ALBANIAN JOURNAL
OF MATHEMATICS
Volume 3, Number 1, Pages 43–48
ISSN 1930-1235: (2009)

CONVEXITY OF FINITE SUMS

Anthony Sofo
Research Group in Mathematical Inequalities and Applications
School of Engineering and Science
Victoria University, PO Box 14428 Melbourne
VIC 8001, Australia.
anthony.sofo@vu.edu.au

Abstract. Convexity and log convexity results are established for sums involving ratios of binomial coefficients. We utilise recent results in which integral identities have been given to represent sums involving ratios of binomial coefficients.

1. Introduction

The integral representation of series of ratios of binomial coefficients have recently been investigated by a number of authors, see in particular Amghibech [1], Batir [2], and Sofo [5, 6, 7]. Recently Purkait and Sury [4], using mainly combinatorial methods, obtained expressions for

$$S=\sum_{n=0}^{p}\frac{(-1)^n\,n^r\binom{p}{n}}{\binom{n+j}{n}}$$

and deduced that for even integer $r\geq 0$ and $p=j>\frac{r}{2}$, S is identically zero or $\frac{1}{2}$ according as to whether $r>0$ or not. In this paper we supplement the results of Purkait and Sury by considering convexity properties of slightly more general forms of S.

In particular, the following theorem was given in [5].

Theorem 1. *for $a>0$, $p\geq 1, t\in\mathbb{R}$ and $j>0$*

$$S(a,j,p,t)=\frac{1}{j}\sum_{n=0}^{p}\frac{t^n\binom{p}{n}}{\binom{an+j}{j}}=\int_0^1(1-x)^{j-1}(1+tx^a)^p\,dx. \tag{1.1}$$

For an integer a we can write

$$\sum_{n=0}^{p}\frac{t^n\binom{p}{n}}{\binom{an+j}{j}}={}_{a+1}F_a\left[\begin{matrix}\frac{1}{a},\frac{2}{a},\frac{3}{a},\ldots,\frac{a}{a},-p\\ \frac{1+j}{a},\frac{2+j}{a},\frac{3+j}{a},\ldots,\frac{a+j}{a}\end{matrix}\middle|-t\right],$$

2000 *Mathematics Subject Classification.* Primary 35E10. Secondary 26A09, 26B20.
Key words and phrases. Convexity, Log convexity, Integral identities.

where the generalised hypergeometric representation ${}_pF_q\ [\cdot,\cdot]$, *is defined as*

$$ {}_pF_q\left[\begin{array}{c} a_1,a_2,...,a_p \\ b_1,b_2,...,b_q \end{array}\middle| t\right] = \sum_{n=0}^{\infty} \frac{(a_1)_n\ (a_2)_n ...(a_p)_n\ t^n}{(b_1)_n\ (b_2)_n ...(b_q)_n\ n!} $$

and $(w)_\alpha = w(w+1)(w+2)...(w+\alpha-1) = \frac{\Gamma(w+\alpha)}{\Gamma(w)}$ *is Pochhammer's symbol.*

The following analysis establishes the monotonicity and convexity properties of $S(a,j,p,t)$, by the use of its integral representation (1.1).

2. Convexity Properties

The following theorem is proved.

Theorem 2. *For* $p \geq 1,\ a \geq 1, t > 0$ *and* $j > 0$ *the function* $a \mapsto S(a,j,p,t)$, *as given in Theorem 1 is strictly decreasing and convex with respect to the parameter* $a \in [1,\infty)$ *for every* $x \in [0,1]$.

Proof. Let

$$ g(x,a) = (1-x)^{j-1}(1+tx^a)^p \tag{2.1} $$

be an integrable function for $x \in [0,1]$ and put

$$ f(a) = \int_0^1 g(x,a)\,dx, $$

so that $f(1) = \frac{1}{j}\ {}_2F_1\left[\begin{array}{c} 1,-p \\ 1+j \end{array}\middle| -t\right]$.

