be the unique integer in $\{1,3,5\}$ such that k is congruent to l or $-l$ modulo 12. Then $b_k=b_l$ by (2.4). If $k\equiv\pm l \pmod 8$, then $k\equiv\pm l \pmod{24}$ and hence $c_k = c_l$ by (2.4). In the case $k \not\equiv \pm l \pmod{24}$, $12 - k \equiv \pm l \pmod{24}$ and hence

$$-c_k = w_{\frac{n+1}{2}}(4-k,4) + w_{\frac{n+1}{2}}(12-k,12) = w_{\frac{n+1}{2}}(l,4) + w_{\frac{n+1}{2}}(l,12) = c_l.$$

Thus

$$c_k = (-1)^{\frac{k^2-l^2}{8}} c_l = (-1)^{\frac{n^2-l^2}{8} - \frac{r(n-r)}{2}} c_l$$

and so

$$12\begin{bmatrix} n \\ r \end{bmatrix}_{12} - 2^n = b_l + (-1)^{\frac{r(n-r)}{2}} \left(\frac{2}{n}\right)(-1)^{\frac{l^2-1}{8}} c_l.$$

Since we have computed b_l and c_l, (1.11) follows immediately. $\square$

3. Applications of Theorems 1 and 2

Theorem 1 implies the following result.

Theorem 3.1. *Let $m, n \in \mathbb{N}$, $m > 2$ and $n \geqslant \delta$ where $\delta \in \{0,1\}$. Then*

$$w_n(2k+\delta, m) = \varphi(m) \sum_{j=0}^{2n-\delta} \frac{\mu(m/(m,j-k-n))}{\varphi(m/(m,j-k-n))} \binom{2n-\delta}{j} \tag{3.1}$$

for all $k \in \mathbb{Z}$. If p is a prime not dividing $2m$, then (1.9) holds.

Proof. Let k be any integer. By Theorem 1,

$$\begin{aligned} &l\begin{bmatrix} 2n-\delta \\ k+n \end{bmatrix}_l - 2^{2n-\delta} - (-1)^{k+n}[2 \mid l \ \&\ 2n = \delta] \\ =&\sum_{\substack{d \mid l \\ d>2}} w_{\lfloor \frac{2n-\delta+1}{2} \rfloor}(2n-\delta-2k-2n, d) = \sum_{\substack{d \mid l \\ d>2}} w_n(2k+\delta, d) \end{aligned}$$

for all $l = 1, 2, 3, \cdots$. Applying the Möbius theorem we then get that

$$w_n(2k+\delta, m) = \sum_{d \mid m} \mu\left(\frac{m}{d}\right)\left(d\begin{bmatrix} 2n-\delta \\ k+n \end{bmatrix}_d - 2^{2n-\delta} - (-1)^{k+n}[2 \mid d \ \&\ 2n = \delta]\right).$$

As $m > 2$, we have $\sum_{d \mid m} \mu(\frac{m}{d}) = 0$ and

$$\sum_{\substack{d \mid m \\ 2 \mid d}} \mu\left(\frac{m}{d}\right) = \begin{cases} \sum_{c \mid m/2} \mu(\frac{m/2}{c}) = 0 & \text{if } 2 \mid m, \\ 0 & \text{if } 2 \nmid m. \end{cases}$$

Therefore

$$w_n(2k+\delta,m)=\sum_{d\mid m}\mu\left(\frac{m}{d}\right)d\begin{bmatrix}2n-\delta\\ k+n\end{bmatrix}_d=\sum_{d\mid m}\mu\left(\frac{m}{d}\right)d\sum_{\substack{j=0\\ d\mid j-(k+n)}}^{2n-\delta}\binom{2n-\delta}{j}$$
$$=\sum_{j=0}^{2n-\delta}\binom{2n-\delta}{j}\sum_{d\mid m}\mu\left(\frac{m}{d}\right)d[d\mid j-k-n].$$

For the equality (3.1), it remains to show that for any $c\in\mathbb{Z}$ we have

$$\sum_{d\mid m}\mu\left(\frac{m}{d}\right)d[d\mid c]=\varphi(m)\frac{\mu(m/(c,m))}{\varphi(m/(c,m))}.$$

This can be verified directly when m is a prime power, also both sides are multiplicative with respect to m. So (3.1) holds.

When n is prime to $2m$, we have

$$w_{\frac{n+1}{2}}\left(2\times\frac{n-1}{2}+1,m\right)=\varphi(m)\sum_{k=0}^{n}\frac{\mu(m/(m,k-\frac{n-1}{2}-\frac{n+1}{2}))}{\varphi(m/(m,k-\frac{n-1}{2}-\frac{n+1}{2}))}\binom{n}{k}$$
$$=\varphi(m)\sum_{k=0}^{n}\frac{\mu(m/(m,n-k))}{\varphi(m/(m,n-k))}\binom{n}{n-k}=\varphi(m)\sum_{k=0}^{n}\frac{\mu(m/(k,m))}{\varphi(m/(k,m))}\binom{n}{k}$$
$$=\varphi(m)\left(\frac{\mu(m/(0,m))}{\varphi(m/(0,m))}+\frac{\mu(m/(n,m))}{\varphi(m/(n,m))}\right)+\varphi(m)\sum_{k=1}^{n-1}\frac{\mu(m/(k,m))}{\varphi(m/(k,m))}\cdot\frac{n}{k}\binom{n-1}{k-1}$$
$$=\varphi(m)+\mu(m)+n\varphi(m)\sum_{k=1}^{n-1}\frac{\mu(m/(k,m))}{\varphi(m/(k,m))}\cdot\frac{1}{k}\binom{n-1}{k-1}.$$

If p is a prime with $p\nmid 2m$, then (1.9) follows from the above since

$$(-1)^l\binom{p-1}{l}=\prod_{0<j\leqslant l}\left(1-\frac{p}{j}\right)\equiv 1-p\sum_{0<j\leqslant l}\frac{1}{j}\pmod{p^2}$$

for any $l=0,1,2,\cdots,p-1$. We are done. □

As examples we apply Theorem 3.1 and Theorem 1 with $m=4,5$.

