The q-binomial theorem

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Introduction

- Many functions and objects in mathematics have natural perturbations, called q-analogues.
- They contain an extra variable q.
- When $q = 1$, everything goes back to normal.

Goals of this talk:

- To see an example of a q-analogue, and how it arises.
- To see that by making a problem more difficult, it sometimes becomes easier to solve.

Question 1 (of 3)

Example: 111001010 is a string, made of 0's and 1's, that contains 9 characters.

How many strings are there, made of 0's and 1's, that contain 9 characters?

Answer: there are two possibilities for each character, so the number of possible strings is

$$2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 = 2^9 (= 512).$$

By similar reasoning, there are 2^n strings, made of 0's and 1's, that contain n characters.

Question 1, continued

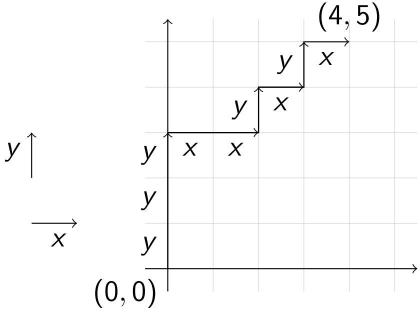
The original question was phrased in terms of 0's and 1's.

Example: 111001010: uses 0's and 1's.

The question could have been phrased in terms of any two symbols, e.g., T and F, x and y, etc.

Example: $yyyxxyxyx$: uses x's and y's.

Code: $x = 0$, $y = 1$.



The string *yyyxxyxyx* can be represented by a lattice path.

There are 9 steps.
The start point is $(0, 0)$ and the end point is $(4, 5)$.

Question 2 (of 3)

The string *yyyxxyxyx* can be represented by a lattice path, starting at $(0,0)$ and ending at $(4,5)$.

How many distinct lattice paths are there, starting at $(0,0)$ and ending at $(4,5)$?

How many distinct lattice paths are there, starting at $(0,0)$ and ending at $(k, n-k)$?

One way of answering it:

Consider strings of length n: $(\; , \; , \; , \; , \; , \; , \; , \; , \;)$

Count the number of ways of choosing k positions to insert x's:
$(\; , \; , \; , x, x, \; , x, \; , x)$

Fill the remaining positions with y's

In this example, $n = 9$ and $k = 4$, and the number of ways of placing the x's is 126

$$\frac{9 \times 8 \times 7 \times 6}{4 \times 3 \times 2 \times 1} = \frac{9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1}{(4 \times 3 \times 2 \times 1) \times (5 \times 4 \times 3 \times 2 \times 1)} = \frac{9!}{4!5!}$$

In general, the number of distinct lattice paths, starting at $(0, 0)$ and ending at $(k, n-k)$, is $\dfrac{n!}{k!(n-k)!}$. "Binomial coefficient"

Another way to view it:

$$(x+y)^9 = (x+\underline{y})(x+\underline{y})(x+\underline{y})(\underline{x}+y)(\underline{x}+y)(x+\underline{y})(\underline{x}+y)(x+\underline{y})(\underline{x}+y)$$

$$= xxxxxxxxx + \cdots + yyyxxyxyx + \cdots + yyyyyyyyy$$

$$= x^9 + \cdots + (\text{how many?})x^4y^5 + \cdots + y^9$$

What is the expansion of $(x+y)^9$?

What is the expansion of $(x+y)^n$?

The binomial theorem

$$
\begin{aligned}
(x+y)^0 &= 1 \\
(x+y)^1 &= x+y \\
(x+y)^2 &= x^2+2xy+y^2 \\
(x+y)^3 &= x^3+3x^2y+3xy^2+y^3 \\
(x+y)^4 &= x^4+4x^3y+6x^2y^2+4xy^3+y^4 \\
(x+y)^5 &= x^5+5x^4y+10x^3y^2+10x^2y^3+5xy^4+y^5 \\
\\
(x+y)^n &= \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}
\end{aligned}
$$

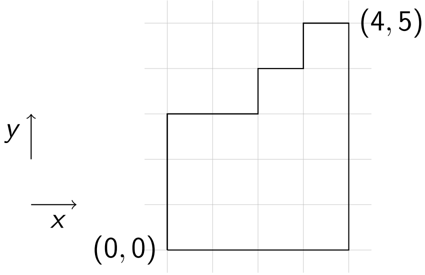
The binomial coefficient $\binom{n}{k} = \frac{n!}{k!(n-k)!}$ is the number of lattice paths of length n, that contain k x's and $(n-k)$ y's.

