# Some new binomial sums related to the Catalan triangle

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**Abstract**

In this paper, we derive many new identities on the classical Catalan triangle $\mathcal{C} = (C_{n,k})_{n\geqslant k\geqslant 0}$, where $C_{n,k} = \frac{k+1}{n+1}\binom{2n-k}{n}$ are the well-known ballot numbers. The first three types are based on the determinant and the fourth is relied on the permanent of a square matrix. It not only produces many known and new identities involving Catalan numbers, but also provides a new viewpoint on combinatorial triangles.

**Keywords:** Catalan number, ballot number, Catalan triangle.

## 1 Introduction

In 1976, by a nice interpretation in terms of pairs of paths on a lattice $\mathbb{Z}^2$, Shapiro [44] first introduced the Catalan triangle $\mathcal{B} = (B_{n,k})_{n\geqslant k\geqslant 0}$ with $B_{n,k} = \frac{k+1}{n+1}\binom{2n+2}{n-k}$ and obtained

$$\sum_{k=0}^{n} B_{n,k} = (2n+1)C_n,$$

$$\sum_{k=0}^{\min\{m,n\}} B_{n,k}B_{m,k} = C_{m+n+1}, \tag{1}$$

where $C_n = \frac{1}{n+1}\binom{2n}{n} = \frac{1}{2n+1}\binom{2n+1}{n}$ is the $n$th Catalan number. Table 1.1 illustrates this triangle for small $n$ and $k$ up to 6. Note that the entries in the first column of the Catalan triangle $\mathcal{B}$ are indeed the Catalan numbers $B_{n,0} = C_{n+1}$, which is the reason why $\mathcal{B}$ is called the Catalan triangle.

| $n/k$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|---|
| 0 | 1 | | | | | | |
| 1 | 2 | 1 | | | | | |
| 2 | 5 | 4 | 1 | | | | |
| 3 | 14 | 14 | 6 | 1 | | | |
| 4 | 42 | 48 | 27 | 8 | 1 | | |
| 5 | 132 | 165 | 110 | 44 | 10 | 1 | |
| 6 | 429 | 572 | 429 | 208 | 65 | 12 | 1 |

Table 1.1. The first values of $B_{n,k}$.

Since then, much attentions have been paid to the Catalan triangle and its generalizations. In 1979, Eplett [20] deduced the alternating sum in the $n$th row of $\mathcal{B}$, namely,

$$\sum_{k=0}^{n}(-1)^k B_{n,k} = C_n.$$

In 1981, Rogers [42] proved that a generalization of Eplett's identity holds in any renewal array. In 2008, Gutiérrez et al. [27] established three summation identities and proposed as one of the open problems to evaluate the moments $\Omega_m = \sum_{k=0}^{n}(k+1)^m B_{n,k}^2$. Using the WZ-theory (see [40, 52]), Miana and Romero computed $\Omega_m$ for $1 \leqslant m \leqslant 7$. Later, based on the symmetric functions and inverse series relations with combinatorial computations, Chen and Chu [13] resolved this problem in general. By using the Newton interpolation formula, Guo and Zeng [26] generalized the recent identities on the Catalan triangle $\mathcal{B}$ obtained by Miana and Romero [37] as well as those of Chen and Chu [13].

Some alternating sum identities on the Catalan triangle $\mathcal{B}$ were established by Zhang and Pang [53], who showed that the Catalan triangle $\mathcal{B}$ can be factorized as the product of the Fibonacci matrix and a lower triangular matrix, which makes them build close connections among $C_n$, $B_{n,k}$ and the Fibonacci numbers. Motivated by a matrix identity related to the Catalan triangle $\mathcal{B}$ [46], Chen et al. [14] derived many nice matrix identities on weighted partial Motzkin paths.

| $n/k$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|---|
| 0 | 1 | | | | | | |
| 1 | 1 | 1 | | | | | |
| 2 | 2 | 3 | 1 | | | | |
| 3 | 5 | 9 | 5 | 1 | | | |
| 4 | 14 | 28 | 20 | 7 | 1 | | |
| 5 | 42 | 90 | 75 | 35 | 9 | 1 | |
| 6 | 132 | 297 | 275 | 154 | 54 | 11 | 1 |

Table 1.2. The first values of $A_{n,k}$.

Aigner [3], in another direction, came up with the admissible matrix, a kind of generalized Catalan triangle, and discussed its basic properties. The numbers in the first column of the admissible matrix are called Catalan-like numbers, which are investigated in [5] from combinatorial views. The admissible matrix $\mathcal{A} = (A_{n,k})_{n \geqslant k \geqslant 0}$ associated to the Catalan triangle $\mathcal{B}$ is defined by $A_{n,k} = \frac{2k+1}{2n+1}\binom{2n+1}{n-k}$, which is considered by Miana and Romero [38] by evaluating the moments $\Phi_m = \sum_{k=0}^{n}(2k+1)^m A_{n,k}^2$. Table 1.2 illustrates this triangle for small $n$ and $k$ up to 6.

