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Integral Identities for Rational Series Involving Binomial Coefficients

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Abstract. We establish integral identities for series involving binomial coefficients. Using the identities, in some cases, we demonstrate they may be represented in closed form of rational type.

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1. Introduction

In this paper we are interested in the integral representation for series of the form

$$\sum_{n=0}^{\infty} t^n \begin{pmatrix} n+m-1 \\ n \end{pmatrix} Q^{(p)}(an,z) = \int f(a,z,m,t;x)\,dx \tag{1.1}$$

for positive integer parameters $a,\ m,\ z,\ \ p$ and real parameter t. The reciprocal binomial coefficient

$$Q(an,z) = \begin{pmatrix} an+z \\ z \end{pmatrix}^{-1} = \frac{(an)!}{(z+1)(z+2)\cdots(z+an)}$$

and $Q^{(p)}(an,z) = \frac{d^p}{dz^p}\left[Q(an,z)\right]$ is the p^{th} derivative of the reciprocal binomial coefficient. Moreover for specific parameter values we obtain closed form representations of (1.1) that include Harmonic numbers. Some related results for Harmonic number series can be seen in [2, 3, 4], however the results presented in this paper are new. The following special functions are referred to in the sequel and are defined as:

The generalized Zeta function

$$\zeta(q,b) = \sum_{k=0}^{\infty} \frac{1}{(k+b)^q}$$

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where any term with $k+b=0$ is excluded and when $b=0$, we have the Riemann Zeta function $\zeta(q)$. $\psi^{(0)}(z)=\psi(z)$, denotes the Psi, or digamma function, defined by

$$\psi(z)=\frac{d}{dz}\log\Gamma(z)=\frac{\Gamma'(z)}{\Gamma(z)},$$

where the Gamma function $\Gamma(z)=\int\limits_0^\infty u^{z-1}e^{-u}du$, for $\mathbb{R}(z)>0$. The generalized Harmonic number in power α is defined as

$$H_n^{(\alpha)}=\sum_{r=1}^{n}\frac{1}{r^\alpha}.$$

2. The main results

The following Lemma deals with the representation of $Q^{(p)}(an,z)$ and will be useful in the forthcoming theorems.

Lemma 2.1. *Let a be a positive real number, $z\geq 0$, n is a positive integer and let $Q(an,z)=\begin{pmatrix} an+z \\ z\end{pmatrix}^{-1}$ be an analytic function of z then,*

$$Q'(an,z)=\frac{dQ}{dz}=\begin{cases} -Q(an,z)P(an,z), & \text{for } z>0 \text{ or,} \\ -Q(an,z)\left[\psi(z+1+an)-\psi(z+1)\right] & \text{for } z>0 \\ -H_n^{(1)}, \text{ for } z=0 \text{ and } a=1 & \end{cases}, \tag{2.1}$$

and for $\lambda\geq 2$

$$Q^{(\lambda)}(an,z)=\frac{d^\lambda Q}{dz^\lambda}= -\sum_{\rho=0}^{\lambda-1}\begin{pmatrix}\lambda-1\\ \rho\end{pmatrix}Q^{(\rho)}(an,z)P^{(\lambda-1-\rho)}(an,z), \tag{2.2}$$

where

$$P(an,z)=\sum_{r=1}^{an}\frac{1}{r+z}=P^{(0)}(an,z).$$

For $i=1,2,3,...$

$$\begin{aligned} P^{(i)}(an,z)=\frac{d^iP}{dz^i}=\frac{d^i}{dz^i}\left(\sum_{r=1}^{an}\frac{1}{r+z}\right)&=(-1)^i i!\sum_{r=1}^{an}\frac{1}{(r+z)^{i+1}} \\ &=(-1)^i i!\left[\zeta(i+1,z+1+an)-\zeta(i+1,z+1)\right]. \end{aligned} \tag{2.3}$$

Proof. Letting

$$Q(an,z)=\begin{pmatrix} an+z\\ z\end{pmatrix}^{-1}=\frac{\Gamma(z+1)\Gamma(an+1)}{\Gamma(an+z+1)}=\frac{\Gamma(an+1)}{\prod\limits_{r=1}^{an}(r+z)}, \tag{2.4}$$

and taking logs of both sides and differentiating with respect to z we obtain the result (2.1).