Corollary 3.1. *Let n be a positive odd integer. Then*

$$\frac{(-1)^{\frac{n^2-1}{8}}2^{\frac{n-1}{2}}-1}{n}=\sum_{\substack{k=1\\ 2\mid k}}^{n-1}\frac{(-1)^{\frac{k}{2}}}{k}\binom{n-1}{k-1}=\sum_{\substack{k=1\\ 2\nmid k}}^{n-1}\frac{(-1)^{\frac{n-k}{2}}}{k}\binom{n-1}{k-1},\tag{3.2}$$

and

$$2\sum_{\substack{k=1\\ 4\mid k-r}}^{n-1}\frac{1}{k}\binom{n-1}{k-1}=q_n(2)+(-1)^{\frac{r(n-r)}{2}}\frac{(-1)^{\frac{n^2-1}{8}}2^{\frac{n-1}{2}}-1}{n}\quad \text{for } r\in\mathbb{Z}. \tag{3.3}$$

Proof. Observe that

$$w_{\frac{n+1}{2}}(n,4)=\begin{cases} w_{\frac{n+1}{2}}(1,4)=2^{\frac{n+1}{2}} & \text{if } n\equiv\pm1\pmod 8,\\ w_{\frac{n+1}{2}}(3,4)=-2^{\frac{n+1}{2}} & \text{if } n\equiv\pm3\pmod 8.\end{cases}$$

Thus, by the proof of Theorem 3.1, we have

$$\begin{aligned}\frac{(-1)^{\frac{n^2-1}{8}}2^{\frac{n-1}{2}}-1}{n}&=\frac{w_{\frac{n+1}{2}}(n,4)-\varphi(4)-\mu(4)}{n\varphi(4)}\\ &=\sum_{k=1}^{n-1}\frac{1}{k}\binom{n-1}{k-1}\frac{\mu(4/(k,4))}{\varphi(4/(k,4))}=\sum_{\substack{k=1\\ 2\mid k}}^{n-1}\frac{(-1)^{\frac{k}{2}}}{k}\binom{n-1}{k-1}\\ &=\sum_{\substack{k=1\\ 2\mid n-k}}^{n-1}\frac{(-1)^{\frac{n-k}{2}}}{n-k}\binom{n-1}{n-k-1}=\sum_{\substack{k=1\\ 2\nmid k}}^{n-1}\frac{(-1)^{\frac{n-k}{2}}}{k}\binom{n-1}{k-1}.\end{aligned}$$

This proves (3.2). Clearly

$$q_n(2)=\frac{1}{2n}\sum_{k=1}^{n-1}\binom{n}{k}=\frac{1}{n}\sum_{\substack{k=1\\ 2\mid k-r}}^{n-1}\binom{n}{k}=\sum_{\substack{k=1\\ 2\mid k-r}}^{n-1}\frac{1}{k}\binom{n-1}{k-1}\quad \text{for } r\in\mathbb{Z},$$

this and (3.2) yield (3.3). □

Corollary 3.2. *Let n be a positive integer not divisible by 2 or 5, and*

$$K_n(r)=\sum_{\substack{k=1\\ 5\mid k-rn}}^{n-1}\frac{1}{k}\binom{n-1}{k-1}\quad \text{for } r\in\mathbb{Z}.$$

Then

$$\frac{F_{n-(\frac{5}{n})}}{n}=K_n(4)-K_n(3) \tag{3.4}$$

and

$$\frac{(\frac{5}{n})F_n-1}{n}=\frac{5}{3}K_n(0)+\frac{1}{3}K_n(3)-\frac{1}{3}K_n(4)-\frac{2}{3}q_n(2). \tag{3.5}$$

Proof. By Theorem 1, for any $r\in\mathbb{Z}$ we have

$$5\begin{bmatrix}n\\ r\end{bmatrix}_5-2^n=w_{\frac{n+1}{2}}(n-2r,5)=\begin{cases}L_{n+1} & \text{if } n-2r\equiv\pm1 \pmod{10},\\ -L_{n-1} & \text{if } n-2r\equiv\pm3 \pmod{10},\\ -2L_n & \text{if } n-2r\equiv\pm5 \pmod{10}.\end{cases}$$

As $5F_j=2L_{j+1}-L_j=L_j+2L_{j-1}$ for $j\in\mathbb{Z}$, $5F_{n-(\frac{5}{n})}=2L_n-(\frac{5}{n})L_{n-(\frac{5}{n})}$ and hence

$$F_{n-(\frac{5}{n})}=\begin{bmatrix}n\\ 4n\end{bmatrix}_5-\begin{bmatrix}n\\ 3n\end{bmatrix}_5=\sum_{k=1}^{n-1}([5\mid k-4n]-[5\mid k-3n])\frac{n}{k}\binom{n-1}{k-1}.$$

So (3.4) follows.