The interlaced combination of the two triangles $\mathcal{A}$ and $\mathcal{B}$ forms the third triangle $\mathcal{C} = (C_{n,k})_{n \geqslant k \geqslant 0}$, defined by the ballot numbers

$$C_{n,k} = \frac{k+1}{2n-k+1}\binom{2n-k+1}{n-k} = \frac{k+1}{n+1}\binom{2n-k}{n}.$$

The triangle $\mathcal{C}$ is also called the "Catalan triangle" in the literature, despite it has the most-standing form $\mathcal{C}' = (C_{n,n-k})_{n \geqslant k \geqslant 0}$ first discovered in 1961 by Forder [24], see for examples [1, 6, 9, 23, 30, 38, 47]. Table 1.3 illustrates this triangle for small $n$ and $k$ up to 7.

| $n/k$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|---|
| 0 | 1 | | | | | | | |
| 1 | 1 | 1 | | | | | | |
| 2 | 2 | 2 | 1 | | | | | |
| 3 | 5 | 5 | 3 | 1 | | | | |
| 4 | 14 | 14 | 9 | 4 | 1 | | | |
| 5 | 42 | 42 | 28 | 14 | 5 | 1 | | |
| 6 | 132 | 132 | 90 | 48 | 20 | 6 | 1 | |
| 7 | 429 | 429 | 297 | 165 | 75 | 27 | 7 | 1 |

Table 1.3. The first values of $C_{n,k}$.

Clearly,

$$A_{n,k} = C_{n+k,2k} \text{ and } B_{n,k} = C_{n+k+1,2k+1}.$$

Three relations, $C_{n,0} = C_n$, $C_{n+1,1} = C_{n+1}$ and $\sum_{k=0}^{n} C_{n,k} = C_{n+1}$ bring the Catalan numbers and the ballot numbers in correlation [4, 29, 43]. Many properties of the Catalan numbers can be generalized easily to the ballot numbers, which have been studied intensively by Gessel [25]. The combinatorial interpretations of the ballot numbers can be found in [5, 8, 10, 11, 14, 18, 19, 21, 22, 28, 31, 35, 39, 41, 44, 50, 51]. It was shown by Ma [33] that the Catalan triangle $\mathcal{C}$ can be generated by context-free grammars in three variables.

The Catalan triangles $\mathcal{B}$ and $\mathcal{C}$ often arise as examples of the infinite matrix associated to generating trees [7, 12, 34, 36]. In the theory of Riordan arrays [45, 46, 49], much interest has been taken in the three triangles $\mathcal{A}, \mathcal{B}$ and $\mathcal{C}$, see [2, 14, 15, 16, 17, 32, 34, 48, 51]. In fact, $\mathcal{A}, \mathcal{B}$ and $\mathcal{C}$ are Riordan arrays

$$\mathcal{A} = (C(t), tC(t)^2), \quad \mathcal{B} = (C(t)^2, tC(t)^2), \text{ and } \mathcal{C} = (C(t), tC(t)),$$

where $C(t) = \frac{1-\sqrt{1-4t}}{2t}$ is the generating function for the Catalan numbers $C_n$.

Recently, Sun and Ma [51] studied the sums of minors of second order of $\mathcal{M} = (M_{n,k}(x,y))_{n\geqslant k\geqslant 0}$, a class of infinite lower triangles related to weighted partial Motzkin paths, and obtained the following theorem.

**Theorem 1.** *For any integers $n, r \geqslant 0$ and $m \geqslant \ell \geqslant 0$, set $N_r = \min\{n+r+1, m+r-\ell\}$. Then there holds*

$$\sum_{k=0}^{N_r} \det \begin{pmatrix} M_{n,k}(x,y) & M_{m,k+\ell+1}(x,y) \\ M_{n+r+1,k}(x,y) & M_{m+r+1,k+\ell+1}(x,y) \end{pmatrix} = \sum_{i=0}^{r} M_{n+i,0}(x,y) M_{m+r-i,\ell}(y,y). \quad (2)$$

Recall that a *partial Motzkin path* is a lattice path from $(0,0)$ to $(n,k)$ in the $XOY$-plane that does not go below the $X$-axis and consists of up steps $\mathbf{u} = (1,1)$, down steps $\mathbf{d} = (1,-1)$ and horizontal steps $\mathbf{h} = (1,0)$. A *weighted partial Motzkin path* (not the same as stated in [14]) is a partial Motzkin path with the weight assignment that all up steps and down steps are weighted by 1, the horizontal steps are endowed with a weight $x$ if they are lying on $X$-axis, and endowed with a weight $y$ if they are not lying on $X$-axis. The *weight* $w(P)$ of a path $P$ is the product of the weight of all its steps. The *weight* of a set of paths is the sum of the total weights of all the paths. Denote by $M_{n,k}(x,y)$ the weight sum of the set $\mathcal{M}_{n,k}(x,y)$ of all weighted partial Motzkin paths ending at $(n,k)$.

| $n/k$ | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| 0 | $1$ | | | | |
| 1 | $x$ | $1$ | | | |
| 2 | $x^2+1$ | $x+y$ | $1$ | | |
| 3 | $x^3+2x+y$ | $x^2+xy+y^2+2$ | $x+2y$ | $1$ | |
| 4 | $x^4+3x^2+2xy+y^2+2$ | $x^3+x^2y+xy^2+3x+y^3+5y$ | $x^2+2xy+3y^2+3$ | $x+3y$ | $1$ |

Table 1.4. The first values of $M_{n,k}(x,y)$.

Table 1.4 illustrates few values of $M_{n,k}(x,y)$ for small $n$ and $k$ up to 4 [51]. The triangle $\mathcal{M}$ can reduce to $\mathcal{A}, \mathcal{B}$ and $\mathcal{C}$ when the parameters $(x,y)$ are specalized, namely,

$$A_{n,k} = M_{n,k}(1,2),\ B_{n,k} = M_{n,k}(2,2) \text{ and } C_{n,k} = M_{2n-k,k}(0,0).$$

In this paper, we derive many new identities on the Catalan triangle $\mathcal{C}$. The first three types are special cases derived from (2) which are presented in Section 2 and 3 respectively. Section 4 is devoted to the fourth type based on the permanent of a square matrix, and gives a general result on the triangle $\mathcal{M}$ in the $x = y$ case. It not only produces many known and new identities involving Catalan numbers, but also provides a new viewpoint on combinatorial triangles.