Now from (2.1) and for $\lambda\geq 2$,

$$Q^{(\lambda)}(an,z)=\frac{d^\lambda Q}{dz^\lambda}=Q^{(\lambda)}(an,z)=\frac{d^{\lambda-1}}{dz^{\lambda-1}}(-QP)=-\sum_{\rho=0}^{\lambda-1}\begin{pmatrix}\lambda-1\\ \rho\end{pmatrix}Q^{(\rho)}P^{(\lambda-1-\rho)}$$

where $P^{(\lambda-1-\rho)}(an,z)$ is given by (2.3). ∎

Now we can state the following theorem.

Theorem 2.1. *Let a be a positive real number, $|t|\le 1$, $m>0$, $p=1,2,3...$ and integer $j\ge m+1$. Then*

$$\sum_{n=0}^{\infty} t^n \binom{n+m-1}{n} Q^{(p)}(an,j) = p\int_0^1 \frac{(1-x)^{j-1}\left[\log(1-x)\right]^{p-1}}{(1-tx^a)^m}dx + j\int_0^1 \frac{(1-x)^{j-1}\left[\log(1-x)\right]^{p}}{(1-tx^a)^m}dx. \tag{2.5}$$

Proof. We have

$$\sum_{n=0}^{\infty}\frac{t^n\binom{n+m-1}{n}}{\binom{an+j}{j}} = \sum_{n=0}^{\infty} t^n \binom{n+m-1}{n}\frac{j\,\Gamma(j)\,\Gamma(an+1)}{\Gamma(an+j+1)}$$
$$= j\sum_{n=0}^{\infty} t^n \binom{n+m-1}{n} B(an+1,j)$$
$$= j\sum_{n=0}^{\infty} t^n \binom{n+m-1}{n}\int_0^1 (1-x)^{j-1}x^{an}dx,$$

where

$$B(\alpha,\beta)=\frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}=\int_0^1 (1-y)^{\alpha-1}y^{\beta-1}dy$$
$$=\int_0^1 (1-y)^{\beta-1}y^{\alpha-1}dy,\ \text{for } \alpha>0 \text{ and } \beta>0$$

is the classical Beta function and $\Gamma(\cdot)$ is the Gamma function.

By an allowable interchange of sum and integral, we have

$$\sum_{n=0}^{\infty}\frac{t^n\binom{n+m-1}{n}}{\binom{an+j}{j}} = j\int_0^1 (1-x)^{j-1}\sum_{n=0}^{\infty}\binom{n+m-1}{n}(tx^a)^n\,dx = j\int_0^1 \frac{(1-x)^{j-1}}{(1-tx^a)^m}dx. \tag{2.6}$$

Now we differentiate, with respect to j, both sides of (2.6), p times and utilize (2.2) in Lemma 2.1, so that

$$\sum_{n=0}^{\infty} t^n \binom{n+m-1}{n}\frac{d^p}{dj^p}\left[Q(an,j)\right] = \frac{\partial^p}{\partial j^p}\left[j\int_0^1 \frac{(1-x)^{j-1}}{(1-tx^a)^m}dx\right],$$

by an allowable change of derivative and integral we obtain the integral in (2.5) follows. ∎

Choosing the values $t=1$ and $a=1$ in (2.5), we obtain the following corollary.

Corollary 2.1. *For $m > 0,\ p = 1, 2, 3, ...$ and integer $j \geq m + 1$ we have*

$$\sum_{n=0}^{\infty} \binom{n+m-1}{n} \frac{d^p}{dj^p} \left[Q(n,j) \right] = \frac{mp!}{(j-m)^{p+1}}$$

and

$$\sum_{j=m+1}^{\infty} \sum_{n=0}^{\infty} \binom{n+m-1}{n} \frac{d^p}{dj^p} \left[Q(n,j) \right] = mp! \zeta(p+1).$$

It is clear that for other values of the parameters a multitude of different particular identities are possible, in particular, consider $a = 2,\ j = 2,\ m = 1,\ p = 2$ and $t = 1$, from (2.5)

$$\sum_{n=0}^{\infty} Q^{(2)}(2n, 2) = 2 \int_0^1 \frac{\log(1-x)\left[1 + \log(1-x)\right]}{1+x} dx.$$

From Lemma 2.1, we can evaluate $Q^{(2)}(2n, 2)$ so that

$$\sum_{n=1}^{\infty} \binom{2n+2}{2}^{-1} \left[\left(\sum_{r=1}^{2n} \frac{1}{r+2} \right)^2 + \sum_{r=1}^{2n} \frac{1}{(r+2)^2} \right]$$
$$= \ln^2(2) - 2\ln(2)\zeta(2) + \frac{7}{2}\zeta(3) + \frac{2}{3}\ln^3(2) - \zeta(2).$$

Finally re-adjusting the left hand side and after some algebraic manipulations, we obtain the following corollary.