Pascal's triangle

$$
\begin{array}{ccccccccccccc}
 & & & & & & 1 & & & & & & \\
 & & & & & 1 & & 1 & & & & & \\
 & & & & 1 & & 2 & & 1 & & & & \\
 & & & 1 & & 3 & & 3 & & 1 & & & \\
 & & 1 & & 4 & & 6 & & 4 & & 1 & & \\
 & 1 & & 5 & & 10 & & 10 & & 5 & & 1 & \\
 & \vdots & & & & & & & & & & \vdots & \\
\binom{n}{0} & & \cdots & & \binom{n}{k-1} & & \binom{n}{k} & & \cdots & & & & \binom{n}{n} \\
\binom{n+1}{0} & & & & & \binom{n+1}{k} & & & & & & & \binom{n+1}{n+1}
\end{array}
$$

Formula: $\binom{n}{k} = \frac{n!}{k!(n-k)!}$

Recurrence relation: $\binom{n}{k-1} + \binom{n}{k} = \binom{n+1}{k}$

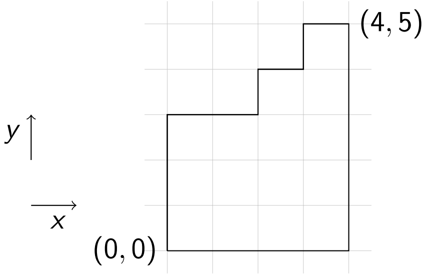


The lattice path *yyyxxyxyx*

How many lattice paths:

- have length 9 — Ans: $2^9 = 512$
- and end at $(4,5)$ — Ans: $\binom{9}{4} = 126$
- and enclose an area of 15 square units? — Ans: ?

Question 3 (of 3)



The lattice path *yyyxxyxyx*

General question: Suppose $0 \leq j \leq k(n-k)$.

How many lattice paths

- have length n — Ans: 2^n
- and end at $(k, n-k)$ — Ans: $\binom{n}{k}$
- and enclose an area of j square units?

How to keep track of the area?

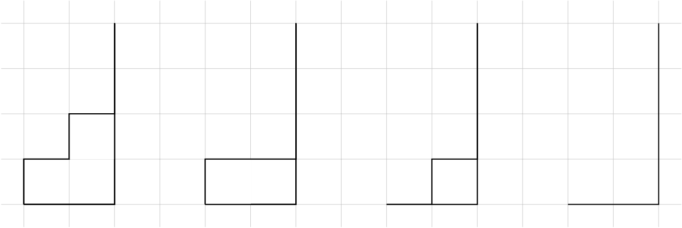
Use a factor of q to record each time a lattice path reduces in area by 1 square unit.

We say that (x, y) is a q-Weyl pair if:

- $yx = qxy$
- $qx = xq$
- $qy = yq$

Example

$$yx = qxy, \qquad qx = xq, \qquad qy = yq$$



$$\begin{aligned} yxyxyy &= yx(yx)yy & &= yx(qxy)yy & &= qyxxyyy \\ &= q(yx)xyyy & &= q(qxy)xyyy & &= q^2xyxyyy \\ &= q^2x(yx)yyy & &= q^2x(qxy)yyy & &= q^3xxyyyy \\ yxyxyy &= q^3x^2y^4. \end{aligned}$$

x^2y^4: the path goes from $(0,0)$ to $(2,4)$.
q^3: the original path encloses an area of 3 square units.

An algorithm

Question

Find the number of lattice paths

- of length n
- that go from $(0, 0)$ to $(k, n - k)$
- and enclose an area of j square units

Solution

Expand $(x + y)^n$ according to the q-Weyl laws

$$yx = qxy, \quad qx = xq, \quad qy = yq.$$

Extract the coefficient of $q^j x^k y^{n-k}$.

Examples

$$(x+y)(x+y) = xx + xy + yx + yy$$

So $$\begin{aligned}(x+y)^2 &= xx + xy + qxy + yy \\ &= x^2 + (1+q)xy + y^2.\end{aligned}$$

$$\begin{aligned}(x+y)^3 &= (x+y)(x+y)^2 \\ &= (x+y)(x^2+(1+q)xy+y^2) \\ &= x^3+(1+q)x^2y+xy^2+yx^2+(1+q)yxy+y^3 \\ &= x^3+(1+q)x^2y+xy^2+q^2x^2y+(1+q)qxy^2+y^3 \\ &= x^3+(1+q+q^2)x^2y+(1+q+q^2)xy^2+y^3.\end{aligned}$$