## 2 The first two operations on the Catalan triangle

Let $\mathcal{X} = (X_{n,k})_{n\geqslant k\geqslant 0}$ and $\mathcal{Y} = (Y_{n,k})_{n\geqslant k\geqslant 0}$ be the infinite lower triangles defined on the Catalan triangle $\mathcal{C}$ respectively by

$$X_{n,k} = \det\begin{pmatrix} C_{n+k,2k} & C_{n+k,2k+1} \\ C_{n+k+1,2k} & C_{n+k+1,2k+1} \end{pmatrix},$$
$$Y_{n,k} = \det\begin{pmatrix} C_{n+k+1,2k+1} & C_{n+k+1,2k+2} \\ C_{n+k+2,2k+1} & C_{n+k+2,2k+2} \end{pmatrix}.$$

Table 2.1 and 2.2 illustrate these two triangles $\mathcal{X}$ and $\mathcal{Y}$ for small $n$ and $k$ up to 5, together with the row sums. It indicates that the two operations contact the row sums of $\mathcal{X}$ and $\mathcal{Y}$ with the first two columns of $\mathcal{C}$.

| $n/k$ | 0 | 1 | 2 | 3 | 4 | 5 | *row sums* |
|---|---|---|---|---|---|---|---|
| 0 | 1 | | | | | | $1 = 1^2$ |
| 1 | 0 | 1 | | | | | $1 = 1^2$ |
| 2 | 0 | 3 | 1 | | | | $4 = 2^2$ |
| 3 | 0 | 14 | 10 | 1 | | | $25 = 5^2$ |
| 4 | 0 | 84 | 90 | 21 | 1 | | $196 = 14^2$ |
| 5 | 0 | 594 | 825 | 308 | 36 | 1 | $1764 = 42^2$ |

Table 2.1. The first values of $X_{n,k}$.

| $n/k$ | 0 | 1 | 2 | 3 | 4 | 5 | *row sums* |
|---|---|---|---|---|---|---|---|
| 0 | 1 | | | | | | $1 = 1 \times 1$ |
| 1 | 1 | 1 | | | | | $2 = 1 \times 2$ |
| 2 | 3 | 6 | 1 | | | | $10 = 2 \times 5$ |
| 3 | 14 | 40 | 15 | 1 | | | $70 = 5 \times 14$ |
| 4 | 84 | 300 | 175 | 28 | 1 | | $588 = 14 \times 42$ |
| 5 | 594 | 2475 | 1925 | 504 | 45 | 1 | $5544 = 42 \times 132$ |

Table 2.2. The first values of $Y_{n,k}$.

More generally, we obtain the first result which is a consequence of Theorem 1.1.

**Theorem 2.** *For any integers $m \geqslant \ell \geqslant 0$ and $n \geqslant 0$, set $N = \min\{n+1, m-\ell\}$. Then there hold*

$$\sum_{k=0}^{N} \det\begin{pmatrix} C_{n+k,2k} & C_{m+k,2k+\ell+1} \\ C_{n+k+1,2k} & C_{m+k+1,2k+\ell+1} \end{pmatrix} = C_n C_{m,\ell}, \tag{3}$$
$$\sum_{k=0}^{N} \det\begin{pmatrix} C_{n+k+1,2k+1} & C_{m+k+1,2k+\ell+2} \\ C_{n+k+2,2k+1} & C_{m+k+2,2k+\ell+2} \end{pmatrix} = C_{n+1} C_{m,\ell}, \tag{4}$$

*or equivalently,*

$$C_{m,\ell}C_n = \sum_{k=0}^{N} \frac{(2k+1)(2k+\ell+2)\lambda_{n,k}(m,\ell)}{(2n+1)_3(2m-\ell)_3}\binom{2n+3}{n-k+1}\binom{2m-\ell+2}{m-k-\ell}, \tag{5}$$

$$C_{m,\ell}C_{n+1} = \sum_{k=0}^{N} \frac{(2k+2)(2k+\ell+3)\mu_{n,k}(m,\ell)}{(2n+2)_3(2m-\ell+1)_3}\binom{2n+4}{n-k+1}\binom{2m-\ell+3}{m-k-\ell}, \tag{6}$$

*where* $\lambda_{n,k}(m,\ell) = (2m-\ell)(2m-\ell+1)(n-k+1)(n+k+2) - (2n+1)(2n+2)(m-\ell-k)(m+k+2)$, $\mu_{n,k}(m,\ell) = (2m-\ell+1)(2m-\ell+2)(n-k+1)(n+k+3) - (2n+2)(2n+3)(m-\ell-k)(m+k+3)$ *and* $(a)_k = a(a+1)\cdots(a+k-1)$ *for* $k \geqslant 1$.

*Proof.* Setting $(x,y) = (0,0)$ and $r = 1$, replacing $n, m$ respectively by $2n, 2m-\ell-1$ in (2), together with the relation $C_{n,k} = M_{2n-k,k}(0,0)$, where

$$M_{n,k}(0,0) = \begin{cases} \frac{k+1}{n+1}\binom{n+1}{\frac{n-k}{2}}, & \text{if } n-k \text{ even,} \\ 0, & \text{otherwise.} \end{cases} \qquad \text{[51, Example (iv)]} \tag{7}$$

we can get (3). After some simple computation, one can easily derive (5) from (3).