Corollary 2.2. *The following summation identity holds*

$$\sum_{n=1}^{\infty} \frac{\left(H_{2n}^{(1)}\right)^2 - 3H_{2n}^{(1)} + H_{2n}^{(2)}}{n(2n-1)}$$
$$= \frac{7}{2}\zeta(3) - \zeta(2) - 2\ln(2)\zeta(2) + \ln^2(2) + \frac{2}{3}\ln^3(2) - 2\ln(2).$$

Remark 2.1. The finite version of (2.5) can be evaluated and in the case of $p = 3, a = 1$ and $t = -1$ we recapture the result of Diaz-Barrero *et al.* [1]:

$$\sum_{n=1}^{q} \frac{(-1)^{n+1} \binom{q}{n}}{\binom{n+j}{j}} \left[\left(\sum_{r=1}^{n} \tfrac{1}{r+j} \right)^3 + 3 \left(\sum_{r=1}^{n} \tfrac{1}{r+j} \right) \left(\sum_{r=1}^{n} \tfrac{1}{(r+j)^2} \right) + 2 \sum_{r=1}^{n} \tfrac{1}{(r+j)^3} \right]$$
$$= \frac{6q}{(q+j)^4}$$

and the new result

$$\sum_{j=1}^{\infty} \sum_{n=1}^{q} \frac{(-1)^{n+1} \binom{q}{n}}{\binom{n+j}{j}} \left[\left(\sum_{r=1}^{n} \tfrac{1}{r+j} \right)^3 + 3 \left(\sum_{r=1}^{n} \tfrac{1}{r+j} \right) \left(\sum_{r=1}^{n} \tfrac{1}{(r+j)^2} \right) + 2 \sum_{r=1}^{n} \tfrac{1}{(r+j)^3} \right]$$
$$= 6q\left(\zeta(4) - H_{q-1}^{(4)}\right).$$

For $j=0$

$$\sum_{n=1}^{q}(-1)^{n+1}\left(\begin{array}{c} q \\ n \end{array}\right)\left[\left(H_n^{(1)}\right)^3+3H_n^{(1)}H_n^{(2)}+2H_n^{(3)}\right]=\frac{6}{q^3},$$

but the method of [1] uses a variety of combinatorial identities.

Integral identities is a useful method of obtaining closed form representation of sums and we can extend the results of the previous section as follows.

3. Extension of results

The previous results can be extended in various ways, we give one such extension.

Theorem 3.1. *Let a be a positive real number, $|t|\le 1$, $m>0$, $p=1,2,3...$ and integer $j\ge m+2$. Then*

$$\begin{aligned}\sum_{n=1}^{\infty} t^n\, n\left(\begin{array}{c} n+m-1 \\ n \end{array}\right)Q^{(p)}(an,j) &= pmt\int_0^1\frac{(1-x)^{j-1}\,x^a\,[\log(1-x)]^{p-1}}{(1-tx^a)^{m+1}}dx \\ &+jmt\int_0^1\frac{(1-x)^{j-1}\,x^a\,[\log(1-x)]^{p}}{(1-tx^a)^{m+1}}dx.\end{aligned} \tag{3.1}$$

Proof. From (2.6) in Theorem 2.1 we have

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{t^n\left(\begin{array}{c} n+m-1 \\ n \end{array}\right)}{\left(\begin{array}{c} an+j \\ j \end{array}\right)} &= j\int_0^1(1-x)^{j-1}\sum_{n=0}^{\infty}\left(\begin{array}{c} n+m-1 \\ n \end{array}\right)(tx^a)^n\,dx \\ &= j\int_0^1\frac{(1-x)^{j-1}}{(1-tx^a)^m}dx\end{aligned} \tag{3.2}$$

and applying the operator $t\frac{d}{dt}(\cdot)$ we may write

$$\sum_{n=1}^{\infty}\frac{t^n\,n\left(\begin{array}{c} n+m-1 \\ n \end{array}\right)}{\left(\begin{array}{c} an+j \\ j \end{array}\right)}=jmt\int_0^1\frac{(1-x)^{j-1}\,x^a}{(1-tx^a)^{m+1}}dx.$$

Differentiating p times, with respect to j and applying Lemma 2.1 so that

$$\sum_{n=0}^{\infty}t^n n\left(\begin{array}{c} n+m-1 \\ n \end{array}\right)\frac{d^p}{dj^p}\left[Q\left(an,j\right)\right]=\frac{\partial^p}{\partial j^p}\left[jmt\int_0^1\frac{(1-x)^{j-1}\,x^a}{(1-tx^a)^{m+1}}dx\right]$$

by an allowable change of integral and derivative we obtain the result (3.1). ∎

Choosing the values $t=1$ and $a=1$ we obtain the following corollary.