Applying the Leibniz rule for differentiation under the integral sign, we have that

$$ \begin{aligned} f'(a) &= \int_0^1 \frac{\partial}{\partial a} g(x,a)\,dx \\ &= pt\int_0^1 x^a(1-x)^{j-1}(1+tx^a)^{p-1}\ln x dx \qquad (2.2) \end{aligned} $$

Since

$$ x^a(1-x)^{j-1}(1+tx^a)^{p-1}\ln x < 0 \quad \text{for} \quad x \in (0,1), $$

then $f'(a) < 0$, so that the sum of the ratio of binomial coefficients (1.1), is a strictly decreasing sum with respect to the parameter a for $x \in [0,1]$. Now

$$ \begin{aligned} f''(a) &= \int_0^1 \frac{\partial^2}{\partial^2 a} g(x,a)\,dx \\ &= pt\int_0^1 (1-x)^{j-1}(1+tx^a)^{p-2} x^a (ptx^a+1)(\ln x)^2\,dx, \qquad (2.3) \end{aligned} $$

and since

$$ (1-x)^{j-1}(1+tx^a)^{p-2}x^a(ptx^a+1)(\ln x)^2 > 0, $$

then $f''(a) > 0$ so that (1.1) is a convex function for $x \in [0,1]$. □

In the following we establish the log convexity of the function $a \mapsto S(a,j,p,t)$ with respect to the positive parameter a. Firstly, we state the Cauchy-Buniakowsky-Schwarz inequality.

Cauchy-Buniakowsky-Schwarz inequality: *Let $p(x)$, $q(x)$ and $r(x)$ be integrable functions for $x \in [\alpha, \beta]$, then*

$$\left(\int_{\alpha}^{\beta} p(x) q^2(x) \, dx\right) \left(\int_{\alpha}^{\beta} p(x) r^2(x) \, dx\right) \geq \left(\int_{\alpha}^{\beta} p(x) q(x) r(x) \, dx\right)^2.$$

A proof of this theorem can be found in [3].

Theorem 3. *For $p \geq 1$, $a \geq 1, t > 0$ and $j > 0$, the function $a \mapsto S(a, j, p, t)$ as given in Theorem 1 is log convex with respect to the parameter $a \in [1, \infty)$ for every $x \in [0, 1]$.*

Proof. Let

$$h(a) = \log\left(\int_0^1 g(x, a) \, dx\right),$$

where $g(x, a)$ is given by (2.1), and applying the Leibniz rule for differentiation under the integral sign, we have that:

$$h'(a) = \frac{pt \int_0^1 x^a (1-x)^{j-1} (1+tx^a)^{p-1} \ln x dx}{\int_0^1 (1-x)^{j-1} (1+tx^a)^p \, dx} < 0.$$

Now,

$$h''(a) = \frac{\left(\int_0^1 \frac{\partial^2}{\partial^2 a} g(x,a) \, dx\right) \left(\int_0^1 g(x,a) \, dx\right) - \left(\int_0^1 \frac{\partial}{\partial a} g(x,a) \, dx\right)^2}{\left(\int_0^1 g(x,a) \, dx\right)^2}, \tag{2.4}$$

where $g(x, a)$ is given by (2.1), $\int_0^1 \frac{\partial}{\partial a} g(x,a) \, dx$ is given by (2.2) and $\int_0^1 \frac{\partial^2}{\partial^2 a} g(x,a) \, dx$ is given by (2.3).

Since we require $h''(a) > 0$ it will suffice to prove that

$$\begin{aligned} &pt \int_0^1 (1-x)^{j-1} (1+tx^a)^{p-2} x^a (ptx^a + 1) (\ln x)^2 \, dx \int_0^1 (1-x)^{j-1} (1+tx^a)^p \, dx \\ &> \left(pt \int_0^1 x^a (1-x)^{j-1} (1+tx^a)^{p-1} \ln x dx\right)^2 > 0 \end{aligned} \tag{2.5}$$

Now,

$$\begin{aligned} &pt \int_0^1 (1-x)^{j-1} (1+tx^a)^{p-2} x^a (ptx^a + 1) (\ln x)^2 \, dx \\ &> pt \int_0^1 (1-x)^{j-1} (1+tx^a)^{p-2} x^a (ptx^a) (\ln x)^2 \, dx \\ &= \int_0^1 (1-x)^{j-1} (1+tx^a)^p \left(\frac{ptx^a \ln x}{1+tx^a}\right)^2 dx > 0, \end{aligned}$$

hence from (2.5),

$$\begin{aligned} &\left(\int_0^1 (1-x)^{j-1} (1+tx^a)^p \left(\frac{ptx^a \ln x}{1+tx^a}\right)^2 dx\right) \left(\int_0^1 (1-x)^{j-1} (1+tx^a)^p \, dx\right) \\ &\qquad > \left(\int_0^1 \left(\frac{ptx^a \ln x}{1+tx^a}\right) \cdot (1-x)^{j-1} (1+tx^a)^p \, dx\right)^2 \end{aligned}$$

is satisfied by application of the Cauchy-Buniakowsky-Schwarz inequality and identifying

$$p(x) = (1-x)^{j-1}(1+tx^a)^p, \quad q(x) = \frac{ptx^a \ln x}{1+tx^a} \quad \text{and} \quad r(x) = 1.$$