Observe that

$$w_{\frac{n+1}{2}}(n,5)=\begin{cases}w_{\frac{n+1}{2}}(1,5)=L_{n+1}=3F_n+F_{n-1} & \text{if } n\equiv\pm1 \pmod{10},\\ w_{\frac{n+1}{2}}(3,5)=-L_{n-1}=-3F_n+F_{n+1} & \text{if } n\equiv\pm3 \pmod{10}.\end{cases}$$

Thus, by the proof of Theorem 3.1, we have

$$\begin{aligned}&\frac{1}{n}\left(3\left(\frac{5}{n}\right)F_n+F_{n-(\frac{5}{n})}-3\right)=4\sum_{k=1}^{n-1}\frac{1}{k}\binom{n-1}{k-1}\frac{\mu(5/(k,5))}{\varphi(5/(k,5))}\\ =&4K_n(0)-(K_n(1)+K_n(2)+K_n(3)+K_n(4))=5K_n(0)-\frac{(1+1)^n-2}{n}.\end{aligned}$$

This, together with (3.4), yields (3.5). $\square$

Remark 3.1. Let p be an odd prime. Various congruences for $F_{p-(\frac{5}{p})}/p \bmod p$ can be found in [W], [SS] and [S3]. In 1995 the author [Su1] showed that

$$-2^{\frac{p+1}{2}}\frac{P_p-2^{\frac{p-1}{2}}}{p}\equiv\sum_{k=1}^{\frac{p-1}{2}}\frac{1}{k2^k}\equiv\sum_{k=1}^{\lfloor\frac{3}{4}p\rfloor}\frac{(-1)^{k-1}}{k}\equiv 2q_p(2)+\sum_{0<k<p/4}\frac{(-1)^k}{k} \pmod p,$$

which was reproved by Z. Shan and Edward T. H. Wang [SW], and extended by W. Kohnen [K]. Therefore $2(2^{\frac{p-1}{2}}P_p-1)/p\equiv\sum_{0<k<p/4}(-1)^{k-1}/k \pmod p$. As

$$\left(\frac{2}{p}\right)Q_{p-(\frac{2}{p})}=4\left(\frac{2}{p}\right)P_p-Q_p\equiv 4-(1+\sqrt{2}+(1-\sqrt{2}))^p\equiv 2 \pmod p,$$

$Q^2_{p-(\frac{2}{p})}-4=8P^2_{p-(\frac{2}{p})}\equiv 0 \pmod{p^2}$ and hence $(\frac{2}{p})P_{p-(\frac{2}{p})}=P_p-\frac{1}{2}Q_{p-(\frac{2}{p})}\equiv P_p-(\frac{2}{p}) \pmod{p^2}$. Thus

(3.6)
$$\frac{P_{p-(\frac{2}{p})}}{p}\equiv\frac{(\frac{2}{p})P_p-1}{p}\equiv\sum_{0<k<\frac{p}{4}}\frac{(-1)^{k-1}}{2k}-\frac{q_p(2)}{2}\equiv\frac{1}{2}\sum_{\frac{p}{4}<k<\frac{p}{2}}\frac{(-1)^k}{k} \pmod p.$$

Theorem 2 has the following consequence.

Theorem 3.2. *Let n be a positive odd integer. Then*

$$\begin{bmatrix} n \\ r \end{bmatrix}_6 = \frac{2^{n-1}-1}{3} + \frac{[3 \nmid n+r]}{2}\left((-1)^{\lfloor \frac{n-2r+1}{6} \rfloor} 3^{\frac{n-1}{2}} + 1\right) \quad \text{for } r \in \mathbb{Z}. \tag{3.7}$$

Providing $n \not\equiv 3 \pmod 6$ we have

$$\frac{(\frac{3}{n})3^{\frac{n-1}{2}} - 1}{n} = \frac{1}{2}\sum_{k=1}^{n-1} \frac{(-1)^{\lfloor \frac{k+1}{3} \rfloor}}{k}\binom{n-1}{k-1} = \frac{1}{3}\sum_{k=1}^{\lfloor n/3 \rfloor} \frac{(-1)^k}{k}\binom{n-1}{3k-1}. \tag{3.8}$$

Proof. As $\begin{bmatrix} n \\ r \end{bmatrix}_6 = \begin{bmatrix} n \\ r \end{bmatrix}_{12} + \begin{bmatrix} n \\ r+6 \end{bmatrix}_{12}$ and $\frac{(r+6)(n-r-6)}{2} - \frac{r(n-r)}{2} \equiv 1 \pmod 2$, (3.7) follows from Theorem 2.