Examples

q-Weyl relations: $\quad yx = qxy, \quad qx = xq, \quad qy = yq$

$$
\begin{aligned}
(x+y)^2 &= x^2 + (1+q)xy + y^2 \\
(x+y)^3 &= x^3 + (1+q+q^2)x^2y + (1+q+q^2)xy^2 + y^3 \\
(x+y)^4 &= x^4 + (1+q+q^2+q^3)x^3y \\
&\quad + (1+q+2q^2+q^3+q^4)x^2y^2 \\
&\quad + (1+q+q^2+q^3)xy^3 + y^4
\end{aligned}
$$

$$(x+y)^n = \sum_{k=0}^{n} c(n,k)x^{n-k}y^k, \qquad c(n,k) = ?$$

$$
\begin{aligned}
&(x+y)^{n+1} \\
&= (x+y)(x+y)^n \\
&= x\sum_{k=0}^{n} c(n,k)x^{n-k}y^k + y\sum_{k=0}^{n} c(n,k)x^{n-k}y^k \\
&= \sum_{k=0}^{n} c(n,k)x^{n+1-k}y^k + \sum_{k=0}^{n} c(n,k)q^{n-k}x^{n-k}y^{k+1}
\end{aligned}
$$

$$
\begin{aligned}
&(x+y)^{n+1} \\
&= (x+y)^n(x+y) \\
&= \sum_{k=0}^{n} c(n,k)x^{n-k}y^k x + \sum_{k=0}^{n} c(n,k)x^{n-k}y^k y \\
&= \sum_{k=0}^{n} c(n,k)q^k x^{n+1-k}y^k + \sum_{k=0}^{n} c(n,k)x^{n-k}y^{k+1}
\end{aligned}
$$

Equate coefficients of $x^{n+1-k}y^k$:

$$c(n,k) + q^{n+1-k}c(n,k-1) = q^k c(n,k) + c(n,k-1)$$

$$c(n,k) = \frac{(1-q^{n+1-k})}{(1-q^k)}c(n,k-1)$$

$$\begin{aligned}
c(4,2) &= \frac{(1-q^3)}{(1-q^2)}c(4,1) \\
&= \frac{(1-q^3)(1-q^4)}{(1-q^2)(1-q)}c(4,0) \\
&= \frac{(1-q^4)(1-q^3)(1-q^2)(1-q)}{(1-q^2)(1-q)(1-q^2)(1-q)}
\end{aligned}$$

cf. $$\binom{4}{2} = \frac{4!}{2!2!} = \frac{4\times 3\times 2\times 1}{2\times 1\times 2\times 1}$$

$$c(4,2) = \frac{(1-q^4)(1-q^3)(1-q^2)(1-q)}{(1-q^2)(1-q)(1-q^2)(1-q)}$$

$$= \frac{\frac{1-q^4}{1-q}\frac{1-q^3}{1-q}\frac{1-q^2}{1-q}\frac{1-q}{1-q}}{\frac{1-q^2}{1-q}\frac{1-q}{1-q}\frac{1-q^2}{1-q}\frac{1-q}{1-q}}$$

$$= \frac{(1+q+q^2+q^3)(1+q+q^2)(1+q)(1)}{(1+q)(1)(1+q)(1)}$$

The q-integer:
$[n]_q = 1 + q + q^2 + q^3 + \cdots + q^{n-1}$ $\qquad$ ($=n$, when $q=1$)

The q-factorial:
$n!_q = [n]_q[n-1]_q \cdots [2]_q[1]_q$ $\qquad$ ($=n!$, when $q=1$)

The solution of

$$c(n,k) = \frac{(1-q^{n+1-k})}{(1-q^k)} c(n,k-1), \qquad c(n,0) = 1$$

is given by the q-binomial coefficient

$$c(n,k) = \left[\begin{array}{c} n \\ k \end{array} \right]_q = \frac{n!_q}{k!_q (n-k)!_q}$$

where

$$n!_q = 1(1+q)(1+q+q^2)\cdots(1+q+q^2+q^3+\cdots+q^{n-1}).$$

The q-binomial theorem

Suppose $yx = qxy$, $qx = xq$ and $qy = yq$. Then

$$(x + y)^n = \sum_{k=0}^{n} \left[\begin{array}{c} n \\ k \end{array} \right]_q x^{n-k} y^k$$

where

$$\left[\begin{array}{c} n \\ k \end{array} \right]_q = \frac{n!_q}{k!_q (n-k)!_q}$$

and

$$n!_q = 1(1+q)(1+q+q^2)\cdots(1+q+q^2+q^3+\cdots+q^{n-1}).$$

Example, again

How many lattice paths:

