The Fibonacci Quarterly 1974 (12,4): 360-362

COMBINATORIAL INTERPRETATION OF AN ANALOG OF GENERALIZED BINOMIAL COEFFICIENTS

M. J. HODEL
Duke University, Durham, North Carolina 27706

1. INTRODUCTION

Defining $f_{j,k}(n;r,s)$ as the number of sequences of nonnegative integers

(1.1) $$\{a_1, a_2, \cdots, a_n\}$$

such that

(1.2) $$-s \leqslant a_{i+1} - a_i \leqslant r \qquad (1 \leqslant i \leqslant n-1),$$

where r and s are arbitrary positive integers, and

(1.3) $$a_1 = j, \qquad a_n = k,$$

the author [2] has shown that the generating function

$$\phi_{j,r,s}(x,y) = \sum_{n=0}^{\infty} \sum_{m=0}^{\min\{n(r+s),j+nr\}} f_{j,j+nr-m}(n+1;r,s)x^n y^m$$

can be expressed in terms of generalized binomial coefficients $c_{r+s}(n,k)$ defined by

(1.4) $$\left(\sum_{h=0}^{r+s} x^h\right)^n = \sum_{k=0}^{\infty} c_{r+s}(n,k)x^k .$$

For the cases $r = 1$ or $s = 1$ we have explicit formulas for $f_{j,k}(n;r,s)$, namely

(1.5) $$f_{j,k}(n+1;1,s) = \sum_{t=0}^{j} c_{s+1}(-t-1,j-t)\left[c_{s+1}(n+t,n+t-k) - \sum_{h=0}^{s-1}(h+1)c_{s+1}(n+t,n+t-k-h-2)\right] .$$

and

(1.6) $$f_{j,k}(n+1;r,1) = \sum_{t=0}^{k} c_{r+1}(-t-1,k-t)\left[c_{r+1}(n+t,n+t-j) - \sum_{h=0}^{r-1}(h+1)c_{r+1}(n+t,n+t-j-h-2)\right].$$

These formulas generalize a result of Carlitz [1] for $r = s = 1$.

We now define an analog of $c_{r+s}(n,k)$, $n > 0$, by

(1.7) $$\prod_{j=1}^{n}\left(\sum_{h=0}^{r+s} q^{(r-h)j}x^h\right) = \sum_{k=0}^{n(r+s)} c_{r+s}(n,k;q)x^k .$$

Letting $f_k(m,n;r,s)$ denote the number of sequences of integers

(1.8) $$\{a_1, a_2, \cdots, a_n\}$$

satisfying

(1.9) $$-s \leqslant a_{i+1} - a_i \leqslant r \qquad (1 \leqslant i \leqslant n-1),$$

where r and s are nonnegative integers,

(1.10) $$a_1 = 0, \qquad a_n = k$$

and

(1.11) $$\sum_{i=1}^{n} a_i = m \ ,$$

we show in this paper that

(1.12) $$c_{r+s}(n,k;q) = \sum_m f_{nr-k}(m, n+1; r,s) q^m \ .$$

From (1.12) we obtain a partition identity.

2. COMBINATORIAL INTERPRETATION OF $c_{r+s}(n,k;q)$

From the definition of $f_k(m,n;r,s)$ it follows that

(2.1) $$f_k(m,1;r,s) = \delta_{k,0}\delta_{m,0}$$

and

(2.2) $$f_k(m,n+1;r,s) = \sum_{h=0}^{r+s} f_{k+s-h}(m-k, n; r,s) \ .$$

Now (2.1) had (2.2) imply respectively

(2.3) $$\sum_m f_k(m,1;r,s) q^m = \delta_{k,0}$$

and

(2.4) $$\sum_m f_k(m,n+1;r,s) q^m = \sum_{h=0}^{r+s} \sum_m f_{k+s-h}(m,n;r,s) q^{m+k} \ .$$

Let

$$\phi(x,y;q) = \sum_{n=0}^{\infty} \sum_{k=0}^{n(r+s)} \sum_m f_{nr-k}(m,n+1;r,s) q^m x^k y^n \ .$$

Using (2.3) and (2.4) we get

$$\phi(x,y;q) = 1 + y \sum_{h=0}^{r+s} \sum_{n=0}^{\infty} \sum_{k=0}^{(n+1)(r+s)} \sum_m f_{nr-k+h}(m,n+1;r,s) q^{m+nr-k} x^k y^r = 1 + y \left(\sum_{h=0}^{r+s} q^{r-h} x^h \right) \phi(xq^{-1}, yq^r; q).$$

By iteration

$$\phi(x,y;q) = \sum_{n=0}^{\infty} \prod_{j=1}^{n} \left(\sum_{h=0}^{r+s} q^{(r-h)j} x^h \right) y^n = \sum_{n=0}^{\infty} \sum_{k=0}^{n(r+s)} c_{r+s}(n,k;q) x^k y^n \ .$$