Similarly, the case $(x,y) = (0,0)$ and $r = 1$, after replacing $n, m$ respectively by $2n+1, 2m-\ell$ in (2), reduces to (4), which, by some routine simplification, leads to (6). □

Taking $m = n$ in (5) and (6), we have

$$\begin{aligned} \lambda_{n,k}(n,\ell) &= (\ell+1)(n+k+2)(2k(2n+1)+\ell(n-k+1)), \\ \mu_{n,k}(n,\ell) &= (\ell+1)(n+k+3)((2k+1)(2n+2)+\ell(n-k+1)), \end{aligned}$$

which yield the following results.

**Corollary 3.** *For any integers* $n \geqslant \ell \geqslant 0$, *there hold*

$$\frac{1}{2n-\ell+1}\binom{2n-\ell+1}{n-\ell}C_n = \sum_{k=0}^{n-\ell} \frac{(2k+1)(2k+\ell+2)\overline{\lambda}_{n,k}(\ell)}{(2n+1)_2(2n-\ell)_3}\binom{2n+2}{n-k+1}\binom{2n-\ell+2}{n-k-\ell}, \tag{8}$$

$$\frac{1}{2n-\ell+1}\binom{2n-\ell+1}{n-\ell}C_{n+1} = \sum_{k=0}^{n-\ell} \frac{(2k+2)(2k+\ell+3)\overline{\mu}_{n,k}(\ell)}{(2n+2)_2(2n-\ell+1)_3}\binom{2n+3}{n-k+1}\binom{2n-\ell+3}{n-k-\ell}, \tag{9}$$

*where* $\overline{\lambda}_{n,k}(\ell) = 2k(2n+1)+\ell(n-k+1)$ *and* $\overline{\mu}_{n,k}(\ell) = (2k+1)(2n+2)+\ell(n-k+1)$.

It should be pointed out that both (8) and (9) are still correct for any integer $\ell \leqslant -1$ if one notices that they hold trivially for any integer $\ell > n$ and both sides of them can be transferred into polynomials in $\ell$. Specially, after shifting $n$ to $n-1$, the case $\ell = -1$ in (8) and (9) generates the following corollary.

**Corollary 4.** *For any integer $n \geqslant 1$, there hold*

$$\binom{2n}{n} C_{n-1} = \sum_{k=0}^{n} \frac{(2k+1)^2(4nk-n-k)}{(2n-1)^2(2n)(2n+1)} \binom{2n}{n-k}\binom{2n+1}{n-k},$$
$$\binom{2n}{n} C_{n} = \sum_{k=0}^{n} \frac{(k+1)^2(4nk+n+k)}{n(n+1)(2n+1)^2} \binom{2n+1}{n-k}\binom{2n+2}{n-k}.$$

# 3 The third operation on the Catalan triangle

Let $\mathcal{Z} = (Z_{n,k})_{n\geqslant k\geqslant 0}$ be the infinite lower triangle defined on the Catalan triangle $\mathcal{C}$ by

$$Z_{2n,2k} = C_{n+k,2k}C_{n+k+1,2k+1}, \quad Z_{2n,2k+1} = C_{n+k+1,2k+1}C_{n+k+1,2k+2}, \quad (n \geqslant 0),$$
$$Z_{2n-1,2k} = C_{n+k,2k}C_{n+k,2k+1}, \quad Z_{2n-1,2k+1} = C_{n+k,2k+1}C_{n+k+1,2k+2}, \quad (n \geqslant 1).$$

Table 3.1 illustrates the triangle $\mathcal{Z}$ for small $n$ and $k$ up to 6, together with the row sums and the alternating sums of rows. It signifies that the sums and the alternating sums of rows of $\mathcal{Z}$ are in direct contact with the first column of $\mathcal{C}$. Generally, we have the second result which is another consequence of Theorem 1.1.

| $n/k$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | *row sums* | *alternating sums of rows* |
|---|---|---|---|---|---|---|---|---|---|
| 0 | 1 | | | | | | | 1 | $1 = 1^2$ |
| 1 | 1 | 1 | | | | | | 2 | 0 |
| 2 | 2 | 2 | 1 | | | | | 5 | $1 = 1^2$ |
| 3 | 4 | 6 | 3 | 1 | | | | 14 | 0 |
| 4 | 10 | 15 | 12 | 4 | 1 | | | 42 | $4 = 2^2$ |
| 5 | 25 | 45 | 36 | 20 | 5 | 1 | | 132 | 0 |
| 6 | 70 | 126 | 126 | 70 | 30 | 6 | 1 | 429 | $25 = 5^2$ |

Table 3.1. The first values of $Z_{n,k}$.