From (2.4) we can claim $h''(a) > 0$ and the theorem is proved. □

Note: The series (1.1) can be represented in the generalised hypergeometric form as:

$$ {}_{a+1}F_a \left[\begin{array}{c} \frac{1}{a}, \frac{2}{a}, \frac{3}{a}, \ldots, \frac{a}{a}, -p \\ \frac{1+j}{a}, \frac{2+j}{a}, \frac{3+j}{a}, \ldots, \frac{a+j}{a} \end{array} \middle| -t \right]. \tag{2.6}$$

We can therefore claim that (2.6) is a log convex function with respect to the parameter $a \geq 1$.

In the paper [5], Sofo further generalised Theorem 1 to obtain the following representation.

Theorem 4. *For $a > 0$, $p \geq 1, t \in \mathbb{R}, r \geq 0$ and $j > 0$ we have*

$$\begin{aligned} \frac{1}{j} S(a,j,p,t,r) &:= \frac{1}{j} \sum_{n=1}^{p} \frac{t^n \ n^r \binom{p}{n}}{\binom{an+j}{j}} \\ &= \int_0^1 (1-x)^{j-1} \frac{(\rho(x))^{(r)}}{a^r} dx \end{aligned}$$

where

$$\begin{cases} (\rho(x))^{(0)} = (1+tx^a)^p \\ \vdots \\ (\rho(x))^{(r)} = x\frac{d}{dx}\left(x\frac{d}{dx}\left(\cdots x\frac{d}{dx}\left((1+tx^a)^p\right)\right)\right) \end{cases}$$

is the consecutive derivative operator of the continuous function $(1+tx^a)^p$ for $x \in (0,1)$.

An example for Theorem 4 is:

$$\frac{1}{j} S(a,j,p,t,1) := \frac{1}{j} \sum_{n=1}^{p} \frac{t^n \ n \binom{p}{n}}{\binom{an+j}{j}} \tag{2.7}$$

$$= pt \int_0^1 (1-x)^{j-1} x^a (1+tx^a)^{p-1} dx \tag{2.8}$$

$$= \frac{pt}{j \begin{pmatrix} a+j \\ j \end{pmatrix}} \ {}_{a+1}F_a \left[\begin{array}{c} \frac{a+1}{a}, \frac{a+2}{a}, \frac{a+3}{a}, \ldots, \frac{a+a}{a}, 1-p \\ \frac{a+1+j}{a}, \frac{a+2+j}{a}, \frac{a+3+j}{a}, \ldots, \frac{a+a+j}{a} \end{array} \middle| -t \right]$$

The following remark claims a convexity and log convexity property for the series $\frac{1}{j} S(a,j,p,t,1)$ by the use of its integral representation (2.8).

Remark 1. *For $a \geq 1$, $p \geq 1$, $t > 0, r \geq 0$ and $j > 0$, the function $a \mapsto \frac{1}{j} S(a,j,p,t,1)$ as given in Theorem 4 is strictly decreasing and convex with respect to the parameter $a \in [1,\infty)$ for every $x \in [0,1]$, it is also log convex.*

As in the proofs of Theorems 2 and 3 we can outline the following steps. From (2.8), let

$$G(x,a) = pt(1-x)^{j-1}x^a(1+tx^a)^{p-1} \tag{2.9}$$

be an integrable function for $x \in [0,1]$ and put

$$F(a) = \int_0^1 G(x,a)\,dx,$$

applying the Leibniz rule for differentiation under the integral sign and using (2.8), we obtain

$$F'(a) = pt\int_0^1 (1-x)^{j-1}x^a(1+tx^a)^{p-2}(1+ptx^a)\ln(x)\,dx < 0, \tag{2.10}$$

and

(2.11)

$$F''(a) = pt\int_0^1 (1-x)^{j-1}x^a(1+tx^a)^{p-3}\left(1+(3p-1)tx^a+(ptx^a)^2\right)(\ln(x))^2\,dx > 0,$$

since $p \geq 1$, so that (2.7) is a monotonic decreasing function for every $x \in [0,1]$.