Equating coefficients we have

(2.5) $$c_{r+s}(n,k;q) = \sum_m f_{nr-k}(m,n+1;r,s) q^m \ .$$

3. APPLICATION TO PARTITIONS

Assuming the parts of a partition to be written in ascending order, let $u_r(k,m,n)$ denote the number of partitions of m into at most n parts with the minimum part at most r, the maximum part k and the difference between consecutive parts at most r. Define $v_r(k,m,n)$ to be the number of partitions of m into k parts with each part at most n and each part occurring at most r times. We show that

(3.1) $$u_r(k,m,n) = v_r(k,m,n) \qquad (r \geqslant 1).$$

Proof. It is easy to see that

(3.2) $$u_r(k,m,n) = f_k(m,n+1;r,0) \ .$$

By (2.5) and (1.7) we have

$$\sum_{k=0}^{nr} \sum_m f_k(m,n+1;r,0) q^m x^k = \sum_{k=0}^{nr} c_r(n,nr-k;q) x^k = \prod_{j=1}^{n} \left(\sum_{h=0}^{r} q^{hj} x^h \right)$$

Thus the generating function for $u_r(k,m,n)$ is

$$\prod_{j=1}^{n} \left(\sum_{h=0}^{r} q^{hj} x^h \right) . \tag{3.3}$$

But it is well known (see for example [3, p. 10] for $r = 1$) that the generating function for $v_r(k,m,n)$ is also (3.3). Hence we have (3.1). This identity is also evident from the Ferrers graph.

To illustrate (3.1) and (3.2) let $m = 7,\ n = 4,\ k = 3$ and $r = 2$. The sequences enumerated by $f_3(7,5;2,0)$ are $0,0,1,3,3$, $0,0,2,2,3$ and $0,1,1,2,3$. The function $u_2(3,7,4)$ counts the corresponding partitions, namely 13^2, 2^23 and 1^223. The partitions which $v_2(3,7,4)$ enumerates are 2^23, 13^2 and 124. From the graphs

```
. . .     . . .     . . . .
. .       . . .     . .
. .       .         .
```

we observe that 13^2 is the conjugate of 2^23, 2^23 is the conjugate of 13^2 and 1^223 is the conjugate of 124.

REFERENCES

1. L. Carlitz, "Enumeration of Certain Types of Sequences," *Mathematische Nachrichten,* Vol. 49 (1971), pp. 125–147.
2. M.J. Hodel, "Enumeration of Sequences of Nonnegative Integers," *Mathematische Nachrichten,* Vol. 59 (1974), pp. 235–252.
3. P.A.M. MacMahon, *Combinatory Analysis,* Vol. 2, Cambridge, 1916.

****

[Continued from page 354.]

SPECIAL CASES

Putting $r = 1,\ s = 0,$ we obtain the generating function for the Fibonacci sequence (see [3] and Riordan [6]). Putting $r = 2,\ s = -1,$ we obtain the generating function for the Lucas sequence (see [3] and Carlitz [1]).

Other results in Riordan [6] carry over to the H-sequence. The H-sequence (and the Fibonacci and Lucas sequences), and the generalized Fibonacci and Lucas sequences are all special cases of the W-sequence studied by the author in [4]. More particularly,

$$\{H_n\} = \{w_n(r, r+s; 1, -1)\}$$

and so

$$\{f_n\} = \{w_n(1, 1; 1, -1)\}, \qquad \{a_n\} = \{w_n(2, 1; 1, -1)\} .$$

Interested readers might consult the article by Kolodner [5] which contains material somewhat similar to that in [3], though the methods of treatment are very different.

REFERENCES

1. L. Carlitz, "Generating Functions for Powers of Certain Sequences of Numbers," *Duke Math. J.* 29 (4) (1962) pp. 521–538.
2. A. Horadam, "A Generalized Fibonacci Sequence," *Amer. Math. Monthly,* 68 (5) (1961), pp. 455–459.
3. A. Horadam, "Generating Functions for Powers of a Certain Generalized Sequence of Numbers," *Duke Math. J.,* 32 (3) (1965), pp. 437–446.
4. A. Horadam, "Basic Properties of a Certain Generalized Sequence of Numbers," *The Fibonacci Quarterly,* Vol. 3, No. 3 (October 1965), pp. 161–176.
5. I. Kolodner, "On a Generating Function Associated with Generalized Fibonacci Sequences," *The Fibonacci Quarterly,* Vol. 3, No. 4 (December 1965), pp. 272–278.
6. J. Riordan, "Generating Functions for Powers of Fibonacci Numbers," *Duke Math. J.,* 29 (1) (1962), pp. 5–12.

****