**Theorem 5.** *For any integers $m, n \geqslant 0$, let $p = m - n + 1$. Then there hold*

$$\sum_{k=0}^{\min\{m,n\}} C_{m+k+1,2k+1}(C_{n+k,2k} + C_{n+k+1,2k+2}) = C_{m+n+1}, \tag{10}$$

$$\sum_{k=0}^{\min\{m,n\}} C_{m+k+1,2k+1}(C_{n+k,2k} - C_{n+k+1,2k+2}) = \begin{cases} \sum_{i=0}^{p-1} C_{n+i}C_{m-i}, & if\ p \geqslant 1, \\ 0, & if\ p = 0, \\ -\sum_{i=1}^{|p|} C_{n-i}C_{m+i}, & if\ p \leqslant -1, \end{cases} \tag{11}$$

*Proof.* The identity (10) is equivalent to (1), if one notices that

$$C_{m+k+1,2k+1} = B_{m,k}, B_{n,k} = C_{n+k+1,2k+1} = C_{n+k,2k} + C_{n+k+1,2k+2}.$$

For the case $p = m - n + 1 \geqslant 0$ in (11), setting $(x, y) = (0, 0)$ in (2), together with the relation $C_{n,k} = M_{2n-k,k}(0, 0)$ and (7), we have

$$\begin{aligned}
&\sum_{k=0}^{\min\{m,n\}} C_{m+k+1,2k+1}(C_{n+k,2k} - C_{n+k+1,2k+2}) \\
&= \sum_{k=0}^{n} \left\{ \det \begin{pmatrix} C_{n+k,2k} & 0 \\ 0 & C_{n+p+k,2k+1} \end{pmatrix} + \det \begin{pmatrix} 0 & C_{n+k+1,2k+2} \\ C_{n+p+k,2k+1} & 0 \end{pmatrix} \right\} \\
&= \sum_{k=0}^{2n} \det \begin{pmatrix} M_{2n,k}(0,0) & M_{2n,k+1}(0,0) \\ M_{2n+2p-1,k}(0,0) & M_{2n+2p-1,k+1}(0,0) \end{pmatrix} \\
&= \sum_{i=0}^{2p-2} M_{2n+i,0}(0,0) M_{2n+2p-i-2,0}(0,0) \\
&= \sum_{i=0}^{p-1} M_{2n+2i,0}(0,0) M_{2n+2p-2i-2,0}(0,0) \\
&= \sum_{i=0}^{p-1} C_{n+i} C_{n+p-i-1} = \sum_{i=0}^{p-1} C_{n+i} C_{m-i},
\end{aligned}$$

as desired.

Similarly, the case $p \leqslant -1$ can be proved, the details are left to interested readers. □

Note that a weighted partial Motzkin path with no horizontal steps is just a partial Dyck path. Then the relation $C_{n,k} = M_{2n-k,k}(0, 0)$ signifies that $C_{n,k}$ counts the set $\mathcal{C}_{n,k}$ of partial Dyck paths of length $2n - k$ from $(0, 0)$ to $(2n - k, k)$ [35]. Such partial Dyck paths have exactly $n$ up steps and $n - k$ down steps. For any step, we say that it is at level $i$ if the $y$-coordinate of its end point is $i$. If $P = L_1 L_2 \dots L_{2n-k-1} L_{2n-k} \in \mathcal{C}_{n,k}$, denote by $\overline{P} = \overline{L}_{2n-k} \overline{L}_{2n-k-1} \dots \overline{L}_2 \overline{L}_1$ the reverse path of $P$, where $\overline{L}_i = \mathbf{u}$ if $L_i = \mathbf{d}$ and $\overline{L}_i = \mathbf{d}$ if $L_i = \mathbf{u}$.

For $k = 0$, a partial Dyck path is an (ordinary) Dyck path. For any Dyck path $P$ of length $2n + 2m + 2$, its $(2n + 1)$-th step $L$ (along the path) must end at odd level, say $2k + 1$ for some $k \geqslant 0$, then $P$ can be uniquely partitioned into $P = P_1 L P_2$, where $(P_1, \overline{P_2}) \in \mathcal{C}_{n+k,2k} \times \mathcal{C}_{m+k+1,2k+1}$ if $L = \mathbf{u}$ and $(P_1, \overline{P_2}) \in \mathcal{C}_{n+k+1,2k+2} \times \mathcal{C}_{m+k+1,2k+1}$ if $L = \mathbf{d}$. Hence, the cases $p = 0, 1$ and $2$, i.e., $m = n - 1, n$ and $n + 1$ in (11) produce the following corollary.

**Corollary 6.** *For any integer $n \geqslant 0$, according to the $(2n + 1)$-th step $\mathbf{u}$ or $\mathbf{d}$, we have*

*(a) The number of Dyck paths of length $4n$ is bisected;*

*(b) The parity of the number of Dyck paths of length $4n + 2$ is $C_n^2$;*

*(c) The parity of the number of Dyck paths of length $4n + 4$ is $2C_n C_{n+1}$.*

# 4 The fourth operation on the Catalan triangle

Let $\mathcal{W} = (W_{n,k})_{n\geqslant k\geqslant 0}$ be the infinite lower triangle defined on the Catalan triangle by

$$W_{2n,k} = \operatorname{per}\begin{pmatrix} C_{n+k,2k} & C_{n+k,2k+1} \\ C_{n+k+1,2k} & C_{n+k+1,2k+1} \end{pmatrix},$$
$$W_{2n+1,k} = \operatorname{per}\begin{pmatrix} C_{n+k,2k} & C_{n+k,2k+1} \\ C_{n+k+2,2k} & C_{n+k+2,2k+1} \end{pmatrix},$$

where $\operatorname{per}(A)$ denotes the permanent of a square matrix $A$. Table 4.1 illustrates the triangle $\mathcal{W}$ for small $n$ and $k$ up to 8, together with the row sums.

| $n/k$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | *row sums* |
|---|---|---|---|---|---|---|---|---|---|---|
| 0 | 1 | | | | | | | | | 1 |
| 1 | 2 | | | | | | | | | 2 |
| 2 | 4 | 1 | | | | | | | | 5 |
| 3 | 10 | 4 | | | | | | | | 14 |
| 4 | 20 | 21 | 1 | | | | | | | 42 |
| 5 | 56 | 70 | 6 | | | | | | | 132 |
| 6 | 140 | 238 | 50 | 1 | | | | | | 429 |
| 7 | 420 | 792 | 210 | 8 | | | | | | 1430 |
| 8 | 1176 | 2604 | 990 | 91 | 1 | | | | | 4862 |

Table 4.1. The first values of $W_{n,k}$.