Now assume that $(6, n) = 1$. Clearly

$$\sum_{k=1}^{\lfloor n/3 \rfloor} \frac{(-1)^k}{3k}\binom{n-1}{3k-1} = \sum_{\substack{k=1 \\ 6 \mid k}}^{n-1} \frac{1}{k}\binom{n-1}{k-1} - \sum_{\substack{k=1 \\ 6 \mid k-3}}^{n-1} \frac{1}{k}\binom{n-1}{k-1}$$

$$= \frac{\begin{bmatrix} n \\ 0 \end{bmatrix}_6 - 1 - \begin{bmatrix} n \\ 3 \end{bmatrix}_6}{n} = \frac{(-1)^{\lfloor \frac{n+1}{6} \rfloor} - (-1)^{\lfloor \frac{n-6+1}{6} \rfloor}}{2n} 3^{\frac{n-1}{2}} - \frac{1}{n} = \frac{(\frac{3}{n})3^{\frac{n-1}{2}} - 1}{n}$$

and

$$\sum_{k=1}^{n-1} \frac{(-1)^{\lfloor \frac{k+1}{3} \rfloor}}{k}\binom{n-1}{k-1} = \sum_{r=-1}^{1}\Bigg(\sum_{\substack{k=1 \\ 6 \mid k-r}}^{n-1} \frac{1}{n}\binom{n}{k} - \sum_{\substack{k=1 \\ 6 \mid k-r-3}}^{n-1} \frac{1}{n}\binom{n}{k}\Bigg)$$

$$= \sum_{r=-1}^{1} \frac{\begin{bmatrix} n \\ r \end{bmatrix}_6 - \begin{bmatrix} n \\ r+3 \end{bmatrix}_6}{n} - \sum_{r=-1}^{1} \frac{[6 \mid r] + [6 \mid n-r]}{n}$$

$$= \sum_{r=-1}^{1} [3 \nmid n+r](-1)^{\lfloor \frac{n-2r+1}{6} \rfloor} \frac{3^{\frac{n-1}{2}}}{n} - \frac{2}{n} = \frac{2}{n}\left(\left(\frac{3}{n}\right)3^{\frac{n-1}{2}} - 1\right).$$

This completes the proof. □

Remark 3.2. For $n \in \mathbb{Z}^+$ and $r \in \mathbb{Z}$, $\begin{bmatrix} n \\ r \end{bmatrix}_m$ in the cases $m = 4, 5, 6$ was also determined by the author's brother Z.-H. Sun [S1] but he did not present unified formulas like (3.3) and (3.7).

From Theorem 2 we can also deduce the following result written in number-theoretic language.

Theorem 3.3. *Let n be a positive integer prime to 6. Set $\bar{n} = (n - (\frac{3}{n}))/2$. For any $r \in \mathbb{Z}$ we have*

$$\begin{aligned} &\sum_{\substack{k=1 \\ k \equiv r \ (\mathrm{mod}\ 6)}}^{n-1} \frac{(-1)^{\frac{k(n-k)}{2}}}{k}\binom{n-1}{k-1} - \left(\frac{2}{n}\right)\frac{2^{\frac{n-1}{2}} - (\frac{2}{n})}{3n} \\ &= \begin{cases} \frac{1+(-1)^{\lfloor \frac{r+1}{3} \rfloor}}{2}(\frac{2}{n})\frac{S_{\bar{n}}}{n} + \frac{1+3(-1)^{\lfloor \frac{r+1}{3} \rfloor}}{2}(\frac{6}{n})\frac{T_{\bar{n}} - 2(\frac{6}{n})}{6n} & \text{if } 3 \nmid n+r, \\ -(\frac{2}{n})\frac{S_{\bar{n}}}{n} - (\frac{6}{n})\frac{T_{\bar{n}} - 2(\frac{6}{n})}{6n} & \text{if } 3 \mid n+r. \end{cases} \end{aligned} \tag{3.9}$$

Proof. Let $\delta_r = [6 \mid r] + [6 \mid n-r] = [n-2r \equiv \pm n \pmod{12}]$, and

$$\Delta_r = 6 \sum_{\substack{k=1\\ k\equiv r \pmod 6}}^{n-1} (-1)^{\frac{k(n-k)}{2}} \frac{n}{k}\binom{n-1}{k-1} - 2\left(\frac{2}{n}\right)\left(2^{\frac{n-1}{2}} - \left(\frac{2}{n}\right)\right).$$

Then

$$\Delta_r + \left(\frac{2}{n}\right)2^{\frac{n+1}{2}} - 2 + 6\delta_r = 6\sum_{\substack{k=0\\ 6\mid k-r}}^{n} (-1)^{\frac{k(n-k)}{2}}\binom{n}{k} = 6(-1)^{\frac{r(n-r)}{2}}\left\{ n \atop r \right\}_6$$

where in the last step we note that

$$\frac{k(n-k)}{2} - \frac{r(n-r)}{2} = \frac{k-r}{2}n - \frac{k^2-r^2}{2} \equiv \frac{k-r}{6} \pmod 2$$

if $k \equiv r \pmod 6$. In view of the above and Theorem 2,

$$\begin{aligned}
&\left(\frac{2}{n}\right)\Delta_r + (6\delta_r - 2)\left(\frac{2}{n}\right)\\
=&6(-1)^{\frac{r(n-r)}{2}}\left(\frac{2}{n}\right)\left(\begin{bmatrix} n\\ r\end{bmatrix}_{12} - \begin{bmatrix} n\\ r+6\end{bmatrix}_{12}\right) - 2^{\frac{n+1}{2}}\\
=&\begin{cases} T_{\frac{n+1}{2}} & \text{if } n-2r \equiv \pm 1 \pmod{12},\\ T_{\frac{n-1}{2}} - T_{\frac{n+1}{2}} & \text{if } n-2r \equiv \pm 3 \pmod{12},\\ -T_{\frac{n-1}{2}} & \text{if } n-2r \equiv \pm 5 \pmod{12}. \end{cases}
\end{aligned}$$