Similarly, if we let

$$H(a) = \log\left(\int_0^1 G(x,a)\,dx\right),$$

$$H'(a) = \frac{\int_0^1 (1-x)^{j-1}x^a(1+tx^a)^{p-2}(1+ptx^a)\ln(x)\,dx}{\int_0^1 (1-x)^{j-1}x^a(1+tx^a)^{p-1}\,dx} < 0,$$

and we require that

$$H''(a) = \frac{\left(\int_0^1 \frac{\partial^2}{\partial^2 a}G(x,a)\,dx\right)\left(\int_0^1 G(x,a)\,dx\right) - \left(\int_0^1 \frac{\partial}{\partial a}G(x,a)\,dx\right)^2}{\left(\int_0^1 G(x,a)\,dx\right)^2} > 0,$$

where $G(x,a)$ is given by (2.9), $\int_0^1 \frac{\partial}{\partial a}G(x,a)\,dx$ is given by (2.10) and $\int_0^1 \frac{\partial^2}{\partial^2 a}G(x,a)\,dx$ is given by (2.11). It is sufficient to investigate

$$\begin{aligned}
&\left(\int_0^1 (1-x)^{j-1}x^a(1+tx^a)^{p-3}\left(1+(3p-1)tx^a+(ptx^a)^2\right)(\ln(x))^2\,dx\right) \\
&\quad\times\left(\int_0^1 (1-x)^{j-1}x^a(1+tx^a)^{p-1}\,dx\right) \\
&\quad> \left(\int_0^1 (1-x)^{j-1}x^a(1+tx^a)^{p-2}(1+ptx^a)\ln(x)\,dx\right)^2.
\end{aligned} \tag{2.12}$$

Now since

$$\begin{aligned}
&\int_0^1 (1-x)^{j-1}x^a(1+tx^a)^{p-3}\left(1+(3p-1)tx^a+(ptx^a)^2\right)(\ln(x))^2\,dx \\
&> \int_0^1 (1-x)^{j-1}x^a(1+tx^a)^{p-1}\left(\frac{(1+ptx^a)\ln(x)}{1+tx^a}\right)^2 dx,
\end{aligned}$$

if we identify

$$p(x) = (1-x)^{j-1}x^a(1+tx^a)^{p-1}, \quad q(x) = \frac{(1+ptx^a)\ln x}{1+tx^a},$$

and $r(x) = 1$ and applying the Cauchy-Buniakowsky-Schwarz inequality we conclude that (2.12) is satisfied, hence (2.7) is log convex with respect to the parameter a for every $x \in [0,1]$.

We make the following observation.

For an integer $a \geq 1$, $p \geq 1$, $t > 0$, and $j > 0$,

(2.13)
$$\sum_{n=1}^{p} \frac{t^n n\binom{p}{n}}{\binom{an+j}{j}} = \frac{pt}{j\begin{pmatrix} a+j \\ j \end{pmatrix}} \; {}_{a+1}F_a\left[\left. \begin{matrix} \frac{a+1}{a}, \frac{a+2}{a}, \frac{a+3}{a}, \ldots, \frac{a+a}{a}, 1-p \\ \frac{a+1+j}{a}, \frac{a+2+j}{a}, \frac{a+3+j}{a}, \ldots, \frac{a+a+j}{a} \end{matrix} \right| -t \right].$$

Hence we make the claim that the generalised hypergeometric function in (2.13) is a log convex function with respect to the parameter $a \geq 1$.

3. Conclusion

We have demonstrated convexity and log convexity properties for a class of series involving ratios of binomial coefficients.

References

[1] Amghibech, S., On sums involving binomial coefficients, *Journal of Integer Sequences* **10**, 2007, article 07.2.1.

[2] Batir, N., On the series $\sum_{k=1}^{\infty} k^{-n} x^k$. *Proc. Indian Acad. Sci.* (Math. Sci.) **115**(4), 2005, 371-381.

[3] Kazarinoff, N.D., *Analytic Inequalities*, Dover Publications, New York, 2003.

[4] Purkait, S. and Sury, B., Some vanishing sums involving binomial coefficients in the denominator. *Albanian Journal of Mathematics* **2**(1), 2008, 27-32.

[5] Sofo, A., Sums of binomial coefficients in integral form, *Accepted Fibonacci Quarterly*, 2007

[6] Sofo, A., General properties involving reciprocals of binomial coefficients, *Journal of Integer Sequences* **9**, 2006, article 06.4.5.

[7] Sofo, A., *Computational Techniques for the Summation of Series*, Kluwer Academic/Plenum Publishers, 2003.