This, in general, motivates us to consider the permanent operation on the triangle $\mathcal{M} = (M_{n,k}(x,y))_{n\geqslant k\geqslant 0}$. Recall that $M_{n,k}(x,y)$ is the weight sum of the set $\mathcal{M}_{n,k}(x,y)$ of all weighted partial Motzkin paths ending at $(n,k)$. For any step of a partial weighted Motzkin path $P$, we say that it is at level $i$ if the $y$-coordinate of its end point is $i$. For $1 \leqslant i \leqslant k$, an up step $\mathbf{u}$ of $P$ at level $i$ is *R-visible* if it is the rightmost up step at level $i$ and there are no other steps at the same level to its right. If $P = L_1L_2\dots L_n \in \mathcal{M}_{n,k}(x,y)$, denote by $\overline{P} = \overline{L}_n \dots \overline{L}_2\overline{L}_1$ the reverse path of $P$, where $\overline{L}_i = \mathbf{u}, \mathbf{h}$ or $\mathbf{d}$ if $L_i = \mathbf{d}, \mathbf{h}$ or $\mathbf{u}$ respectively.

**Theorem 7.** *For any integers $m, n, r$ with $m \geqslant n \geqslant 0$, there holds*

$$\sum_{k=0}^{m} \operatorname{per}\begin{pmatrix} M_{n,k}(y,y) & M_{n+r,k+1}(y,y) \\ M_{m,k}(y,y) & M_{m+r,k+1}(y,y) \end{pmatrix} = M_{m+n+r,1}(y,y) + H_{n,m}(r), \tag{12}$$

*where*

$$H_{n,m}(r) = \begin{cases} \sum_{i=0}^{r-1} M_{n+i,0}(y,y)M_{m+r-i-1,0}(y,y), & if\ r \geqslant 1, \\ 0, & if\ r = 0, \\ -\sum_{i=1}^{|r|} M_{n-i,0}(y,y)M_{m-|r|+i-1,0}(y,y), & if\ r \leqslant -1. \end{cases}$$

*Proof.* We just give the proof of the part when $r \geqslant 0$, the other part can be done similarly and is left to interested readers. Define

$$\begin{aligned}
\mathcal{A}_{n,m,k}^{(r)} &= \{(P,Q)|P \in \mathcal{M}_{n,k}(y,y), Q \in \mathcal{M}_{m+r,k+1}(y,y)\},\\
\mathcal{B}_{n,m,k}^{(r)} &= \{(P,Q)|P \in \mathcal{M}_{n+r,k+1}(y,y), Q \in \mathcal{M}_{m,k}(y,y)\},
\end{aligned}$$

and $\mathcal{C}_{n,m,k}^{(r,i)}$ to be the subset of $\mathcal{A}_{n,m,k}^{(r)}$ such that for any $(P,Q) \in \mathcal{C}_{n,m,k}^{(r,i)}$, $Q = Q_1\mathbf{u}Q_2$ with $Q_1 \in \mathcal{M}_{i,k}(y,y)$ and $Q_2 \in \mathcal{M}_{m+r-i-1,0}(y,y)$ for $k \leqslant i \leqslant r-1$, where the step $\mathbf{u}$ is the last R-visible up step of $Q$. Clearly, $\mathcal{C}_{n,m,k}^{(r,i)}$ is the empty set if $r = 0$.

It is easily to see that the weights of the sets $\mathcal{A}_{n,m,k}^{(r)}$ and $\mathcal{B}_{n,m,k}^{(r)}$ are

$$\begin{aligned}
w(\mathcal{A}_{n,m,k}^{(r)}) &= M_{n,k}(y,y)M_{m+r,k+1}(y,y),\\
w(\mathcal{B}_{n,m,k}^{(r)}) &= M_{n+r,k+1}(y,y)M_{m,k}(y,y).
\end{aligned}$$

For $0 \leqslant i < r$, the weight of the set $\bigcup_{k=0}^{i}\mathcal{C}_{n,m,k}^{(r,i)}$ is $M_{n+i,0}(y,y)M_{m+r-i-1,0}(y,y)$. This claim can be verified by the following argument. For any $(P,Q) \in \mathcal{C}_{n,m,k}^{(r,i)}$, we have $Q = Q_1\mathbf{u}Q_2$ as mentioned above with $Q_1 \in \mathcal{M}_{i,k}(y,y)$ and $Q_2 \in \mathcal{M}_{m+r-i-1,0}(y,y)$, then $P\overline{Q}_1 \in \mathcal{M}_{n+i,0}(y,y)$ such that the last $(i+1)$-th step of $P\overline{Q}_1$ is at level $k$. Summing $k$ for $0 \leqslant k \leqslant i$, all $P\overline{Q}_1 \in \mathcal{M}_{n+i,0}(y,y)$ contribute the total weight $M_{n+i,0}(y,y)$ and all $Q_2 \in \mathcal{M}_{m+r-i-1,0}(y,y)$ contribute the total weight $M_{m+r-i-1,0}(y,y)$. Hence, $w(\bigcup_{k=0}^{i}\mathcal{C}_{n,m,k}^{(r,i)}) = M_{n+i,0}(y,y)M_{m+r-i-1,0}(y,y)$, and then

$$w(\bigcup_{i=0}^{r-1}\bigcup_{k=0}^{i}\mathcal{C}_{n,m,k}^{(r,i)}) = w(\bigcup_{k=0}^{r-1}\bigcup_{i=k}^{r-1}\mathcal{C}_{n,m,k}^{(r,i)}) = \sum_{i=0}^{r-1} M_{n+i,0}(y,y)M_{m+r-i-1,0}(y,y) = H_{n,m}(r).$$