Observe that

$$6S_{\frac{n+1}{2}} - T_{\frac{n+1}{2}} = T_{\frac{n+1}{2}} - T_{\frac{n-1}{2}} = 3T_{\frac{n-1}{2}} - T_{\frac{n-3}{2}} = 6S_{\frac{n-1}{2}} + T_{\frac{n-1}{2}}.$$

If $n-2r \equiv \pm 1 \pmod{12}$, then $\delta_r = [n \equiv \pm 1 \pmod{12}]$, therefore

$$\begin{aligned}
\left(\frac{2}{n}\right)\Delta_r =&T_{\frac{n+1}{2}} + (2-6\delta_r)\left(\frac{2}{n}\right)\\
=&\begin{cases} 6S_{\frac{n-1}{2}} + 2T_{\frac{n-1}{2}} - 4(\frac{2}{n}) & \text{if } (\frac{3}{n}) = 1,\\ T_{\frac{n+1}{2}} + 2(\frac{2}{n}) & \text{if } (\frac{3}{n}) = -1, \end{cases}\\
=&3\left(1+\left(\frac{3}{n}\right)\right)S_{\bar n} + \frac{3+(\frac{3}{n})}{2}\left(T_{\bar n} - 2\left(\frac{6}{n}\right)\right).
\end{aligned}$$

If $n-2r\equiv\pm3\pmod{12}$ (i.e. $3\mid n+r$), then $\delta_r=[n\equiv\pm3\pmod{12}]=0$ and hence

$$\begin{aligned}\left(\frac{2}{n}\right)\Delta_r=&T_{\frac{n-1}{2}}-T_{\frac{n+1}{2}}+(2-6\delta_r)\left(\frac{2}{n}\right)\\=&-6S_{\frac{n-1}{2}}-T_{\frac{n-1}{2}}+2\left(\frac{2}{n}\right)=-6S_{\frac{n+1}{2}}+T_{\frac{n+1}{2}}+2\left(\frac{2}{n}\right)\\=&-6S_{\bar n}-\left(\frac{3}{n}\right)\left(T_{\bar n}-2\left(\frac{6}{n}\right)\right).\end{aligned}$$

If $n-2r\equiv\pm5\pmod{12}$, then $\delta_r=[n\equiv\pm5\pmod{12}]$ and so

$$\begin{aligned}\left(\frac{2}{n}\right)\Delta_r=&-T_{\frac{n-1}{2}}+(2-6\delta_r)\left(\frac{2}{n}\right)\\=&\begin{cases}-T_{\frac{n-1}{2}}+2(\frac{2}{n})&\text{if }(\frac{3}{n})=1,\\6S_{\frac{n+1}{2}}-2T_{\frac{n+1}{2}}-4(\frac{2}{n})&\text{if }(\frac{3}{n})=-1,\end{cases}\\=&3\left(1-\left(\frac{3}{n}\right)\right)S_{\bar n}-\frac{3-(\frac{3}{n})}{2}\left(T_{\bar n}-2\left(\frac{6}{n}\right)\right).\end{aligned}$$

When $3\nmid n(n-2r)$, we have $\{\frac{n+3}{6}\}\geqslant\{\frac{r+1}{3}\}$ (otherwise $6\mid n+1$ and $3\mid r-1$, which implies that $3\mid n-2r$), thus $\lfloor\frac{n+1}{6}\rfloor-\lfloor\frac{r+1}{3}\rfloor=\lfloor\frac{n+3}{6}-\frac{r+1}{3}\rfloor=\lfloor\frac{n-2r+1}{6}\rfloor$ and hence

$$(-1)^{\lfloor\frac{r+1}{3}\rfloor}\left(\frac{3}{n}\right)=(-1)^{\lfloor\frac{n-2r+1}{6}\rfloor}=\begin{cases}1&\text{if }n-2r\equiv\pm1\pmod{12},\\-1&\text{if }n-2r\equiv\pm5\pmod{12}.\end{cases}$$

In view of the above, (3.9) can be easily verified. $\square$

Proof of Theorem 3. Applying (3.9) with $r=3,-n$ we obtain that

$$\begin{aligned}&\sum_{k=1}^{n-1}\frac{(-1)^{\frac{k(n-k)}{2}}}{k}\binom{n-1}{k-1}([6\mid k-3]-[6\mid k+n])\\=&-\left(\frac{6}{n}\right)\frac{T_{\bar n}-2(\frac{6}{n})}{6n}+\left(\frac{2}{n}\right)\frac{S_{\bar n}}{n}+\left(\frac{6}{n}\right)\frac{T_{\bar n}-2(\frac{6}{n})}{6n}=\left(\frac{2}{n}\right)\frac{S_{\bar n}}{n}.\end{aligned}$$

If $k\equiv3\pmod 6$ then $\frac{k(n-k)}{2}\equiv\frac{n-k}{2}\equiv\frac{n-1}{2}-\frac{k+3}{6}\pmod 2$; if $k\equiv-n\pmod 6$ then $\frac{k(n-k)}{2}\equiv\frac{k+n}{2}-k\equiv\frac{k+n}{6}-1\pmod 2$. Thus (1.12) follows.