- have length 9 — Ans: $2^9 = 512$
- and end at $(4, 5)$ — Ans: $\binom{9}{4} = 126$
- and enclose an area of 15 square units? — Ans: ?

$$(x+y)^9 = \sum_{k=0}^{9} \begin{bmatrix} 9 \\ k \end{bmatrix}_q x^k y^{n-k}$$

$$\begin{bmatrix} 9 \\ 4 \end{bmatrix}_q = \frac{(1-q^9)(1-q^8)(1-q^7)(1-q^6)}{(1-q^4)(1-q^3)(1-q^2)(1-q)}$$

$$= q^{20} + q^{19} + 2q^{18} + 3q^{17} + 5q^{16} + 6q^{15} + 8q^{14} + 9q^{13}$$
$$+ 11q^{12} + 11q^{11} + 12q^{10} + 11q^9 + 11q^8 + 9q^7 + 8q^6$$
$$+ 6q^5 + 5q^4 + 3q^3 + 2q^2 + q + 1$$

q-binomial coefficients; also called Gaussian polynomials

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{n!_q}{k!_q(n-k)!_q}$$

$$\begin{aligned} \begin{bmatrix} n+1 \\ k \end{bmatrix}_q &= \begin{bmatrix} n \\ k \end{bmatrix}_q + q^{n+1-k} \begin{bmatrix} n \\ k-1 \end{bmatrix}_q \\ &= q^k \begin{bmatrix} n \\ k \end{bmatrix}_q + \begin{bmatrix} n \\ k-1 \end{bmatrix}_q \end{aligned}$$

The q-binomial coefficient $\begin{bmatrix} n \\ k \end{bmatrix}_q$ is a polynomial in q of degree $k(n-k)$.

$$\begin{bmatrix} n \\ k \end{bmatrix}_q \to \binom{n}{k} \quad \text{as } q \to 1$$

Examples of q-Weyl pairs

What are x and y if $yx = qxy$, $qx = xq$, $qy = yq$?

Obviously, x and y are not numbers (real or complex).

$$x = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & q & 0 & \cdots & 0 \\ 0 & 0 & q^2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ldots & \vdots \\ 0 & 0 & 0 & \cdots & q^{n-1} \end{pmatrix}, \qquad y = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ldots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}.$$

Define operators x and y by

$$x(f(t)) = tf(t), \quad y(f(t)) = f(qt).$$

Then $yx(f(t)) = qxy(f(t))$.

Exercises

Prove that

$$(a+b)(a+qb)(a+q^2b)\cdots(a+q^{n-1}b) = \sum_{k=0}^{n} \left[\begin{array}{c} n \\ k \end{array} \right]_q q^{k(k-1)/2} a^{n-k} b^k,$$

assuming all variables commute.

Prove that

$$\prod_{k=1}^{\infty}(1+xq^{2k-1})(1+x^{-1}q^{2k-1})(1-q^{2k}) = \sum_{k=-\infty}^{\infty} q^{k^2} x^k$$

Hint for Exercise 1

Hints: Write

$$(a+b)(a+qb)(a+q^2b)\cdots(a+q^{n-1}b) = \sum_{k=0}^{n} f(n,k)a^k b^{n-k}$$

Then

$$\begin{aligned}&(a+b)(a+qb)(a+q^2b)\cdots(a+q^{n-1}b)(a+q^nb)\\ &=(a+b)(a+qb)(a+q^2b)\cdots(a+q^{n-1}b)(a+q^nb)\end{aligned}$$

So

$$\sum_{k=0}^{n} f(n,k)a^k b^{n-k}(a+q^nb) = (a+b)\sum_{k=0}^{n} f(n,k)a^k(qb)^{n-k}$$

See where this leads. (When $q=1$, it slips though our fingers...)

Hint for Exercise 2

Start with

$$(a+b)(a+qb)(a+q^2b)\cdots(a+q^{n-1}b)$$
$$=\sum_{k=0}^{n}\left[\begin{array}{c}n\\k\end{array}\right]_q q^{k(k-1)/2}a^{n-k}b^k.$$

Replace q with q^2, n with $2n$, b with xq^{1-2n} and let $a=1$.

Then take the limit as $n\to\infty$.

The result is

$$\prod_{k=1}^{\infty}(1+xq^{2k-1})(1+x^{-1}q^{2k-1})(1-q^{2k})=\sum_{k=-\infty}^{\infty}q^{k^2}x^k$$

Summary

We have seen q-analogues of:

- integers
- factorials
- binomial coefficients
- the binomial theorem

The q-binomial coefficients have combinatorial significance.

The extra variable q allowed us to *deduce* the q-binomial theorem instead of just *verifying* it.

The end!