Let $\mathcal{A}_{n,m}^{(r)} = \bigcup_{k=0}^{n+r-1}\mathcal{A}_{n,m,k}^{(r)}$, $\mathcal{B}_{n,m}^{(r)} = \bigcup_{k=0}^{m+r-1}\mathcal{B}_{n,m,k}^{(r)}$ and $\mathcal{C}_{n,m,k}^{(r)} = \bigcup_{i=k}^{r-1}\mathcal{C}_{n,m,k}^{(r,i)}$. To prove (12), it suffices to construct a bijection $\varphi$ between $\mathcal{B}_{n,m}^{(r)}\bigcup\big(\mathcal{A}_{n,m}^{(r)} - \bigcup_{k=0}^{r-1}\mathcal{C}_{n,m,k}^{(r)}\big)$ and $\mathcal{M}_{m+n+r,1}(y,y)$.

For any $(P,Q) \in \mathcal{B}_{n,m,k}^{(r)}$, $P\overline{Q}$ is exactly an element of $\mathcal{M}_{m+n+r,1}(y,y)$. Note that in this case, the first R-visible up step of $P$ is still the one of $P\overline{Q}$ and it is at most the $(n+r)$-th step of $P\overline{Q}$.

For any $(P,Q) \in \mathcal{A}_{n,m,k} - \mathcal{C}_{n,m,k}^{(r)}$, find the last R-visible up step $\mathbf{u}^*$ of $Q$, $Q$ can be uniquely partitioned into $Q = Q_1\mathbf{u}^*Q_2$, where $Q_1 \in \mathcal{M}_{j,k}(y,y)$ for some $j \geqslant r$, then $P\overline{Q}_1\mathbf{u}^*Q_2$ forms an element of $\mathcal{M}_{m+n+r,1}(y,y)$. Note that in this case, the last R-visible up step $\mathbf{u}^*$ of $Q$ is still the one of $P\overline{Q}_1\mathbf{u}^*Q_2$. Moreover, the $\mathbf{u}^*$ step is at least the $(n+r+1)$-th step of $P\overline{Q}_1\mathbf{u}^*Q_2$.

Conversely, for any path in $\mathcal{M}_{m+n+r,1}(y,y)$, it can be partitioned uniquely into $PQ$, where $P \in \mathcal{M}_{n+r,k}(y,y)$ for some $k \geqslant 0$. If the unique R-visible up step $\mathbf{u}^*$ of $PQ$ is lying in $P$, then $k \geqslant 1$ and $(P,\overline{Q}) \in \mathcal{B}_{n,m,k-1}^{(r)}$; If the $\mathbf{u}^*$ step is lying in $Q$, $PQ$ can be repartitioned into $P_1P_2\mathbf{u}^*Q_1$ with $P_1 \in \mathcal{M}_{n,j}(y,y)$ for some $j \geqslant 0$, then $(P_1,\overline{P_2}\mathbf{u}^*Q_1) \in \mathcal{A}_{n,m,j} - \mathcal{C}_{n,m,j}^{(r)}$.

Clearly, the above procedure is invertible. Hence, $\varphi$ is indeed a bijection as desired and (12) is proved. $\square$

**Theorem 8.** *For any integers $m, n, p$ with $m \geqslant n \geqslant 0$, there hold*

$$\sum_{k=0}^{m} \operatorname{per}\begin{pmatrix} C_{n+k,2k} & C_{n+p+k,2k+1} \\ C_{m+k,2k} & C_{m+p+k,2k+1} \end{pmatrix} = C_{m+n+p,1} + F_{n,m}(p), \tag{13}$$

$$\sum_{k=0}^{m} \operatorname{per}\begin{pmatrix} C_{n+k,2k+1} & C_{n+p+k+1,2k+2} \\ C_{m+k,2k+1} & C_{m+p+k+1,2k+2} \end{pmatrix} = C_{m+n+p,1} + F_{n,m}(p), \tag{14}$$

*where*

$$F_{n,m}(p) = \begin{cases} \sum_{i=0}^{p-1} C_{n+i} C_{m+p-i-1}, & if \ \ p \geqslant 1, \\ 0, & if \ \ p = 0, \\ -\sum_{i=1}^{|p|} C_{n-i} C_{m-|p|+i-1}, & if \ \ p \leqslant -1. \end{cases}$$

*Proof.* To prove (13), replacing $n, m, r$ respectively by $2n, 2m, 2p-1$ and setting $(y, y) = (0, 0)$ in (12), together with the relation $C_{n,k} = M_{2n-k,k}(0, 0)$ and (7), we have

$$\begin{aligned}
&\sum_{k=0}^{m} \operatorname{per}\begin{pmatrix} C_{n+k,2k} & C_{n+p+k,2k+1} \\ C_{m+k,2k} & C_{m+p+k,2k+1} \end{pmatrix} \\
&= \sum_{k=0}^{m} \operatorname{per}\begin{pmatrix} M_{2n,2k}(0,0) & M_{2n+2p-1,2k+1}(0,0) \\ M_{2m,2k}(0,0) & M_{2m+2p-1,2k+1}(0,0) \end{pmatrix} \\
&= \sum_{k=0}^{2m} \operatorname{per}\begin{pmatrix} M_{2n,k}(0,0) & M_{2n+2p-1,k+1}(0,0) \\ M_{2m,k}(0,0) & M_{2m+2p-1,k+1}(0,0) \end{pmatrix} \\
&= M_{2n+2m+2p-1,1}(0,0) + H_{2n,2m}(2p-1) \\
&= C_{m+n+p,1} + F_{n,m}(p),
\end{aligned}$$

as desired.