Now suppose that p is a prime greater than 3. Applying (3.9) with $r=3$, we find that p divides $T_{\bar p}-2(\frac{6}{p})$. Observe that

$$\begin{aligned}12S_{\bar p}^2=&\left((2+\sqrt3)^{\bar p}+(2-\sqrt3)^{\bar p}\right)^2-4(2+\sqrt3)^{\bar p}(2-\sqrt3)^{\bar p}\\=&T_{\bar p}^2-4=\left(T_{\bar p}-2\left(\frac{6}{p}\right)\right)^2+4\left(\frac{6}{p}\right)\left(T_{\bar p}-2\left(\frac{6}{p}\right)\right).\end{aligned}$$

So $p \mid S_{\bar p}$ and $p^2 \mid T_{\bar p} - 2(\frac{6}{p})$.

Notice that

$$\begin{aligned}
&6S_{\frac{p+1}{2}} - T_{\frac{p+1}{2}} = 6S_{\frac{p-1}{2}} + T_{\frac{p-1}{2}} = 6S_{\bar p} + \left(\frac{3}{p}\right)T_{\bar p}\\
=&\frac{6}{2\sqrt3}\left((2+\sqrt3)^{\frac{p-1}{2}} - (2-\sqrt3)^{\frac{p-1}{2}}\right) + (2+\sqrt3)^{\frac{p-1}{2}} + (2-\sqrt3)^{\frac{p-1}{2}}\\
=&(1+\sqrt3)(2+\sqrt3)^{\frac{p-1}{2}} + (1-\sqrt3)(2-\sqrt3)^{\frac{p-1}{2}}\\
=&2^{-\frac{p-1}{2}}\left((1+\sqrt3)^{1+2\cdot\frac{p-1}{2}} + (1-\sqrt3)^{1+2\cdot\frac{p-1}{2}}\right)\\
=&2^{-\frac{p-1}{2}}\sum_{\substack{k=0\\2\mid k}}^{p}\binom{p}{k}\left((\sqrt3)^k + (-\sqrt3)^k\right)\\
=&2\cdot 2^{-\frac{p-1}{2}} + 2^{-\frac{p-1}{2}}p\sum_{k=1}^{\frac{p-1}{2}}\frac{2\cdot 3^k}{2k}\binom{p-1}{2k-1}.
\end{aligned}$$

Therefore

$$\begin{aligned}
&-\sum_{k=1}^{\frac{p-1}{2}}\frac{3^k}{k} \equiv \sum_{k=1}^{\frac{p-1}{2}}\frac{3^k}{k}\binom{p-1}{2k-1} = \frac{1}{p}\left(2^{\frac{p-1}{2}}\left(6S_{\bar p} + \left(\frac{3}{p}\right)T_{\bar p}\right) - 2\right)\\
\equiv&6\cdot 2^{\frac{p-1}{2}}\frac{S_{\bar p}}{p} + \left(\frac{3}{p}\right)T_{\bar p}\frac{2^{\frac{p-1}{2}} - (\frac{2}{p})}{p} + \left(\frac{6}{p}\right)\frac{T_{\bar p} - 2(\frac{6}{p})}{p}\\
\equiv&6\left(\frac{2}{p}\right)\frac{S_{\bar p}}{p} + \left(2^{\frac{p-1}{2}} + \left(\frac{2}{p}\right)\right)\frac{2^{\frac{p-1}{2}} - (\frac{2}{p})}{p} = 6\left(\frac{2}{p}\right)\frac{S_{\bar p}}{p} + q_p(2) \pmod p.
\end{aligned}$$

Taking $r = 0, 3$ in (3.9) we then have

$$\sum_{0<k<p/6}\frac{(-1)^k}{6k}\binom{p-1}{6k-1} - 2\left(\frac{2}{p}\right)\frac{2^{\frac{p-1}{2}} - (\frac{2}{p})}{6p} = \left(\frac{2}{p}\right)\frac{S_{\bar p}}{p} + 2\left(\frac{6}{p}\right)\frac{T_{\bar p} - 2(\frac{6}{p})}{6p}$$

and

$$\sum_{k=1}^{\lfloor\frac{p+1}{6}\rfloor}\frac{(-1)^{\frac{p-1}{2}-k}}{6k-3}\binom{p-1}{6k-4} - 2\left(\frac{2}{p}\right)\frac{2^{\frac{p-1}{2}} - (\frac{2}{p})}{6p} = -\left(\frac{6}{p}\right)\frac{T_{\bar p} - 2(\frac{6}{p})}{6p}.$$

Consequently,

$$-\frac{1}{6}\sum_{0<k<p/6}\frac{(-1)^k}{k} - \frac{1}{6}q_p(2) \equiv \left(\frac{2}{p}\right)\frac{S_{\bar p}}{p} \pmod p$$

and (1.14) holds. This completes the proof. □

Remark 3.3. Let $p>3$ be a prime and $\bar p=(p-(\frac{3}{p}))/2$. By the proof of Theorem 3,

$$\left(\frac{6}{p}\right)\frac{T_{\bar p}-2(\frac{6}{p})}{p^2}=3\left(\frac{S_{\bar p}}{p}\right)^2-\left(\frac{T_{\bar p}-2(\frac{6}{p})}{2p}\right)^2\equiv 3\left(\frac{S_{\bar p}}{p}\right)^2 \pmod{p^2}.$$