Corollary 3.1. *For $m>0$, $p=1,2,3,...$and integer $j\ge m+2$, we have*

$$\sum_{n=1}^{\infty}n\left(\begin{array}{c} n+m-1 \\ n \end{array}\right)Q^{(p)}(n,j)=(-1)^p mp!\left[\frac{m+1}{(j-m-1)^{p+1}}-\frac{m}{(j-m)^{p+1}}\right],$$

and we may also extrapolate the interesting sum

$$\sum_{j=m+2}^{\infty}\sum_{n=1}^{\infty} n \binom{n+m-1}{n} Q^{(p)}(n,j) = (-1)^p \, mp! \left[\zeta(p+1)+m\right].$$

The following corollary is also noted.

Corollary 3.2. *Let $a=2,\ j=3,\ m=1,\ p=1$ and $t=-1$, after some algebraic manipulations and by the use of Lemma 2.1, we obtain*

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1} n \, H_{2n+3}^{(1)}}{(2n+3)(2n+1)(n+1)}$$
$$= \frac{13}{4} - G - \frac{\pi}{4} - \frac{5}{8}\zeta(2) - \frac{5}{4}\ln(2) + \frac{1}{4}\ln^2(2) + \frac{\pi}{8}\ln(2)$$

where G is Catalan's constant.

Noting that

$$H_{2n+3}^{(1)} = H_{2n}^{(1)} + \frac{12n^2+24n+11}{(2n+3)(2n+2)(2n+1)}$$

and that

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1} 2n \left(12n^2+24n+11\right)}{\left((2n+3)(2n+2)(2n+1)\right)^2} = 1 - G - \frac{\pi}{4} - \frac{1}{8}\zeta(2) + \frac{1}{2}\ln(2),$$

then we may extract the following result:

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1} n \, H_{2n}^{(1)}}{(2n+3)(2n+1)(n+1)}$$
$$= \frac{9}{4} - \frac{\pi}{4} - \frac{3}{8}\zeta(2) - \frac{7}{4}\ln(2) + \frac{1}{4}\ln^2(2) + \frac{\pi}{8}\ln(2).$$

Remark 3.1. The finite version of (3.1) can be evaluated and in the case of $p=3,\ a=1$ and $t=-1$ we obtain the result

$$\sum_{n=1}^{q} \frac{(-1)^n \, n \binom{q}{n}}{\binom{n+j}{j}} \left[\begin{array}{c} \left(\sum_{r=1}^{n} \frac{1}{r+j}\right)^3 + 3\left(\sum_{r=1}^{n} \frac{1}{r+j}\right)\left(\sum_{r=1}^{n} \frac{1}{(r+j)^2}\right) \\ +2\sum_{r=1}^{n} \frac{1}{(r+j)^3} \end{array} \right]$$
$$= \frac{6q\left(3q^4+2q^3(4j-3)+2q^2(3j^2-6j+2)-q(6j^2-4j+1)-j^4\right)}{(q+j)^4(q+j-1)^4}$$

and the new result

$$\sum_{j=1}^{\infty}\sum_{n=1}^{q} \frac{(-1)^n \, n \binom{q}{n}}{\binom{n+j}{j}} \left[\begin{array}{c} \left(\sum_{r=1}^{n} \frac{1}{r+j}\right)^3 + 3\left(\sum_{r=1}^{n} \frac{1}{r+j}\right)\left(\sum_{r=1}^{n} \frac{1}{(r+j)^2}\right) \\ +2\sum_{r=1}^{n} \frac{1}{(r+j)^3} \end{array} \right]$$
$$= \frac{6}{q^2} + 6qH_{q-1}^{(4)} - 6q\zeta(4),\ q \geq 1,$$

we define $H_0^{(4)} \equiv 0$.

For $j = 0$, $q > 1$

$$\sum_{n=1}^{q} (-1)^n \; n \binom{q}{n} \left[\left(H_n^{(1)} \right)^3 + 3H_n^{(1)} H_n^{(2)} + 2H_n^{(3)} \right] = \frac{6\left(3q^2 - 3q + 1\right)}{q^2 \left(q-1\right)^3}.$$

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