Similarly, replacing $n, m, r$ respectively by $2n-1, 2m-1, 2p+1$ and setting $(y, y) = (0, 0)$ in (12), together with the relation $C_{n,k} = M_{2n-k,k}(0, 0)$ and (7), one can prove (14), the details are left to interested readers. □

The case $p = 0$ in (13) and (14), after some routine computation, gives

**Corollary 9.** *For any integers $m \geqslant n \geqslant 1$, there hold*

$$C_{n+m} = \sum_{k=0}^{n} \frac{(2k+1)(2k+2)(4mn - 2(m+n)k)}{(2n)(2n+1)(2m)(2m+1)} \binom{2n+1}{n-k} \binom{2m+1}{m-k},$$

$$C_{n+m} = \sum_{k=0}^{n-1} \frac{(2k+2)(2k+3)(4mn + 4m + 4n + 2(m+n)k)}{(2n)(2n+1)(2m)(2m+1)} \binom{2n+1}{n-k-1} \binom{2m+1}{m-k-1}.$$

*Specially, the $m = n$ case produces*

$$C_{2n} = \sum_{k=0}^{n-1} \frac{(2k+1)(2k+2)}{n(2n+1)} \binom{2n}{n-k-1} \binom{2n+1}{n-k},$$
$$C_{2n} = \sum_{k=0}^{n-1} \frac{(2k+2)(2k+3)}{n(2n+1)} \binom{2n}{n-k-1} \binom{2n+1}{n-k-1}.$$

The cases $p = 1$ in (13) and $p = -1$ in (14), replacing $n$ and $m$ in (14) by $n+1$ and $m+1$, after some routine computation, yield

**Corollary 10.** *For any integers $m \geqslant n \geqslant 0$, there hold*

$$C_{n+m+1} + C_n C_m = \sum_{k=0}^{n} \frac{(2k+1)(2k+2)\eta_{n,m}(k)}{(2n+1)(2n+2)(2m+1)(2m+2)} \binom{2n+2}{n-k} \binom{2m+2}{m-k}, \quad (15)$$
$$C_{n+m+1} - C_n C_m = \sum_{k=0}^{n} \frac{(2k+2)(2k+3)\rho_{n,m}(k)}{(2n+1)(2n+2)(2m+1)(2m+2)} \binom{2n+2}{n-k} \binom{2m+2}{m-k}, \quad (16)$$

*where $\eta_{n,m}(k) = 4mn + 5(m+n) + 2k(m+n+1) + 4$ and $\rho_{n,m}(k) = 4mn + m + n - 2k(m+n+1)$. Specially, the $m = n$ case produces*

$$C_{2n+1} + C_n^2 = \sum_{k=0}^{n} \frac{(2k+1)(2k+2)}{(n+1)(2n+1)} \binom{2n+1}{n-k} \binom{2n+2}{n-k},$$
$$C_{2n+1} - C_n^2 = \sum_{k=0}^{n} \frac{(2k+2)(2k+3)}{(n+1)(2n+1)} \binom{2n+1}{n-k-1} \binom{2n+2}{n-k}.$$

Subtracting (16) from (15), after some routine simplification, one gets

$$C_n C_m = \sum_{k=0}^{n} \frac{(2k+2)\Big((2k+1)(2k+3)(m+n+1) - (2n+1)(2m+1)\Big)}{(2n+1)(2n+2)(2m+1)(2m+2)} \binom{2n+2}{n-k} \binom{2m+2}{m-k},$$

which, in the case $n = m$, reduces to Corollary 3.7 in [51].

In the case $y = 2$ and $r = p$ in (12), together with the relations $B_{n,k} = M_{n,k}(2,2)$ and $B_{n,0} = C_{n+1}$, similar to the proof of (13), we obtain a result on Shapiro's Catalan triangle.

**Theorem 11.** *For any integers $m, n, p$ with $m \geqslant n \geqslant 0$, there holds*

$$\sum_{k=0}^{m} \operatorname{per} \begin{pmatrix} B_{n,k} & B_{n+p,k+1} \\ B_{m,k} & B_{m+p,k+1} \end{pmatrix} = B_{m+n+p,1} + F_{n+1,m+1}(p). \quad (17)$$

The case $p = 0$ in (17), after some routine computation, generates

**Corollary 12.** *For any integers $m \geqslant n \geqslant 0$, there holds*

$$\frac{2}{n+m+1}\binom{2n+2m+2}{n+m-1} = \sum_{k=0}^{n} \frac{(2k+2)(2k+4)\nu_{n,k}(m)}{(2n+2)_2(2m+2)_2}\binom{2n+3}{n-k}\binom{2m+3}{m-k},$$

*where $\nu_{n,k}(m) = 2mn + 3m + 3n - 6k - 2k^2$. Specially, the $m = n$ case produces*

$$\frac{1}{2n+1}\binom{4n+2}{2n-1} = \sum_{k=0}^{n-1} \frac{(k+1)(k+2)}{(n+1)^2}\binom{2n+2}{n-k-1}\binom{2n+2}{n-k}.$$


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