Since $2S_{\frac{p-1}{2}}=4S_{\frac{p+1}{2}}-T_{\frac{p+1}{2}}$ and $2S_{\frac{p+1}{2}}=8S_{\frac{p-1}{2}}-2S_{\frac{p-3}{2}}=4S_{\frac{p-1}{2}}+T_{\frac{p-1}{2}}$,

$$\frac{S_{(p+(\frac{3}{p}))/2}-(\frac{2}{p})}{p}=2\frac{S_{\bar p}}{p}+\left(\frac{3}{p}\right)\frac{T_{\bar p}-2(\frac{6}{p})}{2p}\equiv 2\frac{S_{\bar p}}{p} \pmod p.$$

As $S_{p-(\frac{3}{p})}=S_{2\bar p}=S_{\bar p}T_{\bar p}$, we have

$$\frac{S_{p-(\frac{3}{p})}}{p}-2\left(\frac{6}{p}\right)\frac{S_{\bar p}}{p}=\frac{S_{\bar p}}{p}\cdot\frac{T_{\bar p}-2(\frac{6}{p})}{p^2}p^2\equiv 3\left(\frac{6}{p}\right)\left(\frac{S_{\bar p}}{p}\right)^3p^2 \pmod{p^4}.$$

Note also that

$$\frac{S_p-(\frac{3}{p})}{p}\equiv 4\left(\frac{6}{p}\right)\frac{S_{\bar p}}{p} \pmod p$$

because $S_p=S_{\frac{p+1}{2}}^2-S_{\frac{p-1}{2}}^2=(\frac{3}{p})(S_{(p+(\frac{3}{p}))/2}^2-S_{\bar p}^2)\equiv(\frac{3}{p})((\frac{2}{p})+2S_{\bar p})^2 \pmod{p^2}$.

In [SS] Z.-H. Sun and Z.-W. Sun employed the sum $\left[{p\atop r}\right]_{10}$ to determine when $p\mid F_{(p-1)/4}$ if p is a prime with $p\equiv 1\pmod 4$.

Let $p>3$ be a prime. We assert that

$$(3.10)\quad p\mid S_{\lfloor\frac{p+1}{4}\rfloor}\iff p\equiv 1,19 \pmod{24};\ \ p\mid T_{\lfloor\frac{p+1}{4}\rfloor}\iff p\equiv 7,13 \pmod{24}.$$

Put $n=\lfloor\frac{p+1}{4}\rfloor$. Clearly

$$T_{2n}=\Big((2+\sqrt3)^n-(2-\sqrt3)^n\Big)^2+2(2+\sqrt3)^n(2-\sqrt3)^n=12S_n^2+2.$$

If $p\equiv 5,11 \pmod{12}$, then $p+(\frac{3}{p})=4n$, hence $p\nmid S_n$ and $p\nmid T_n$ because $S_nT_n=S_{2n}\equiv(\frac{2}{p}) \pmod p$ by Remark 3.3. When $p\equiv 1,7 \pmod{12}$, clearly $4n=p-(\frac{3}{p})=2\bar p$, therefore

$$\begin{aligned}p\mid S_n&\iff T_{\bar p}=12S_n^2+2\equiv 2 \pmod p,\ \text{i.e. } p\ \Big|\ 2\left(\frac{6}{p}\right)-2\\&\iff\left(\frac{2}{p}\right)=\left(\frac{3}{p}\right),\ \text{i.e. } p\equiv 1,19 \pmod{24},\end{aligned}$$

and

$$T_n=S_{\bar p}/S_n\equiv 0 \pmod p\iff p\nmid S_n\iff p\equiv 7,13 \pmod{24}$$

since $S_{\bar p}\equiv 0\pmod p$ and $T_n^2-12S_n^2=4\not\equiv 0 \pmod p$.

Corollary 3.3. *Let $p>3$ be a prime. Let $r\in\mathbb{Z}$,*

$$K_p(r,12)=\sum_{\substack{0<k<p\\ 12\mid k-rp}}\frac{1}{k}\quad\text{and}\quad \varepsilon_r=\begin{cases}1 & \text{if } r\equiv 0,1 \pmod 6,\\ -1 & \text{if } 3\mid r+1,\\ 0 & \text{otherwise.}\end{cases}\tag{3.11}$$

Then

$$\begin{aligned}(-1)^{r-1}K_p(r,12)\equiv&\frac{2+(-1)^{\lfloor r/2\rfloor}}{12}q_p(2)+[3\nmid r+1](-1)^{\lfloor r/3\rfloor}\frac{q_p(3)}{8}\\ &+\varepsilon_r(-1)^{\lfloor r/2\rfloor}\left(\frac{2}{p}\right)\frac{S_{(p-(\frac{3}{p}))/2}}{2p}\pmod p.\end{aligned}\tag{3.12}$$

Proof. By Theorem 3.2,

$$\begin{aligned}&[6\mid rp]+[6\mid p-rp]+p\sum_{\substack{0<k<p\\ 6\mid k-rp}}\frac{1}{k}\binom{p-1}{k-1}\\ =&\begin{bmatrix}p\\ rp\end{bmatrix}_6=\frac{2^{p-1}-1}{3}+\frac{[3\nmid p+rp]}{2}\left((-1)^{\lfloor\frac{p-2rp+1}{6}\rfloor}3^{\frac{p-1}{2}}+1\right).\end{aligned}$$

Since $\binom{p-1}{l}\equiv(-1)^l \pmod p$ for $l=0,1,\cdots,p-1$, and

$$q_p(a)=\left(a^{\frac{p-1}{2}}+\left(\frac{a}{p}\right)\right)\frac{a^{\frac{p-1}{2}}-(\frac{a}{p})}{p}\equiv 2\left(\frac{a}{p}\right)\frac{a^{\frac{p-1}{2}}-(\frac{a}{p})}{p}\pmod p$$

for any integer $a\not\equiv 0 \pmod p$, we have

$$\begin{aligned}&\sum_{\substack{0<k<p\\ 6\mid k-rp}}\frac{(-1)^{k-1}}{k}-\frac{q_p(2)}{3}\\ \equiv&\frac{[3\nmid r+1]}{2p}\left((-1)^{\lfloor\frac{p+1-2rp}{6}\rfloor}3^{\frac{p-1}{2}}+1-2[r\equiv 0,1 \pmod 6]\right)\\ \equiv&\frac{[3\nmid r+1]}{2p}(-1)^{\lfloor\frac{r}{3}\rfloor}\left((-1)^{\lfloor\frac{p+1}{6}\rfloor}3^{\frac{p-1}{2}}-1\right)\equiv[3\nmid r+1](-1)^{\lfloor\frac{r}{3}\rfloor}\frac{q_p(3)}{4}\pmod p.\end{aligned}$$

Set $\bar p=(p-(\frac{3}{p}))/2$. As $T_{\bar p}\equiv 2(\frac{6}{p}) \pmod{p^2}$, Theorem 3.3 implies that

$$\begin{aligned}&\sum_{\substack{0<k<p\\ 6\mid k-rp}}\frac{(-1)^{\frac{k(p-k)}{6}}}{k}(-1)^{k-1}-\frac{q_p(2)}{6}\\ \equiv&\left(\frac{2}{p}\right)\frac{S_{\bar p}}{p}\left(\frac{1+(-1)^{\lfloor\frac{rp+1}{3}\rfloor}}{2}[3\nmid r+1]-[3\mid r+1]\right)\pmod p.\end{aligned}$$

Clearly $\lfloor\frac{rp+1}{3}\rfloor\equiv\lfloor\frac{r}{3}\rfloor \pmod 2$ if $3\nmid r+1$, so

$$\frac{1+(-1)^{\lfloor\frac{rp+1}{3}\rfloor}}{2}[3\nmid r+1]-[3\mid r+1]=\left[2\mid\left\lfloor\frac{r}{3}\right\rfloor\ \&\ 3\nmid r+1\right]-[3\mid r+1]=\varepsilon_r.$$

If $k\equiv rp \pmod 6$, then $\frac{k(p-k)}{2}\equiv\frac{k-rp}{2}p+\frac{rp(p-rp)}{2}\equiv\frac{k-rp}{6}-\lfloor\frac{r}{2}\rfloor \pmod 2$. Thus

$$\begin{aligned}2(-1)^{rp-1}K_p(r,12)&=\sum_{\substack{0<k<p\\6\mid k-rp}}\frac{(-1)^{k-1}}{k}\left(1+(-1)^{\lfloor\frac{r}{2}\rfloor+\frac{k(p-k)}{2}}\right)\\ &\equiv\frac{q_p(2)}{3}+[3\nmid r+1](-1)^{\lfloor\frac{r}{3}\rfloor}\frac{q_p(3)}{4}+(-1)^{\lfloor\frac{r}{2}\rfloor}\left(\frac{q_p(2)}{6}+\varepsilon_r\left(\frac{2}{p}\right)\frac{S_{\bar p}}{p}\right)\pmod p,\end{aligned}$$

which is equivalent to (3.12). $\square$

Remark 3.4. Let $p>3$ be a prime and r be an integer. Clearly

$$\sum_{\frac{r}{12}p<j<\frac{r+1}{12}p}\frac{1}{j}=\sum_{\substack{rp<l<(r+1)p\\12\mid l}}\frac{12}{l}=\sum_{\substack{k=1\\12\mid k+rp}}^{p-1}\frac{12}{k+rp}\equiv 12K_p(-r,12)\pmod p.$$

Thus, for $a=1,5,7,11$ we can also deduce the congruence

$$B_{p-1}\left(\frac{a}{12}\right)-B_{p-1}\equiv\left(\frac{3}{a}\right)\frac{3}{p}S_{p-(\frac{3}{p})}+3q_p(2)+\frac{3}{2}q_p(3)\pmod p$$

given in [GS] from our Corollary 3.3, where $B_{p-1}=B_{p-1}(0)$, and $B_{p-1}(x)$ denotes the Bernoulli polynomial of degree $p-1$. If $0\leqslant r<12$ then we can determine $\binom{p-1}{\lfloor\frac{r}{12}p\rfloor}$ mod p^2 since

$$(-1)^{\lfloor\frac{r}{12}p\rfloor}\binom{p-1}{\lfloor\frac{r}{12}p\rfloor}\equiv 1-p\sum_{0<j<\frac{r}{12}p}\frac{1}{j}\pmod{p^2}.$$

The reader may consult [Su2] for $\prod_{0<k<n/2}\binom{p-1}{\lfloor\frac{k}{n}p\rfloor}$ mod p^2 where n is any positive integer not divisible by p.

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